Shape optimization in superconductivity

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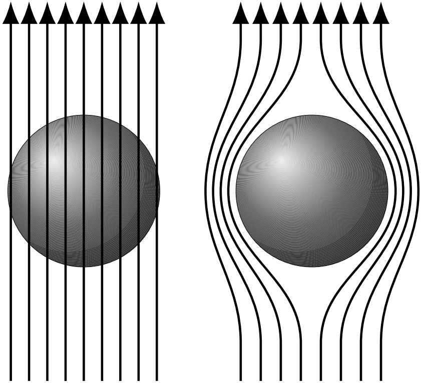
Open-Minded

Physical Background – Superconductivity



Maxwell's equations

$$\begin{cases} \varepsilon \partial_t \boldsymbol{E} - \mathbf{curl}\, \boldsymbol{H} + \boldsymbol{J} = \boldsymbol{f} & \text{in } \Omega \times (0,T), \\ \mu \partial_t \boldsymbol{H} + \mathbf{curl}\, \boldsymbol{E} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{in } \partial\Omega \times (0,T), \\ \boldsymbol{E}(\cdot,0) = \boldsymbol{E}_0 & \text{in } \Omega, \\ \boldsymbol{H}(\cdot,0) = \boldsymbol{H}_0 & \text{in } \Omega, \end{cases}$$

Bean's critical state model:

$$\begin{cases} \boldsymbol{J} \cdot \boldsymbol{E} = j_c(\theta)|\boldsymbol{E}| & \text{a.e. in } \Omega_{sc} \times (0,T), \\ |\boldsymbol{J}| \leq j_c(\theta) & \text{a.e. in } \Omega_{sc} \times (0,T), \\ \boldsymbol{J} = 0 & \text{a.e. in } \Omega \backslash \overline{\Omega_{sc}} \times (0,T). \end{cases}$$

Variational inequality

Combining Maxwell's equations and Bean's model $\rightsquigarrow$ Variational inequality

$$\begin{cases} \int_\Omega \epsilon \frac{d}{dt}\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) + \mu^{-1}\frac{d}{dt}\boldsymbol{B}(t)\cdot(\boldsymbol{w}-\boldsymbol{B}(t))\,dx \\ \quad + \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}(t)\cdot\boldsymbol{w} - \mu^{-1}\boldsymbol{B}(t)\cdot\mathbf{curl}\,\boldsymbol{v}\,dx \\ \quad + \varphi(\theta(t),\boldsymbol{v}) - \varphi(\theta(t),\boldsymbol{E}(t)) \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,dx \\ \text{for a.e. } t\in(0,T) \text{ and every } (\boldsymbol{v},\boldsymbol{w})\in \boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{L}^2(\Omega), \\ (\boldsymbol{E}(0),\boldsymbol{B}(0)) = (\boldsymbol{E}_0,\boldsymbol{B}_0), \end{cases} \tag{VI}$$

with $\varphi\colon L^\infty(\Omega)\times\boldsymbol{L}^2(\Omega)\to\mathbb{R}$ defined by $\varphi(y,\boldsymbol{v}) := \int_\Omega j_c(x,y(x))|\boldsymbol{v}(x)|\,dx$

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Variational inequality

Implicit Euler in time

$$\begin{cases} \displaystyle\int_\Omega \epsilon\delta\boldsymbol{E}^n \cdot (\boldsymbol{v} - \boldsymbol{E}^n) + \mu^{-1}\delta\boldsymbol{B}^n \cdot (\boldsymbol{w} - \boldsymbol{B}^n)\,dx \\ \quad + \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}^n \cdot \boldsymbol{w} - \mu^{-1}\boldsymbol{B}^n \cdot \mathbf{curl}\,\boldsymbol{v}\,dx \\ \quad + \varphi(\theta^n, \boldsymbol{v}) - \varphi(\theta^n, \boldsymbol{E}^n) \geq \displaystyle\int_\Omega \boldsymbol{f}^n \cdot (\boldsymbol{v} - \boldsymbol{E}^n)\,dx \\ \quad \text{for } n \in \{1, \ldots, N\} \text{ and every } (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega), \\ \quad (\boldsymbol{E}^0, \boldsymbol{B}^0) = (\boldsymbol{E}_0, \boldsymbol{B}_0), \end{cases} \tag{VI$_N$}$$

where

$$\delta\boldsymbol{E}^n = \frac{\boldsymbol{E}^n - \boldsymbol{E}^{n-1}}{\tau} \quad \text{and} \quad \delta\boldsymbol{B}^n = \frac{\boldsymbol{B}^n - \boldsymbol{B}^{n-1}}{\tau}$$

Variational inequality

Implicit Euler in time

$$
\begin{cases}
\int_\Omega \epsilon \delta \boldsymbol{E}^n \cdot (\boldsymbol{v} - \boldsymbol{E}^n) + \mu^{-1} \delta \boldsymbol{B}^n \cdot (\boldsymbol{w} - \boldsymbol{B}^n) \, dx \\
\quad + \int_\Omega \mu^{-1} \, \mathbf{curl} \, \boldsymbol{E}^n \cdot \boldsymbol{w} - \mu^{-1} \boldsymbol{B}^n \cdot \mathbf{curl} \, \boldsymbol{v} \, dx \\
\quad + \varphi(\theta^n, \boldsymbol{v}) - \varphi(\theta^n, \boldsymbol{E}^n) \geq \int_\Omega \boldsymbol{f}^n \cdot (\boldsymbol{v} - \boldsymbol{E}^n) \, dx \\
\quad \text{for } n \in \{1, \dots, N\} \text{ and every } (\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{H}_0(\mathbf{curl}) \times \boldsymbol{L}^2(\Omega), \\
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\end{cases}
\tag{VI$_N$}
$$

where

$$
\delta \boldsymbol{E}^n = \frac{\boldsymbol{E}^n - \boldsymbol{E}^{n-1}}{\tau} \quad \text{and} \quad \delta \boldsymbol{B}^n = \frac{\boldsymbol{B}^n - \boldsymbol{B}^{n-1}}{\tau}
$$

Decoupled System

Inserting $\boldsymbol{v} = \boldsymbol{E}^n \rightsquigarrow$ **discrete Faraday's law**

$$\boldsymbol{B}^n = \boldsymbol{B}^{n-1} - \tau \,\mathbf{curl}\, \boldsymbol{E}^n \tag{Fara$_N$}$$

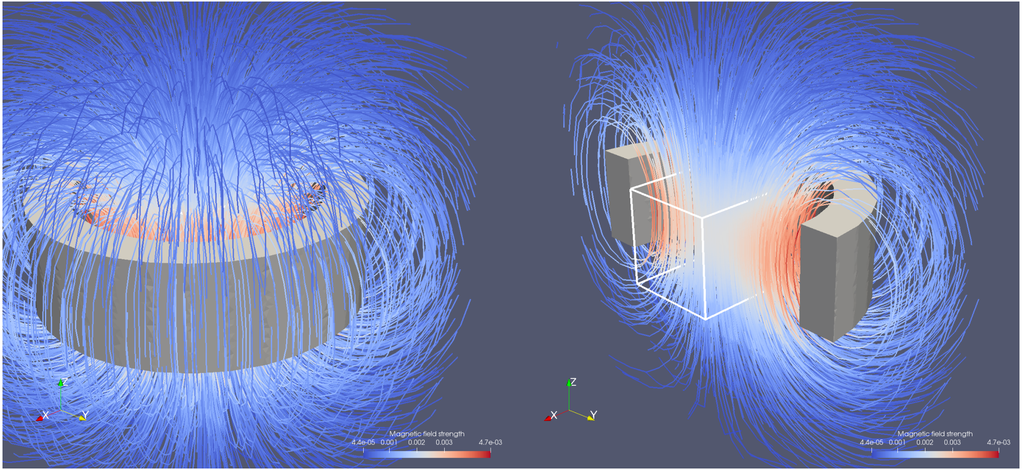
Inserting (Fara$_N$) into (VI$_N$) yields elliptic curl-curl VI

$$\int_\Omega \epsilon\tau^{-1} \boldsymbol{E}^n \cdot (\boldsymbol{v} - \boldsymbol{E}^n)\, dx + \int_\Omega \tau\mu^{-1}\, \mathbf{curl}\, \boldsymbol{E}^n \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}^n)\, dx + \varphi(\boldsymbol{v}) \\ - \varphi(\boldsymbol{E}^n) \geq \int_\Omega \boldsymbol{f}^n \cdot (\boldsymbol{v} - \boldsymbol{E}^n) + (\mu^{-1}\boldsymbol{B}^{n-1} + \epsilon\tau^{-1}\boldsymbol{E}^{n-1}) \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{E}^n)\, dx$$

$\rightsquigarrow$ Fully discrete scheme with convergence analysis yields well-posedness for (VI) with temperature effects[1]

[1] M. W. and I. Yousept, Fully discrete scheme for Bean's critical-state model with temperature effects in superconductivity, minor revision in *SIAM J. Numer. Anal.*, 2019.

Motivation for shape optimization



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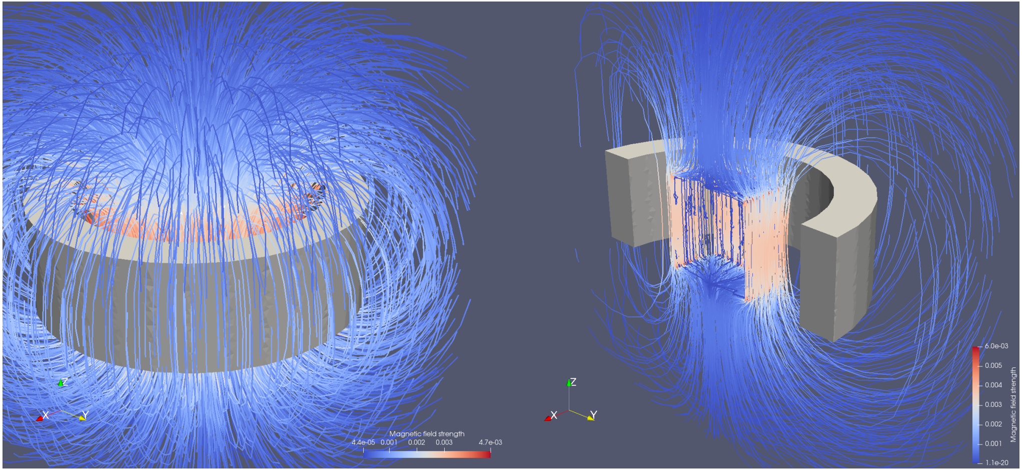

Shape optimization problem

The minimization problem for some $B \subset \Omega$

$$\min_{\omega \in \mathcal{O}} J(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx \tag{P}$$

Set of admissible shapes $\mathcal{O} = \{\omega \subset B \mid \omega \text{ is open, } L\text{-Lipschitz}\}$ where $\boldsymbol{E}(\omega)$ is the unique solution to

$$a(\boldsymbol{E}(\omega), \boldsymbol{v} - \boldsymbol{E}(\omega)) + \int_\omega j_c |\boldsymbol{v}| \, dx - \int_\omega j_c |\boldsymbol{E}(\omega)| \, dx \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E}(\omega)) \, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \tag{VI$_\omega$}$$

with

$$a(\boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \nu \, \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{w} \, dx + \int_\Omega \varepsilon \boldsymbol{v} \cdot \boldsymbol{w} \, dx,$$

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Set of admissible shapes $\mathcal{O} = \{\omega \subset B \mid \omega \text{ is open, } L\text{-Lipschitz}\}$ where $\boldsymbol{E}(\omega)$ is the unique solution to

$$\begin{cases} a(\boldsymbol{E}, \boldsymbol{v}) + \int_\omega \boldsymbol{\lambda} \cdot \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx & \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ |\boldsymbol{\lambda}(x)| \leq j_c(x), & \text{for a.e. } x \in \omega \\ \boldsymbol{\lambda}(x) \cdot \boldsymbol{E}(x) = j_c(x) |\boldsymbol{E}(x)| & \text{for a.e. } x \in \omega \end{cases}$$

Problem: **Not differentiable!** $\rightsquigarrow$ Regularization technique.

Regularization

Regularized minimization problem

$$\min_{\omega \in \mathcal{O}} J_\gamma(\omega) := \frac{1}{2} \int_B \kappa |\boldsymbol{E}^\gamma(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx \qquad (\mathsf{P}_\gamma)$$

subject to

$$a(\boldsymbol{E}^\gamma, \boldsymbol{v} - \boldsymbol{E}^\gamma) + \varphi_\gamma(\boldsymbol{v}) - \varphi_\gamma(\boldsymbol{E}^\gamma) \geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{E}^\gamma) \, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$$

with $\varphi_\gamma \colon \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ given by

$$\varphi_\gamma(\boldsymbol{v}) := \int_\Omega j_c \psi_\gamma(\boldsymbol{v}) \, dx, \quad \psi_\gamma(x) := \begin{cases} |x| - \dfrac{1}{2\gamma} & , \text{ if } |x| \geq \dfrac{1}{\gamma} \\ \dfrac{\gamma}{2}|x|^2 & , \text{ else.} \end{cases}$$

Regularization

Regularized minimization problem

$$\min_{\omega \in \mathcal{O}} J_\gamma(\omega) := \frac{1}{2}\int_B \kappa |\boldsymbol{E}^\gamma(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega dx \qquad (\mathsf{P}_\gamma)$$

subject to

$$\begin{cases} a(\boldsymbol{E}^\gamma, \boldsymbol{v}) + \int_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \cdot \boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma)(x) = \dfrac{j_c \gamma \boldsymbol{E}^\gamma(x)}{\max\{1, \gamma |\boldsymbol{E}^\gamma(x)|\}} \text{ for a.e. } x \in \omega. \end{cases}$$

Problem: **Still not differentiable!** $\rightsquigarrow$ Another regularization technique.

Regularization

Regularized minimization problem

$$\min_{\omega \in \mathcal{O}} J_\gamma(\omega) := \frac{1}{2}\int_B \kappa |\boldsymbol{E}^\gamma(\omega) - \boldsymbol{E}_d|^2 \, dx + \int_\omega \, dx \qquad (\mathsf{P}_\gamma)$$

subject to

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$\rightsquigarrow$ Well-posedness via **compactness result** for $\mathcal{O}$.

Shape sensitivity analysis

- Shape derivative at $\omega \in \mathcal{O}$:

$$dJ(\omega)(\boldsymbol{\theta}) := \lim_{t \searrow 0} \frac{J(\omega_t) - J(\omega)}{t},$$

where $\omega_t = \boldsymbol{T}_t(\omega)$ with $\boldsymbol{T}_t : \Omega \to \Omega$ the flow of a vector field $\boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(B, \mathbb{R}^3)$

- Lagrangian approach:

$$\mathcal{L}(\omega, \boldsymbol{e}, \boldsymbol{v}) := \frac{1}{2} \int_\Omega \kappa |\boldsymbol{e} - \boldsymbol{E}_d|^2 \, dx + \int_\omega \, dx + a(\boldsymbol{e}, \boldsymbol{v}) + \int_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{e}) \cdot \boldsymbol{v} \, dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx$$

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- Pull back: $\int_{\omega_t} \to \int_\omega$ with $x \mapsto \boldsymbol{T}_t(x)$ $\rightsquigarrow$ terms like $\boldsymbol{e} \circ \boldsymbol{T}_t$ with $\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl})$

Shape-Lagrangian

- Reparametrization $\boldsymbol{H}_0^1(\Omega) = \{\boldsymbol{e} \circ \boldsymbol{T}_t^{-1} \, : \, \boldsymbol{e} \in \boldsymbol{H}_0^1(\Omega)\} \rightsquigarrow$ **Not** for $\boldsymbol{H}_0(\mathbf{curl})$!
- **Covariant transformation**

$$\Psi_t \colon \boldsymbol{H}(\mathbf{curl}, \omega) \to \boldsymbol{H}(\mathbf{curl}, \omega_t), \qquad \Psi_t(\boldsymbol{e}) := (D\boldsymbol{T}_t^{-\mathsf{T}} \boldsymbol{e}) \circ \boldsymbol{T}_t^{-1}.$$

  with important identity

$$\big(\mathbf{curl}\, \Psi_t(\boldsymbol{e})\big) \circ \boldsymbol{T}_t = \xi(t)^{-1} D\boldsymbol{T}_t \, \mathbf{curl}\, \boldsymbol{e},$$

- Shape-Lagrangian

$$G(t, \boldsymbol{e}, \boldsymbol{v}) := \mathcal{L}(\omega_t, \Psi_t(\boldsymbol{e}), \Psi_t(\boldsymbol{v})) = \frac{1}{2} \int_B \kappa |\Psi_t(\boldsymbol{e}) - \boldsymbol{E}_d|^2 \, dx + \int_{\omega_t} dx$$
$$+ a(\Psi_t(\boldsymbol{e}), \Psi_t(\boldsymbol{v})) + \int_{\omega_t} \boldsymbol{\Lambda}_\gamma(\Psi_t(\boldsymbol{e})) \cdot \Psi_t(\boldsymbol{v}) \, dx - \int_\Omega \boldsymbol{f} \cdot \Psi_t(\boldsymbol{v}) \, dx.$$

Averaged adjoint method

Shape-Lagrangian after change of variables $x \mapsto \boldsymbol{T}_t(x)$:

$$G(t,\boldsymbol{e},\boldsymbol{v}) = \frac{1}{2}\int_B \kappa \circ \boldsymbol{T}_t |D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e} - \boldsymbol{E}_d \circ \boldsymbol{T}_t|^2 \xi(t)\,dx + \int_\omega \xi(t)\,dx$$
$$+ \int_\Omega \mathbb{M}_1(t)\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\boldsymbol{v} + \mathbb{M}_2(t)\boldsymbol{e}\cdot\boldsymbol{v}\,dx$$
$$+ \int_\omega \mathbb{M}_3(t,\boldsymbol{e})\cdot\boldsymbol{v}\,dx - \int_\Omega (\boldsymbol{f}\circ\boldsymbol{T}_t)\cdot(D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{v})\xi(t)\,dx.$$

with the notations

$$\mathbb{M}_1(t) := \xi(t)^{-1} D\boldsymbol{T}_t^{\mathsf{T}}(\nu\circ\boldsymbol{T}_t)D\boldsymbol{T}_t,$$
$$\mathbb{M}_2(t) := \xi(t)D\boldsymbol{T}_t^{-1}(\varepsilon\circ\boldsymbol{T}_t)D\boldsymbol{T}_t^{-\mathsf{T}},$$
$$\mathbb{M}_3(t,\boldsymbol{e}) := \xi(t)D\boldsymbol{T}_t^{-1}\boldsymbol{\Lambda}_\gamma(D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e}).$$

Averaged adjoint method

Shape-Lagrangian after change of variables $x \mapsto \boldsymbol{T}_t(x)$:

$$G(t,\boldsymbol{e},\boldsymbol{v}) = \frac{1}{2}\int_B \kappa\circ\boldsymbol{T}_t|D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e} - \boldsymbol{E}_d\circ\boldsymbol{T}_t|^2\xi(t)\,dx + \int_\omega \xi(t)\,dx$$
$$+ \int_\Omega \mathbb{M}_1(t)\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\boldsymbol{v} + \mathbb{M}_2(t)\boldsymbol{e}\cdot\boldsymbol{v}\,dx$$
$$+ \int_\omega \mathbb{M}_3(t,\boldsymbol{e})\cdot\boldsymbol{v}\,dx - \int_\Omega (\boldsymbol{f}\circ\boldsymbol{T}_t)\cdot(D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{v})\xi(t)\,dx.$$

⇝ Compute shape derivative as $\partial_t G(t,\boldsymbol{E}_t^\gamma,\boldsymbol{v})$ via AAM[2,3].

[2] K. Sturm. Minimax Lagrangian approach to the differentiability of nonlinear PDE constrained shape functions without saddle point assumption. *SIAM J. Control Optim.*, 53(4):2017–2039, 2015.

[3] A. Laurain and K. Sturm. Distributed shape derivative *via* averaged adjoint method and applications. *ESAIM Math. Model. Numer. Anal.*, 50(4):1241–1267, 2016.

Averaged adjoint method

Shape-Lagrangian after change of variables $x \mapsto \boldsymbol{T}_t(x)$:

$$G(t, \boldsymbol{e}, \boldsymbol{v}) = \frac{1}{2}\int_B \kappa \circ \boldsymbol{T}_t |D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e} - \boldsymbol{E}_d \circ \boldsymbol{T}_t|^2 \xi(t)\, dx + \int_\omega \xi(t)\, dx$$
$$+ \int_\Omega \mathbb{M}_1(t)\, \mathbf{curl}\, \boldsymbol{e} \cdot \mathbf{curl}\, \boldsymbol{v} + \mathbb{M}_2(t)\boldsymbol{e} \cdot \boldsymbol{v}\, dx$$
$$+ \int_\omega \mathbb{M}_3(t, \boldsymbol{e}) \cdot \boldsymbol{v}\, dx - \int_\Omega (\boldsymbol{f} \circ \boldsymbol{T}_t) \cdot (D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{v})\xi(t)\, dx.$$

(H0) the mapping $[0,1] \ni s \mapsto G(t, s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma, \boldsymbol{v})$ is absolutely continuous;

(H1) the mapping $[0,1] \ni s \mapsto \partial_{\boldsymbol{e}} G(t, s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma, \boldsymbol{v}; \hat{\boldsymbol{e}})$ belongs to $L^1(0,1)$ for every $\hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl})$;

Averaged adjoint method

Shape-Lagrangian after change of variables $x \mapsto \boldsymbol{T}_t(x)$:

$$G(t,\boldsymbol{e},\boldsymbol{v}) = \frac{1}{2}\int_B \kappa \circ \boldsymbol{T}_t |D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{e} - \boldsymbol{E}_d \circ \boldsymbol{T}_t|^2 \xi(t)\,dx + \int_\omega \xi(t)\,dx$$
$$+ \int_\Omega \mathbb{M}_1(t)\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\boldsymbol{v} + \mathbb{M}_2(t)\boldsymbol{e}\cdot\boldsymbol{v}\,dx$$
$$+ \int_\omega \mathbb{M}_3(t,\boldsymbol{e})\cdot\boldsymbol{v}\,dx - \int_\Omega (\boldsymbol{f}\circ\boldsymbol{T}_t)\cdot(D\boldsymbol{T}_t^{-\mathsf{T}}\boldsymbol{v})\xi(t)\,dx.$$

(H2) there exists a unique $\boldsymbol{P}_t^\gamma \in \boldsymbol{H}_0(\mathbf{curl})$ that solves the averaged adjoint equation

$$\int_0^1 \partial_{\boldsymbol{e}} G(t, s\boldsymbol{E}_t^\gamma + (1-s)\boldsymbol{E}_0^\gamma, \boldsymbol{P}_t^\gamma; \hat{\boldsymbol{e}})\,ds = 0 \quad \forall \hat{\boldsymbol{e}} \in \boldsymbol{H}_0(\mathbf{curl});$$

Averaged adjoint method

Shape-Lagrangian after change of variables $x \mapsto \boldsymbol{T}_t(x)$:

$$G(t, \boldsymbol{e}, \boldsymbol{v}) = \frac{1}{2} \int_B \kappa \circ \boldsymbol{T}_t | D\boldsymbol{T}_t^{-\mathsf{T}} \boldsymbol{e} - \boldsymbol{E}_d \circ \boldsymbol{T}_t |^2 \xi(t) \, dx + \int_\omega \xi(t) \, dx$$
$$+ \int_\Omega \mathbb{M}_1(t) \, \mathbf{curl} \, \boldsymbol{e} \cdot \mathbf{curl} \, \boldsymbol{v} + \mathbb{M}_2(t) \boldsymbol{e} \cdot \boldsymbol{v} \, dx$$
$$+ \int_\omega \mathbb{M}_3(t, \boldsymbol{e}) \cdot \boldsymbol{v} \, dx - \int_\Omega (\boldsymbol{f} \circ \boldsymbol{T}_t) \cdot (D\boldsymbol{T}_t^{-\mathsf{T}} \boldsymbol{v}) \xi(t) \, dx.$$

(H3) the sequence $\{\boldsymbol{P}_t^\gamma\}_{t \geq 0}$ satisfies

$$\lim_{t \searrow 0} \frac{G(t, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_t^\gamma) - G(0, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_t^\gamma)}{t} = \partial_t G(0, \boldsymbol{E}_0^\gamma, \boldsymbol{P}_0^\gamma).$$

Shape derivative

Via averaged adjoint method:

$$dJ_\gamma(\omega)(\boldsymbol{\theta}) = \partial_t G(0, \boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma) = \int_\Omega \boldsymbol{S}_1^\gamma : D\boldsymbol{\theta} + \boldsymbol{S}_0^\gamma \cdot \boldsymbol{\theta}\, dx,$$

with

$$\begin{aligned}
\boldsymbol{S}_1^\gamma = {} & [\frac{\kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 + \chi_\omega - \nu\, \mathbf{curl}\, \boldsymbol{E}^\gamma \cdot \mathbf{curl}\, \boldsymbol{P}^\gamma + \varepsilon \boldsymbol{E}^\gamma \cdot \boldsymbol{P}^\gamma + \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \cdot \boldsymbol{P}^\gamma \\
& - \boldsymbol{f} \cdot \boldsymbol{P}^\gamma] \boldsymbol{I}_3 - \kappa \boldsymbol{E}^\gamma \otimes (\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + \nu\, \mathbf{curl}\, \boldsymbol{E}^\gamma \otimes \mathbf{curl}\, \boldsymbol{P}^\gamma \\
& + \nu^\mathsf{T}\, \mathbf{curl}\, \boldsymbol{P}^\gamma \otimes \mathbf{curl}\, \boldsymbol{E}^\gamma - \boldsymbol{P}^\gamma \otimes \varepsilon \boldsymbol{E}^\gamma - \boldsymbol{E}^\gamma \otimes \varepsilon^\mathsf{T} \boldsymbol{P}^\gamma + \boldsymbol{P}^\gamma \otimes \boldsymbol{f} \\
& - \chi_\omega \boldsymbol{\Lambda}_\gamma(\boldsymbol{E}^\gamma) \otimes \boldsymbol{P}^\gamma - \boldsymbol{E}^\gamma \otimes \boldsymbol{\psi}^\gamma(\boldsymbol{E}^\gamma) \boldsymbol{P}^\gamma, \\
\boldsymbol{S}_0^\gamma = {} & \frac{\nabla\kappa}{2}|\boldsymbol{E}^\gamma - \boldsymbol{E}_d|^2 - \kappa D\boldsymbol{E}_d^\mathsf{T}(\boldsymbol{E}^\gamma - \boldsymbol{E}_d) + (D\nu^\mathsf{T}\, \mathbf{curl}\, \boldsymbol{E}^\gamma)\, \mathbf{curl}\, \boldsymbol{P}^\gamma \\
& + (D\epsilon^\mathsf{T} \boldsymbol{E}^\gamma) \boldsymbol{P}^\gamma - D\boldsymbol{f}^\mathsf{T} \boldsymbol{P}^\gamma.
\end{aligned}$$

Stability analysis

Theorem

Let $\omega \in \mathcal{O}$. Then, the following stability estimate holds

$$|dJ_\gamma(\omega)(\boldsymbol{\theta})| \le C\|\boldsymbol{\theta}\|_{\boldsymbol{\mathcal{C}}^{0,1}(\Omega)} \quad \forall \boldsymbol{\theta} \in \boldsymbol{\mathcal{C}}_c^{0,1}(\Omega)$$

with a constant $C = C(j_c, \kappa, \epsilon, \nu, \boldsymbol{f}, \Omega, \boldsymbol{E}_d)$ (given explicitly).

- ⇝ Estimate $\boldsymbol{E}^\gamma, \boldsymbol{P}^\gamma$ via calculations with state and adjoint equation.
- ⇝ Then estimate $\boldsymbol{S}_1^\gamma$ and $\boldsymbol{S}_0^\gamma$.

Convergence analysis

Theorem

For $\gamma_n \to \infty$ as $n \to \infty$ we have (for a subsequence) that there exists $\omega^* \in \mathcal{O}$

$$\omega^{\gamma_n} \to \omega^*$$

and ω^* is an optimal solution to (P). Moreover,

$$\lim_{n\to\infty} \|\boldsymbol{E}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{E}(\omega^*)\|_{\boldsymbol{H}(\mathbf{curl})} = 0$$

$$\lim_{n\to\infty} \|\boldsymbol{\lambda}^{\gamma_n}(\omega^{\gamma_n}) - \boldsymbol{\lambda}(\omega^*)\|_{\boldsymbol{H}_0(\mathbf{curl})^*} = 0$$

Numerics

Algorithm 1 Level set algorithm

1: Set $k = 0$ and choose an initial level-set function ϕ_0 and $\omega_0 = \{x \in B \,|\, \phi_0(x) < 0\}$
2: Solve state equation and adjoint equation
3: Compute descent direction $\boldsymbol{\theta}_k$ by solving

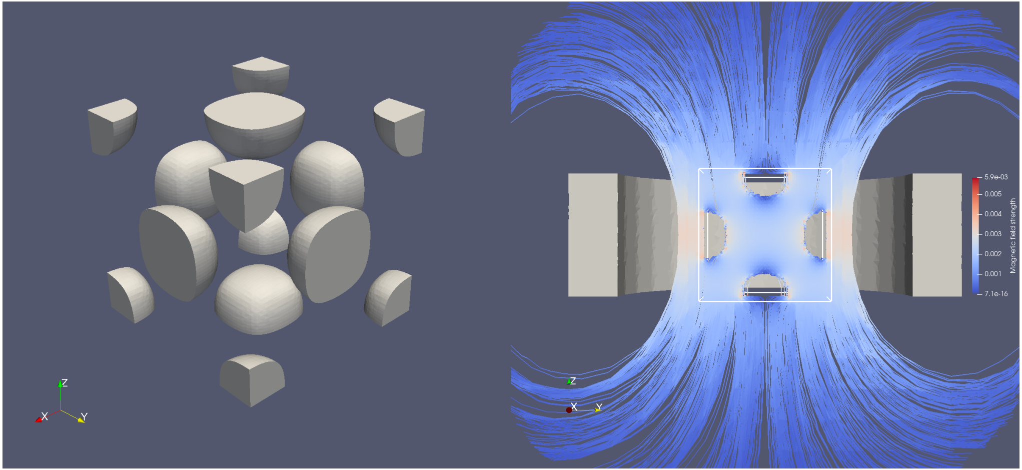
$$\mathcal{B}(\boldsymbol{\theta}, \boldsymbol{\xi}) = -dJ_\gamma(\omega_k)(\boldsymbol{\xi})$$

4: Solve Hamilton-Jacobi equations

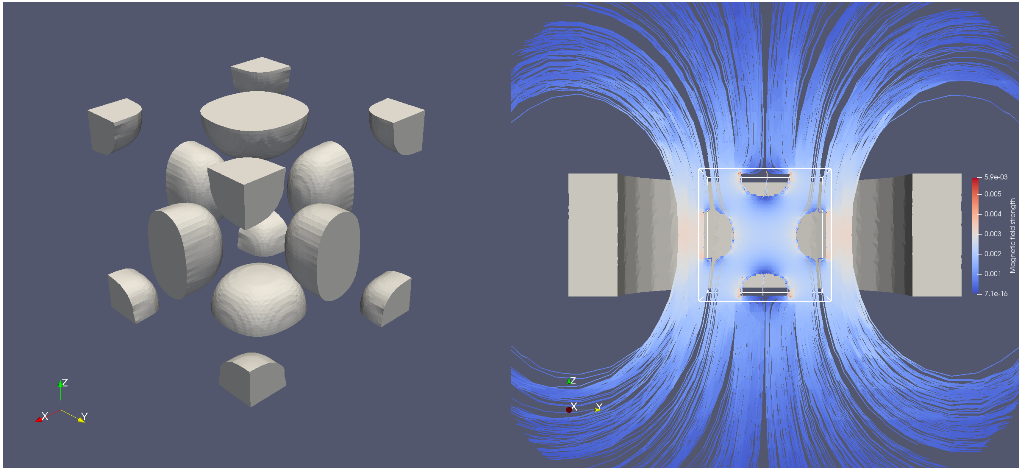
$$\partial_t \phi(x,t) + \boldsymbol{\theta}_k(x)\nabla\phi(x,t) = 0 \quad \text{in } B \times \mathbb{R}^+, \qquad \phi(x,0) = \phi_k(x)$$

5: Update $\phi_{k+1}(x) = \phi(x, \Delta t_k)$ and $\omega_{k+1} = \{x \in B \,|\, \phi_{k+1}(x) < 0\}$.
6: Set $k = k + 1$ and go to step 2 unless some stopping criterion is satisfied.

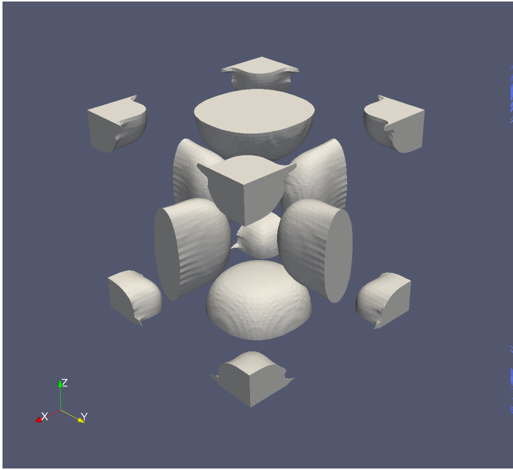
Numerical experiments – Iteration 0

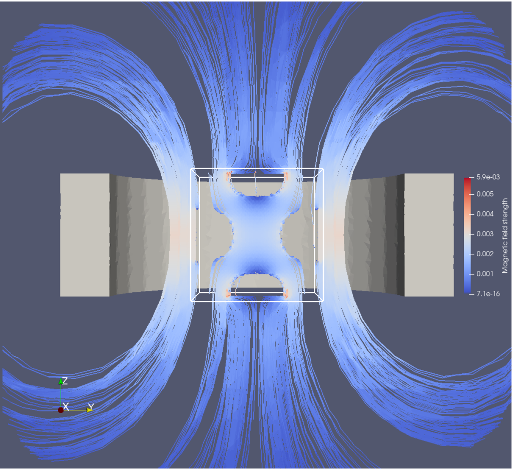


Numerical experiments – Iteration 10

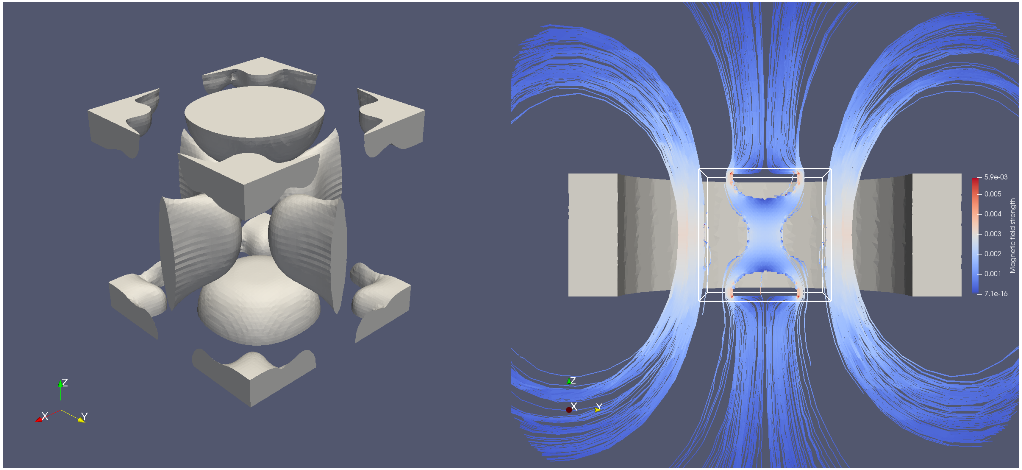


Numerical experiments – Iteration 20

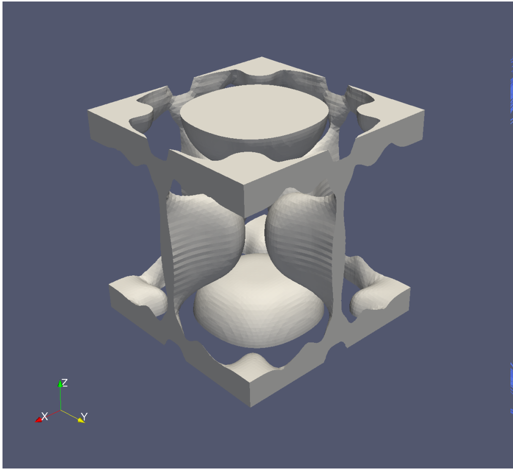


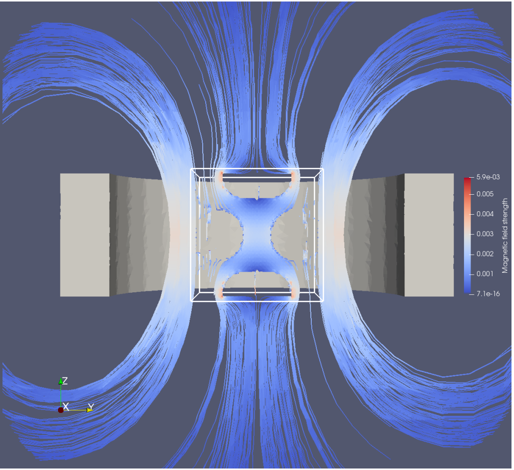


Numerical experiments – Iteration 35

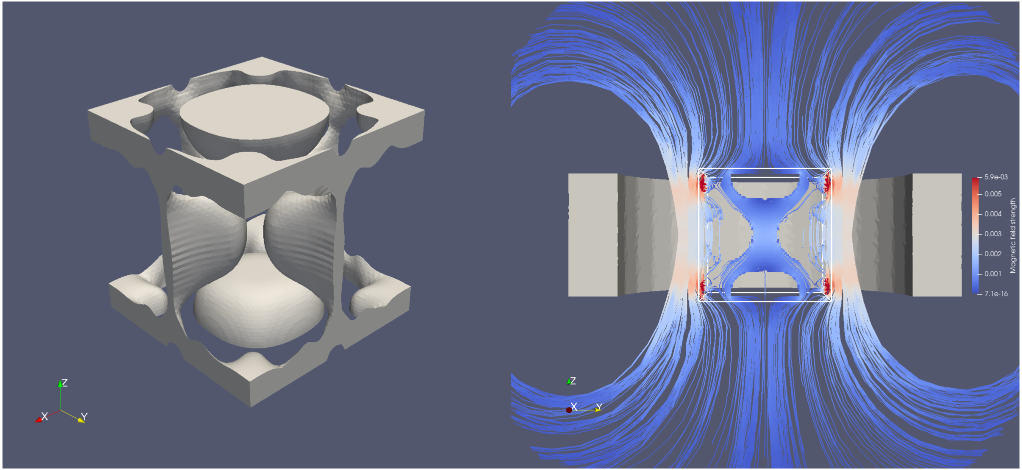


Numerical experiments – Iteration 45

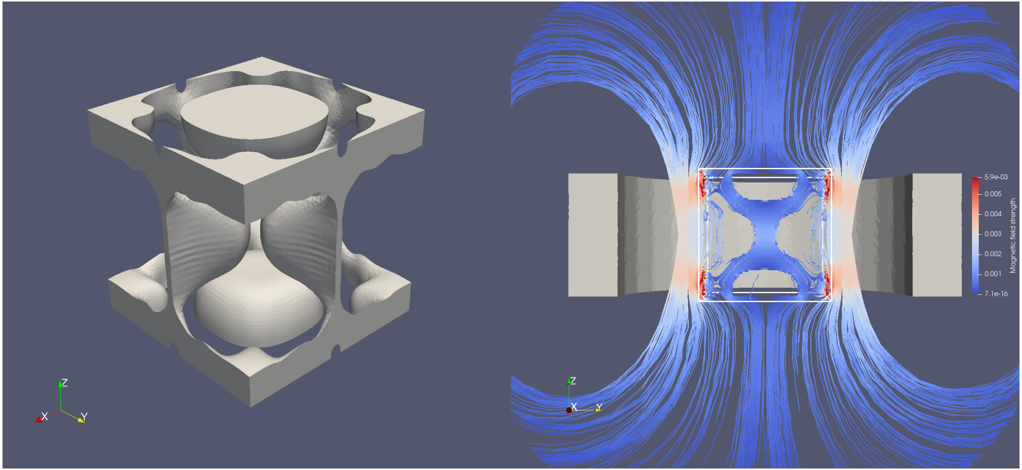




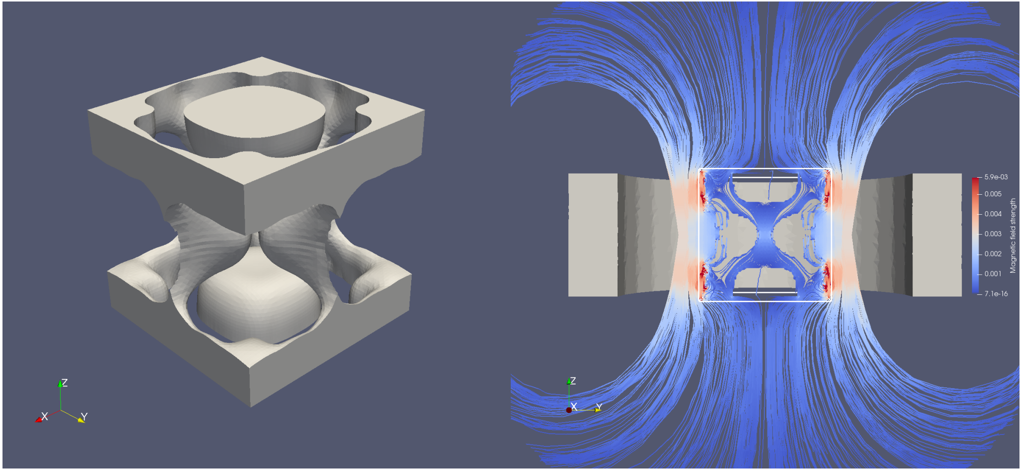
Numerical experiments – Iteration 50



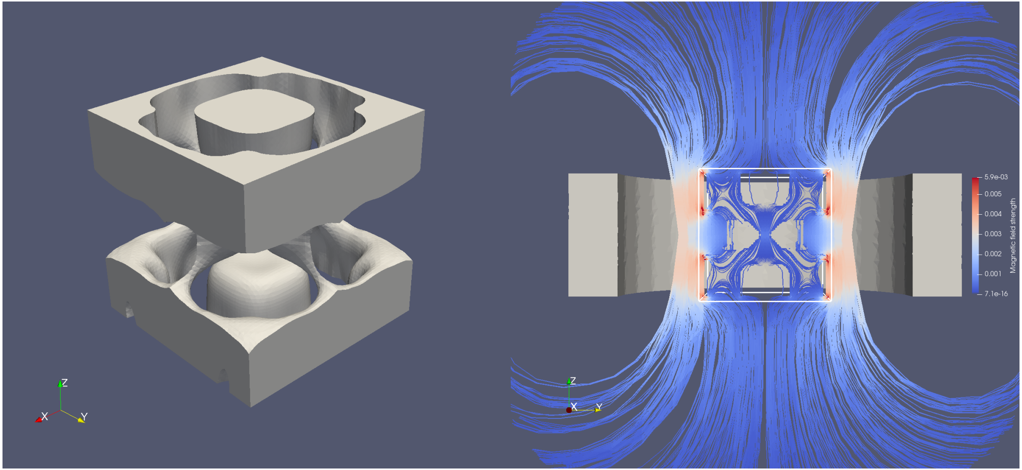
Numerical experiments – Iteration 60



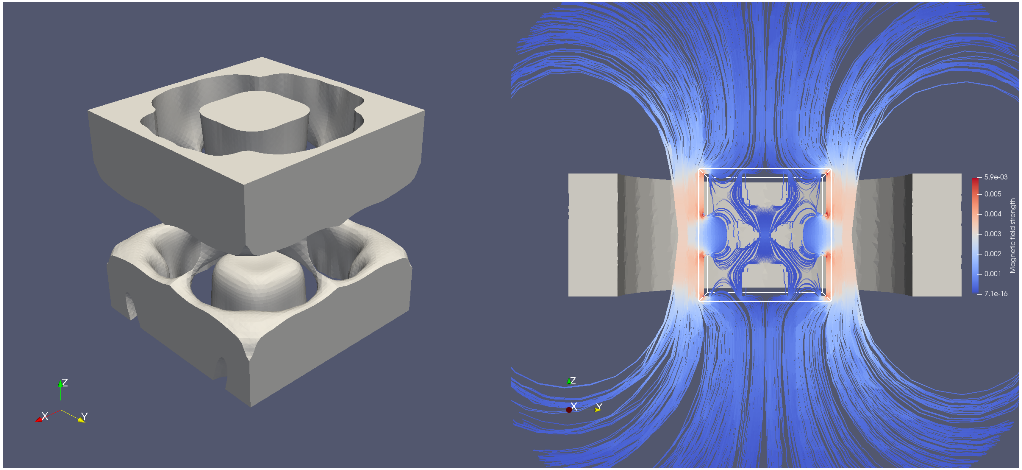
Numerical experiments – Iteration 75



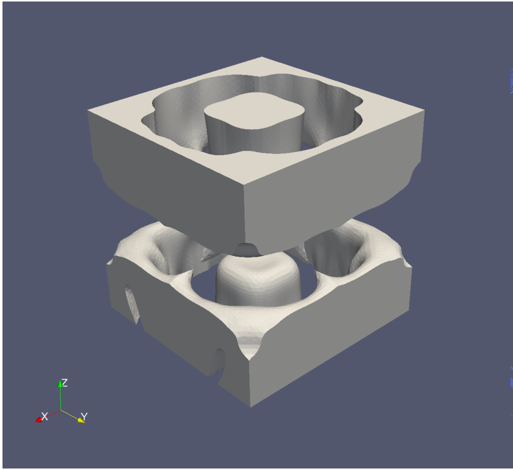
Numerical experiments – Iteration 100

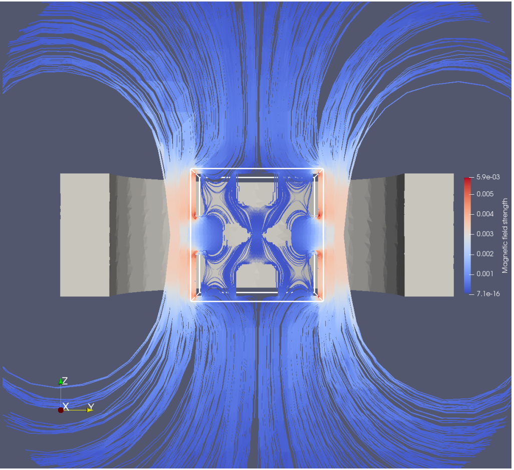


Numerical experiments – Iteration 110

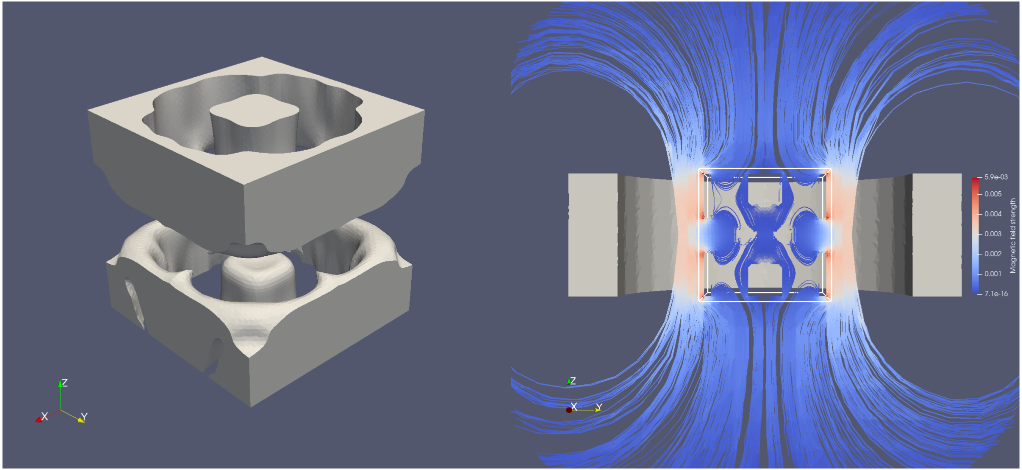


Numerical experiments – Iteration 120

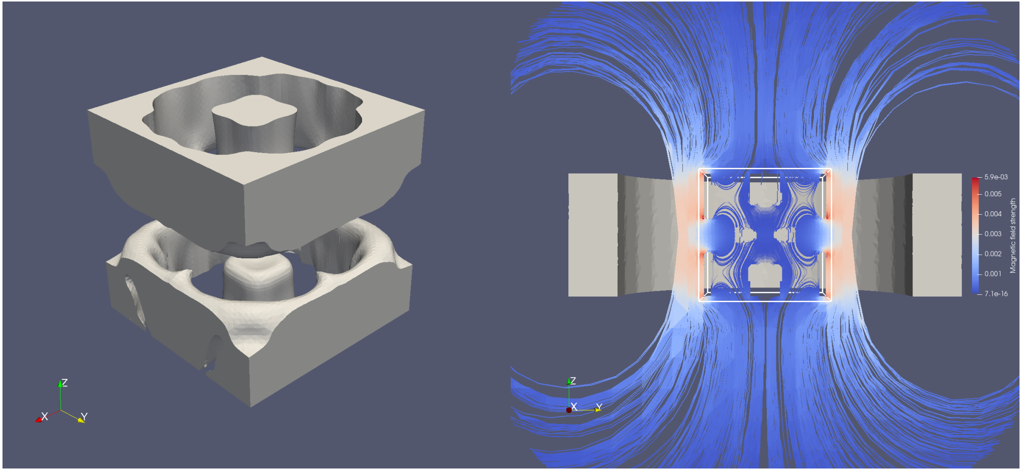




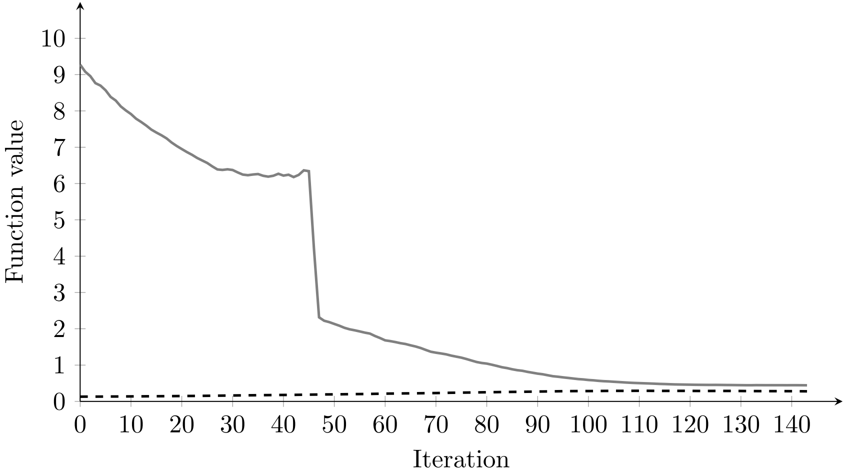
Numerical experiments – Iteration 135



Numerical experiments – Iteration 143



Functional value



Thank you!