# A Priori Error Analysis for an Optimal Control Problem Governed by a Variational Inequality of the Second Kind

Monika Weymuth

Universität der Bundeswehr München
Institut für Mathematik und Computergestützte Simulation

joint work with Christian Meyer (TU Dortmund)

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# Problem Formulation

## Problem (P)

$$\min_{(y,u)\in H_0^1(\Omega)\times L^2(\Omega)} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

$$\text{s.t.} \int_\Omega \nabla y\cdot\nabla(v-y)\,\mathrm{d}x + \|v\|_{L^1(\Omega)} - \|y\|_{L^1(\Omega)} \geq \langle u, v-y\rangle \quad \forall v\in H_0^1(\Omega)$$

- $\Omega \subset \mathbb{R}^d$ $(d = 1, 2)$ a bounded domain with $C^{1,1}$-boundary
- $y_d \in L^2(\Omega)$, $\alpha > 0$

The control-to-state operator $S : H^{-1}(\Omega) \to H_0^1(\Omega)$, $u \mapsto y$ is in general nonlinear and not Gâteaux-differentiable

## Lemma (Existence and Uniqueness, Lipschitz Continuity of $S$)

*For every $u \in H^{-1}(\Omega)$ the VI has a unique solution $y \in H_0^1(\Omega)$.*
*The solution operator $S$ is globally Lipschitz continuous.*

# Known Results

## Lemma (Complementarity)

*A function $y \in H_0^1(\Omega)$ solves VI, iff there exists a $q \in L^\infty(\Omega)$ such that*

$$\int_\Omega \nabla y \cdot \nabla v \, \mathrm{d}x + \int_\Omega qv \, \mathrm{d}x = \langle u, v \rangle \quad \forall v \in H_0^1(\Omega)$$
$$q(x)y(x) = |y(x)|, \quad |q(x)| \leq 1 \quad a.e. \ in \ \Omega.$$

*Hence, if $u \in L^p(\Omega)$, $p \in (1, \infty)$, then $y \in W_0^{2,p}(\Omega)$.*

## Proposition (Existence of Global Optima)

*There exists a globally optimal solution of (P) which is in general not unique.*

## Proposition (Regularity of Optimal Solutions)

*Every locally optimal solution satisfies $\overline{u} \in H^1(\Omega)$.*

# Variational Discretization

## Problem $(P_h)$

$$\min_{(y_h,u)\in V_h\times L^2(\Omega_h)} J(y_h,u) := \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega_h)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega_h)}$$

$$\text{s.t.} \int_{\Omega_h} \nabla y_h \cdot \nabla(v_h - y_h)\,\mathrm{d}x + \|v_h\|_{L^1(\Omega_h)} - \|y_h\|_{L^1(\Omega_h)} \geq \langle u, v_h - y_h\rangle \;\; \forall v_h \in V_h$$

- $\mathcal{T}_h$ shape-regular and quasi-uniform triangulation with mesh size $h$
- $\Omega_h = \bigcup_{T\in\mathcal{T}_h} T \subseteq \Omega, \quad \max_{x\in\partial\Omega_h} \mathrm{dist}(x,\partial\Omega) \leq Ch^2$
- $V_h := \{v_h \in H^1_0(\Omega_h) : \; v_h|_T \in \mathbb{P}_1(T)\; \forall T \in \mathcal{T}_h\}$
- No discretization of the control!

## Lemma (Existence and Uniqueness)

*For all $u \in H^{-1}(\Omega)$ the discrete VI has a unique solution $y_h \in V_h$.*

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# Properties of the Discrete Problem

Solution operator of discrete VI: $S_h : H^{-1}(\Omega) \to V_h \subset H_0^1(\Omega)$, $u \mapsto y_h$

**Lemma (Lipschitz Continuity of $S_h$)**

*The solution operator $S_h$ is globally Lipschitz continuous.*

**Proposition (Existence of Global Optima)**

*Problem $(P_h)$ has a solution which is in general not unique.*

**Proposition (Variational Discretization = Full Discretization)**

*If $\overline{u}_h$ is a local optimal solution of $(P_h)$, then $\overline{u}_h \in V_h$.*

# $L^\infty$-Error Estimates for the State

- No classical Nitsche-trick in $L^2$ due to lack of regularity of the dual problem
- Use $L^\infty$-error estimates to circumvent this difficulty

**Theorem**

*If $u \in L^p(\Omega)$, $p \in (1, \infty)$, then there exists a constant $C > 0$ such that*

$$\|y - y_h\|_{L^\infty(\Omega)} \leq C|\log(h)|h^{2-d/p}(\|u\|_{L^p(\Omega)} + 1).$$

*If $u \in L^\infty(\Omega)$, then there exists a constant $C > 0$ such that*

$$\|y - y_h\|_{L^\infty(\Omega)} \leq C(h|\log(h)|)^2(\|u\|_{L^\infty(\Omega)} + 1).$$

*$C$ is independent of $h$.*

The proof is based on Nochetto 1988.

# Quadratic Growth Condition and Strong Convergence of $\overline{u}_h$

Let $\overline{u} \in L^2(\Omega)$ be a fixed local optimum of (P).

## Quadratic growth condition (QGC)

A local solution $\overline{u} \in L^2(\Omega)$ fulfills the quadratic growth condition, if there are $\epsilon,\ \delta > 0$ such that

$$J(S(\overline{u}),\overline{u}) \leq J(S(u),u) - \delta\|u-\overline{u}\|^2_{L^2(\Omega)} \quad \forall u \in L^2(\Omega):\ \|u-\overline{u}\|_{L^2(\Omega)} \leq \epsilon.$$

## Lemma (Strong Convergence in $L^2(\Omega)$)

*Suppose that $\overline{u}$ satisfies (QGC). Then there is a sequence $\{\overline{u}_h\}$ of locally optimal solutions to $(P_h)$ with $\overline{u}_h \to \overline{u}$ in $L^2(\Omega)$ as $h \to 0$.*

# Strong Convergence and Uniform Boundedness of $\overline{u}_h$

Proof of strong convergence:

- Consider localized problem $(P_h^\epsilon)$

$$\min J(S_h(u), u) = \frac{1}{2}\|S_h(u) - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \quad \text{s.t. } \|u - \overline{u}\|_{L^2(\Omega)} \leq \epsilon$$

- Global solutions $\overline{u}_h$ of $(P_h^\epsilon)$ converge strongly in $L^2(\Omega)$
- Local optimality of $\overline{u}_h$ for $(P_h)$ for $h > 0$ sufficiently small (Casas/Tröltzsch 2002)

**Lemma (Uniform Boundedness in $H^1(\Omega)$)**

*The sequence $\{\overline{u}_h\}$ is uniformly bounded in $H^1(\Omega)$.*

# Convergence Rates

## Theorem (1D)

*Let $\Omega \subset \mathbb{R}$. If $\overline{u}$ satisfies the quadratic growth condition, then there exists a constant $C > 0$ such that, for $h > 0$ sufficiently small,*

$$\|\overline{u} - \overline{u}_h\|_{L^2(\Omega)} \leq Ch|\log(h)|.$$

## Theorem (2D)

*Let $\Omega \subset \mathbb{R}^2$ be sufficiently regular. If $\overline{u}$ satisfies the quadratic growth condition, then, for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for $h > 0$ sufficiently small,*

$$\|\overline{u} - \overline{u}_h\|_{L^2(\Omega)} \leq C_\epsilon h^{1-\epsilon}.$$

- Convergence rate in 3D: $\sqrt{|\log(h)|}h^{3/2}$
- The same results as for the obstacle problem (Meyer/Thoma 2013)

# Proof of Convergence Rates I

We set $f(u) := J(S(u), u)$ and $f_h(u) := J(S_h(u), u)$.

- $\{\overline{u}_h\}$ sequence of locally optimal solutions to $(P_h)$ with $\overline{u}_h \to \overline{u}$ in $L^2(\Omega)$
- These local solutions are global solutions of $(P_h^\epsilon)$ for $h > 0$ sufficiently small and thus
$$f_h(\overline{u}_h) \leq f_h(\overline{u}) \tag{1}$$
- For $h$ sufficiently small $\overline{u}_h \in \{u \in L^2(\Omega) : \|u - \overline{u}\|_{L^2(\Omega)} \leq \epsilon\}$
- QGC and (1) imply
$$\begin{aligned}\delta\|\overline{u}_h - \overline{u}\|^2_{L^2(\Omega)} &\leq f(\overline{u}_h) - f_h(\overline{u}_h) + f_h(\overline{u}) - f(\overline{u}) + f_h(\overline{u}_h) - f_h(\overline{u}) \\ &\leq |f(\overline{u}_h) - f_h(\overline{u}_h)| + |f(\overline{u}) - f_h(\overline{u})|\end{aligned}$$

# Proof of Convergence Rates II

- $$|f(\overline{u}_h) - f_h(\overline{u}_h)| \leq \frac{1}{2}\|S_h(\overline{u}_h) - S(\overline{u}_h)\|^2_{L^2(\Omega)} + \|S_h(\overline{u}_h) - S(\overline{u}_h)\|_{L^2(\Omega)}\|S(\overline{u}_h) - y_d\|_{L^2(\Omega)}$$
- Use the continuous embeddings $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ in 1D respectively $H^1(\Omega) \hookrightarrow L^p(\Omega)\ \forall p < \infty$ in 2D and the $L^\infty$-error estimates for the states in order to estimate
$$\|S_h(\overline{u}_h) - S(\overline{u}_h)\|_{L^2(\Omega)}.$$
- The uniform boundedness of $\{\overline{u}_h\}$ in $H^1(\Omega)$ and the Lipschitz continuity of $S$ imply the uniform boundedness of $\|S(\overline{u}_h) - y_d\|_{L^2(\Omega)}$.
- We end up with
$$|f(\overline{u}_h) - f_h(\overline{u}_h)| \leq C(h\,|\log(h)|)^2, \quad \text{respectively} \ \leq Ch^{2(1-\epsilon)}.$$
- Apply the same argument for $|f(\overline{u}) - f_h(\overline{u})|$.

# Derivation of Optimality Conditions

Control problems governed by VIs exhibit a lack of regularity since the solution operator $S$ is in general not Gâteaux-differentiable

$\Longrightarrow$ Derivation of necessary and sufficient optimality conditions is very challenging

Approaches for the derivation of optimality conditions:

- Regularization techniques (e.g. de los Reyes 2011)
  - Optimality conditions for the original problem are obtained as a limit of the regularized ones
  - Loss of information by passage to the limit $\Longrightarrow$ less rigorous optimality system
- Use differentiability properties of $S$ (e.g. de los Reyes/Meyer 2016)
  - Sharp optimality system
  - Assumptions on the active set in order to prove directional differentiability of $S$

# Directional Differentiability in 1D

Let $u, h \in L^2(a,b)$. $\eta := S'(u,h)$ solves the following variational inequality:
Find $\eta \in \mathcal{K}(\overline{y})$ such that

$$\int_a^b \eta' \cdot (v-\eta)' \,\mathrm{d}x + 2 \sum_{x\in\mathcal{M}} \frac{\eta(x)(v(x)-\eta(x))}{|\overline{y}'(x)|} \geq \langle h, v-\eta\rangle \quad \forall v \in \mathcal{K}(\overline{y})$$

with

$$\begin{aligned}\mathcal{K}(\overline{y}) := \{v \in W_{\overline{y}} :\ & v(x) \leq 0\ \forall x \in (a,b) : \overline{y}(x) = 0 \wedge -1 \leq \overline{u}(x) < 1,\\ & v(x) \leq 0\ \forall x \in (a,b) : \overline{y}'(x) = 0 \wedge x \in \partial\{\overline{y}(x) < 0\},\\ & v(x) \geq 0\ \forall x \in (a,b) : \overline{y}(x) = 0 \wedge -1 < \overline{u}(x) \leq 1,\\ & v(x) \geq 0\ \forall x \in (a,b) : \overline{y}'(x) = 0 \wedge x \in \partial\{\overline{y}(x) > 0\}\}\end{aligned}$$

$$W_{\overline{y}} := \left\{ v \in H_0^1(a,b) : \sum_{x\in\mathcal{M}} \frac{v(x)^2}{|\overline{y}'(x)|} < \infty \right\}$$

$$\mathcal{M} := \{x \in (a,b) : \overline{y}(x) = 0,\ \overline{y}'(x) \neq 0\}$$

(cf. [Christof/Meyer 2018]).

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# Strong Stationarity Conditions in 1D

## Theorem

*There exists an adjoint state $\overline{p} \in H_0^1(a,b)$ and a multiplier $\mu \in H^{-1}(a,b)$ such that the following strong stationarity system is fullfilled:*

$$\int_a^b \overline{y}' \cdot v' \,\mathrm{d}x + \int_a^b \overline{q} v \,\mathrm{d}x = \langle \overline{u}, v \rangle \quad \forall v \in \; H_0^1(a,b)$$

$$\overline{q}(x)\overline{y}(x) = |\overline{y}(x)|, \; |\overline{q}(x)| \leq 1 \textit{ a.e. in } (a,b)$$

$$\int_a^b \overline{p}' \cdot v' \,\mathrm{d}x + 2 \sum_{x \in \mathcal{M}} \frac{\overline{p}(x) v(x)}{|y'(x)|} + \langle \mu, v \rangle_{W_{\overline{y}}^*, W_{\overline{y}}} = \int_a^b (\overline{y} - y_d) \cdot v \,\mathrm{d}x \; \forall v \in W_{\overline{y}}$$

$$\overline{p} \in \mathcal{K}(\overline{y}), \quad \langle \mu, w \rangle_{W_{\overline{y}}^*, W_{\overline{y}}} \geq 0 \quad \forall w \in \mathcal{K}(\overline{y})$$

$$\overline{p} + \alpha \overline{u} = 0 \quad \textit{a.e. in } (a,b)$$

# Conclusion

Summary:

- Error analysis for optimal control problems subject to VIs is quite challenging
  - No classical Nitsche-trick in $L^2$
  - Derivation of necessary and sufficient optimality conditions is complicated
- Proof of nearly optimal a priori error estimates for the FE-discretization is based on
  - $L^\infty$-estimates for the state
  - Quadratic growth condition
  - Uniform boundedness of $\overline{u}_h$ in $H^1(\Omega)$
- Strong stationarity conditions in 1D

Open problems:

- Identify cases with low regularity based on the problem data
- Higher order of convergence in case of higher regularity?
- Practicable second-order sufficient conditions with minimal gap

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Thank you for your attention!

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