Stability of Galerkin discretizations of parabolic IVPs

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Contents

- Today: heat equation (cf. [SW19] for linear parabolic IBVPs)
- We'll look at **two** space-time variational formulations
- and investigate the properties of their discretizations

Heat equation (strong form)

Time domain $I := (0, T)$, space domain $\Omega \subset \mathbb{R}^d$, space-time cylinder $I \times \Omega$. Given functions u_0 and f, find $u : I \times \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \text{on } I \times \Omega, \\ u = 0 & \text{on } I \times \partial\Omega, \quad \text{(bdr condition)} \\ u = u_0 & \text{on } \{0\} \times \Omega. \quad \text{(initial condition)} \end{cases}$$

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Solving linear parabolic evolution equations numerically

- Typical approach: method of lines ("time-marching")
  1. Discretize space (using eg FEM) $\Longrightarrow$ system of coupled ODEs
  2. Numerically solve ODEs (using e.g. BW Euler)
- Alternative: simultaneous space-time discretization [BJ89]
  - Galerkin on space-time cylinder
  - Massively parallel implementation possible
  - Can hope for **uniform quasi-optimality** of discrete solutions
  - $\Rightarrow$ Better suited for space-time adaptive refinement
- Def *solution space* U; consider family $(U^\delta)_{\delta \in \Delta}$ of *trial spaces*. Discrete solutions $u^\delta \in U^\delta$ are **uniformly quasi-optimal** when

$$\|u - u^\delta\|_U \leq C_\Delta \inf_{w^\delta \in U^\delta} \|u - w^\delta\|_U \quad (u \in U, \quad \delta \in \Delta).$$

  - Akin to Céa's Lemma.
  - Gives us certainty about error reduction.

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$$(Bu)(v) := \int_I \int_\Omega \frac{\partial u}{\partial t} v \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_I \int_\Omega \nabla_{\boldsymbol{x}} u \cdot \nabla_{\boldsymbol{x}} v \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$f(v) := \int_I \int_\Omega f v \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

Space-time variational form of heat equation

Take $U := L_2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$ and $V := L_2(I; H_0^1(\Omega))$. Given $u_0 \in L_2(\Omega)$, $f \in V'$, find $u \in U$ s.t.

$$(Bu)(v) + \langle \gamma_0 u, \sigma \rangle_{L_2(\Omega)} = f(v) + \langle u_0, \sigma \rangle_{L_2(\Omega)} \quad (v \in V,\ \sigma \in L_2(\Omega)).$$

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- [And13]: minimal residual Petrov-Galerkin discretizations
$$u^\delta := \underset{w^\delta \in U^\delta}{\arg\min} \|Bw^\delta - f\|_{V'}.$$
- Equivalent to Galerkin discretization of. . .

'Andreev' self-adjoint saddle-point formulation [And13]

$$\text{Find } \begin{bmatrix} \mu \\ \sigma \\ u \end{bmatrix} \in V \times L_2(\Omega) \times U \text{ s.t. } \begin{bmatrix} A & 0 & B \\ 0 & \mathrm{Id} & \gamma_0 \\ B' & \gamma_0' & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \sigma \\ u \end{bmatrix} = \begin{bmatrix} f \\ u_0 \\ 0 \end{bmatrix} \quad (1)$$

where $\mu, \sigma = 0$.

- Schur complement: $(B'A^{-1}B + \gamma_0'\gamma_0)u = B'A^{-1}f + \gamma_0' u_0$.

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Second alternative space-time formulation

$$B = \partial_t + A; \quad (B' A^{-1} B + \gamma_0' \gamma_0) u = B' A^{-1} f + \gamma_0' u_0.$$

- Operator is self-adjoint, coercive, invertible w/ bdd inverse!
  - However, factor A^{-1} unsuitable for computation
  - Possible to replace $P \approx A^{-1}$ in Schur complement equation
- Int. by parts $\implies B' A^{-1} B + \gamma_0' \gamma_0 = \partial_t' A^{-1} \partial_t + (A + \gamma_T' \gamma_T)$;
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New self-adjoint saddle-point formulation

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where $\lambda = u$.

- Advantages over (1):
  - Quasi-optimality under milder conditions
  - Sparser matrix (∂_t in off-diagonal instead of $B = \partial_t + A$)

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Uniform quasi-optimality of Galerkin discretizations

- Given some family $(U^\delta, V^\delta)_{\delta \in \Delta}$ of closed subspaces of $V \times U$,
- want **uniform quasi-optimality** of discrete sol u^δ of (1), (2).
- (1) and (2) are well-posed, so inf-sup condition satisfied:

$$\inf_{w \in U} \sup_{v \in V} \frac{(\partial_t w)(v)}{\|v\|_V \|\partial_t w\|_{V'}} = \alpha > 0.$$

  - Tells us something about 'degree' of well-posedness.
- Key step: show **uniform stability** discrete inf-sup constants:

$$\alpha_\Delta := \inf_{\delta \in \Delta} \inf_{w^\delta \in U^\delta} \sup_{v^\delta \in V^\delta} \frac{(\partial_t w^\delta)(v^\delta)}{\|v^\delta\|_V \|\partial_t w^\delta\|_{V'}} > 0.$$

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Generating uniformly stable subspaces

$$\alpha_\Delta := \inf_{\delta \in \Delta} \inf_{w^\delta \in U^\delta} \sup_{v^\delta \in V^\delta} \frac{(\partial_t w^\delta)(v^\delta)}{\|v^\delta\|_V \|\partial_t w^\delta\|_{V'}} > 0.$$

- Take Ω polygonal in 2D or connected in 1D
- Use NVB for conforming refinements $\mathbb{T} = \{\mathcal{T}\}$ of $\mathcal{T}_\perp$ over Ω
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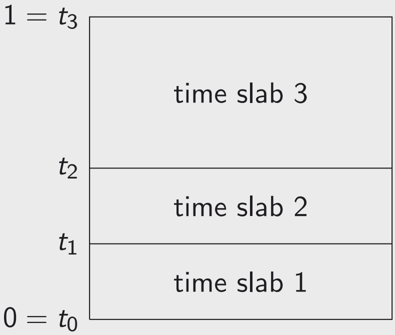
Example U^δ, V^δ for $I \times \Omega = [0,1]^2$, linear FEM in space



(a) Partition of I and time slabs.

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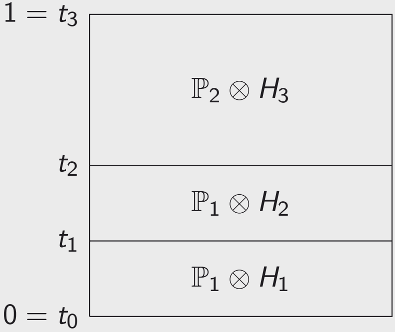

(a) U^δ: function spaces on slabs.

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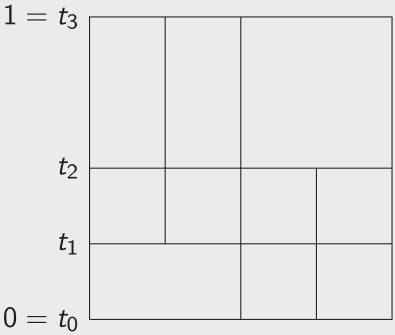

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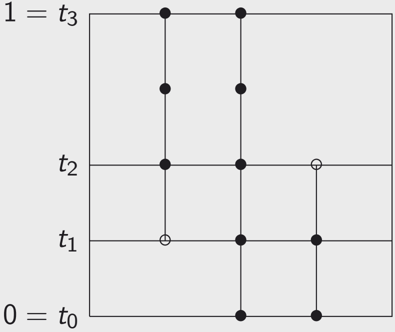

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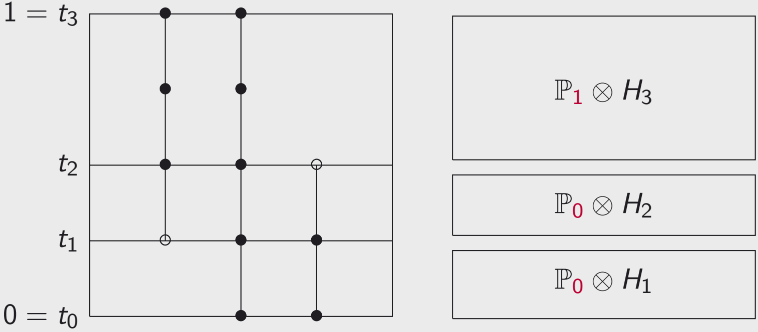

(a) U^δ: degrees of freedom. (b) V^δ: function spaces on slabs.

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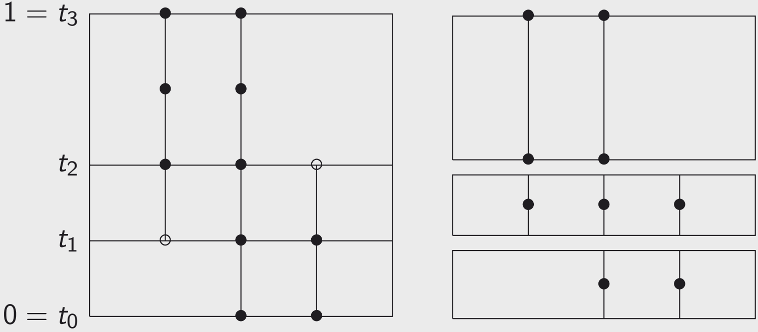

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Numerical results (i)

- We take $I \times \Omega := [0,1]^2$; uniform meshes with $h_t = h_x$.

U^δ	*continuous piecewise linears* in time $\otimes$ continuous piecewise linears in space
$V^\delta_{\mathbf{Andr}}$	*discont. piecewise linears* in time $\otimes$ continuous piecewise linears in space
$V^\delta_{\mathbf{new}}$	*discont. piecewise constants* in time $\otimes$ continuous piecewise linears in space

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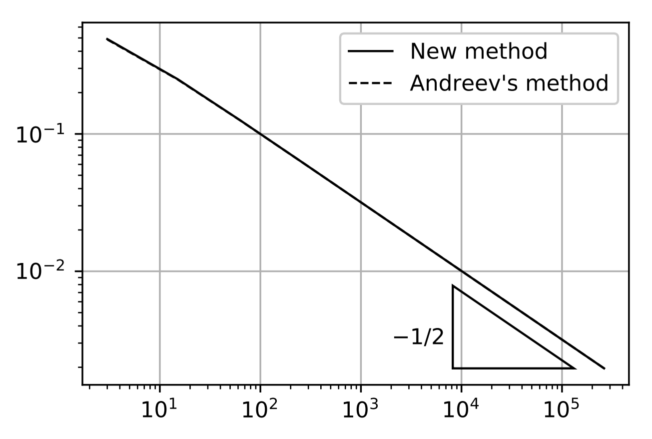

Figure: $\|u - u^\delta\|_U$ vs. dim U^δ for $u(t,x) = e^{-2t}\sin \pi x$.

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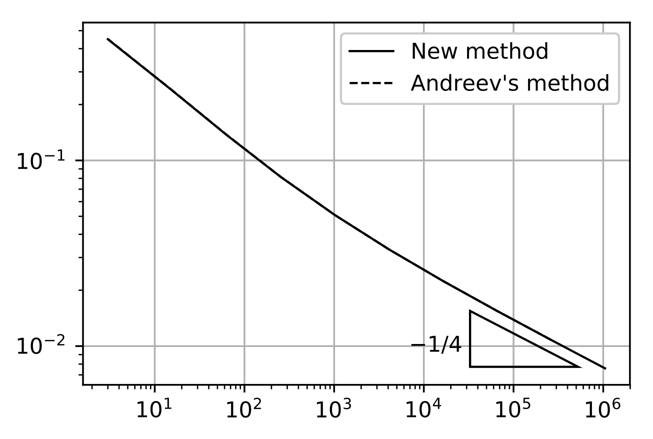

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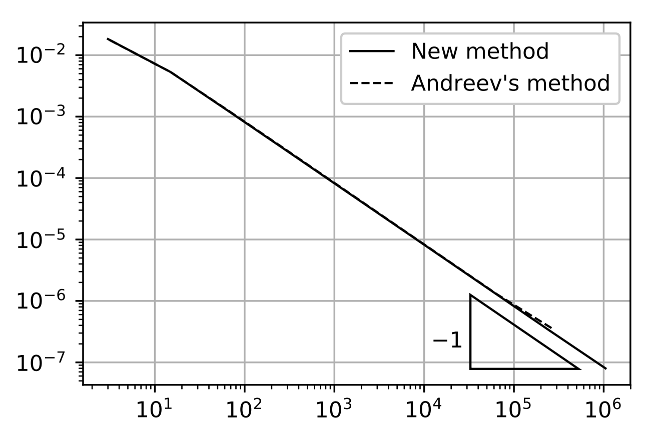
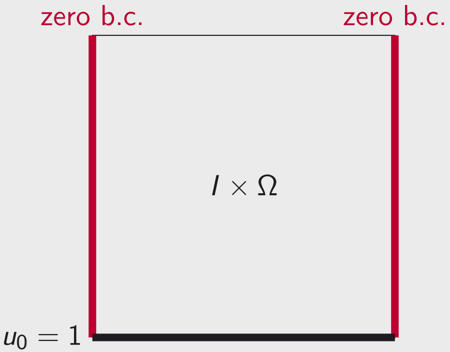


Figure: $\|u(T,\cdot) - u^{\delta}(T,\cdot)\|_{L_2(\Omega)}$ vs. $\dim U^{\delta}$.

Outlook

- Example: corner singularity
  - For optimal error reduction, refine corners at $t = 0$
  - Impossible in slab-framework
- In [RS19], optimal rate space-time adaptivity using wavelets
  - Main disadvantage: software complexity
- Current research direction: achieving similar performance without space-time wavelets



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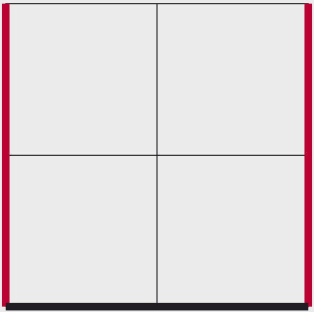

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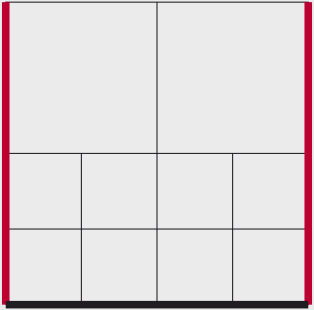

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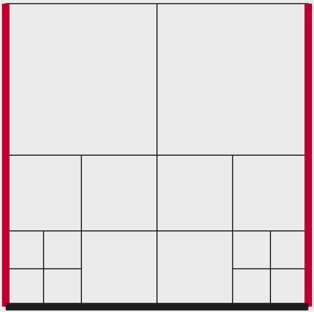

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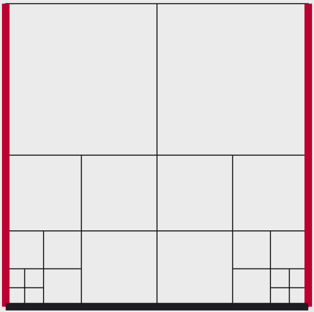

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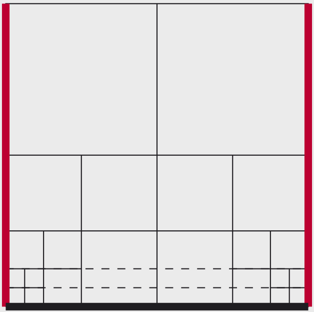

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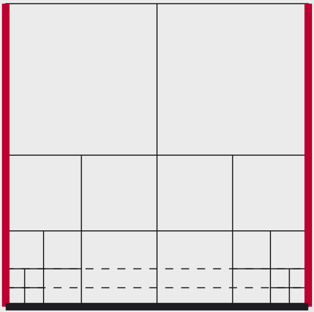

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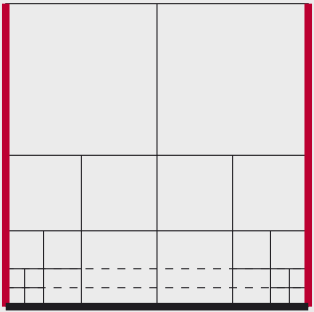

Conclusion

- We saw two space-time variational formulations of heat eqn
- Andreev's minimal residual discretization yields quasi-optimal approximation in U^δ
- Equivalent to self-adjoint saddle-point formulation (1)
- By taking Schur complements, find 'reduced' formulation (2)
  - with quasi-optimality under milder assumptions;
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- Outlook: full space-time adaptivity at optimal rates.

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References

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Generating uniformly stable subspaces: addendum

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- Suppose $\Omega \subset \mathbb{R}^d$ is a polytope.
- For $H \subset H_0^1(\Omega)$, define $L_2(\Omega)$-orth proj $Q_H : H_0^1(\Omega) \to H$.
- If $\mathcal{O} := \{H\}$ is such that the operator norms are unif bdd,

$$\sup_{H \in \mathcal{O}} \|Q_H\|_{\mathcal{L}(H_0^1(\Omega), H_0^1(\Omega))} =: M < \infty$$

  then the theorem holds with $\alpha_\Delta \geq 1/M > 0$.
- Example spaces:
  - $\Omega \subset \mathbb{R}^d$: FEM-space over quasi-uniform partition of Ω
  - $\Omega \subset \mathbb{R}^2$: FEM-space over local refinements (Carstensen 2001)