University of Amsterdam

Operator Preconditioning

Uniform preconditioners for problems of negative order

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July 3, 2019

Korteweg-de Vries Institute for Mathematics

Contents

Introduction: preconditioning BEM

- Boundary Element Method (BEM)
- Single Layer operator (bounded & coercive):

$$A\colon H^{-1/2}(\Omega) \to H^{1/2}(\Omega)$$

- Galerkin matrix $\boldsymbol{A}_\mathcal{T}$ for piecewise constants $V_\mathcal{T}$
- Solve $\boldsymbol{A}_\mathcal{T} x = y$ using Conjugate Gradients
- Condition number $\kappa(\boldsymbol{A}_\mathcal{T}) = \mathcal{O}(h^{-1})$

$\Longrightarrow$ Number of CG iterations grows to ∞ as $h \downarrow 0$

Solution: consider a preconditioned system $\boldsymbol{G}_\mathcal{T} \boldsymbol{A}_\mathcal{T} x = \boldsymbol{G}_\mathcal{T} y$

Problem

How to construct the preconditioner $\boldsymbol{G}_\mathcal{T}$ ($\approx \boldsymbol{A}_\mathcal{T}^{-1}$), such that

$$\kappa(\boldsymbol{G}_\mathcal{T} \boldsymbol{A}_\mathcal{T}) = \mathcal{O}(1) \quad \text{for all meshes } \mathcal{T}$$

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On a continuous level we find

- Single Layer operator $A\colon H^{-1/2} \to H^{1/2}$
- Hypersingular operator $B\colon H^{1/2} \to H^{-1/2}$
- Combined $BA\colon H^{-1/2} \to H^{-1/2}$

This suggests that B can serve as a preconditioner, **informally**:

- Suppose we have a basis for $H^{-1/2}$
- Inducing a bijection $T\colon \mathbb{R}^\infty \to H^{-1/2}$
- Matrix representation is $\boldsymbol{BA} = T^{-1}\, BA\, T$
- For $\rho(\cdot)$ the spectral radius:

$$\kappa(\boldsymbol{BA}) = \rho(\boldsymbol{BA})\rho((\boldsymbol{BA})^{-1}) = \rho(BA)\rho((BA)^{-1}) \leq \|AB\|\|(BA)^{-1}\|$$

$\Longrightarrow$ $\boldsymbol{BA}$ is well-conditioned!

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Discretized we have

- Piecewise constants $V_\mathcal{T} = \operatorname{span} \Xi_\mathcal{T} \subset H^{-1/2}$, with basis $\xi_T := \mathbb{1}_T$
- Single Layer operator $A_\mathcal{T} \colon V_\mathcal{T} \to V'_\mathcal{T}$, with $\boldsymbol{A}_\mathcal{T} := (A\Xi_\mathcal{T})(\Xi_\mathcal{T})$
- Family $\mathbb{T}$ of triangulations of Ω

Operator preconditioning (Steinbach & Wendland [SW98], Hiptmair [Hip06]):

- Given a *suitable* 'dual' space $W_\mathcal{T} = \operatorname{span} \Psi_\mathcal{T} \subset H^{1/2}$
- Boundedly invertible $B_\mathcal{T} \colon W_\mathcal{T} \to W'_\mathcal{T}$ (e.g. Hypersingular)
- $L_2(\Omega)$-duality pairing $D_\mathcal{T} \colon V_\mathcal{T} \to W'_\mathcal{T}$

For matrices $\boldsymbol{B}_\mathcal{T} := (B\Psi_\mathcal{T})(\Psi_\mathcal{T})$, $\boldsymbol{D}_\mathcal{T} := \langle \Xi_\mathcal{T}, \Psi_\mathcal{T} \rangle_{L_2}$

$$\text{Preconditioner:} \quad \boldsymbol{D}_\mathcal{T}^{-1} \boldsymbol{B}_\mathcal{T} \boldsymbol{D}_\mathcal{T}^{-\top}$$

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$$\dim W_\mathcal{T} = \dim V_\mathcal{T}, \quad \inf_{\mathcal{T}\in\mathbb{T}} \inf_{v\in V_\mathcal{T}} \sup_{w\in W_\mathcal{T}} \frac{\langle v, w\rangle_{L_2}}{\|v\|_{H^{-1/2}}\|w\|_{H^{1/2}}} > 0 \qquad (1)$$

then the preconditioner yields a uniformly bounded condition number:

$$\kappa(\boldsymbol{D}_\mathcal{T}^{-1}\boldsymbol{B}_\mathcal{T}\boldsymbol{D}_\mathcal{T}^{-\top}\boldsymbol{A}_\mathcal{T}) = \mathcal{O}(1) \quad (\mathcal{T}\in\mathbb{T})$$

Finding 'dual' spaces $W_\mathcal{T}$ that satisfy (1) is *difficult*.

Buffa & Christiansen [BC07] constructed $W_\mathcal{T}$, however:

- Matrix $\boldsymbol{D}_\mathcal{T}$ is *not* diagonal: inverse has to be approximated (costly)
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Dual mesh approach

Construction $\Psi_\mathcal{T}$ on a *barycentric* refined mesh [BC07]:

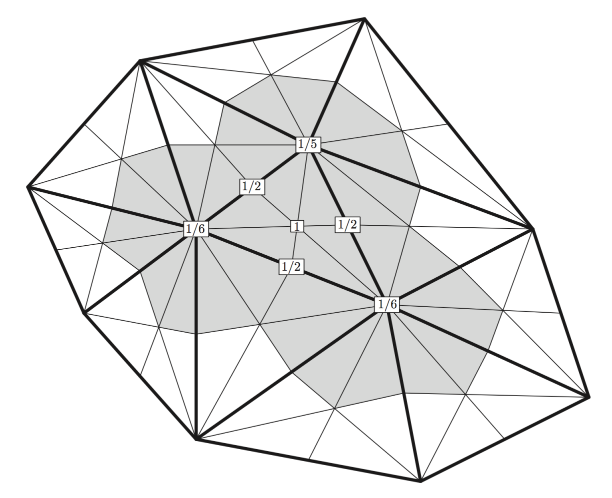
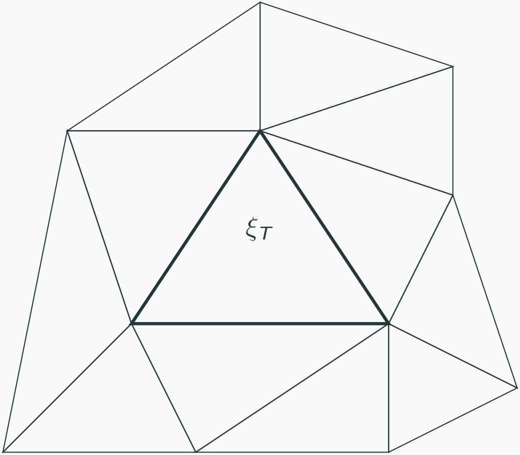


Figure 1: A basis function $\psi_T \in W_\mathcal{T}$ associated with ξ_T. Picture from [BC07].

Our approach [Sv18] for $d = 2$

Construct $\Psi_{\mathcal{T}} \subset H^{1/2}$ such that $\boldsymbol{D}_{\mathcal{T}} = \langle \Xi_{\mathcal{T}}, \Psi_{\mathcal{T}} \rangle_{L_2}$ *diagonal*.

$$\psi_T := \sum_{\nu \in N_T} d_\nu^{-1} \phi_\nu + \theta_T - \sum_{T' \in \mathcal{T}} \left(\frac{1}{3} \sum_{\nu \in N_T \cap N_{T'}} d_\nu^{-1} \right) \theta_{T'}$$



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Bubbles $\theta_T \in H_0^1(T)$ s.t. $\langle \theta_T, \xi_{T'} \rangle_{L_2} = \delta_{TT'} |T|$

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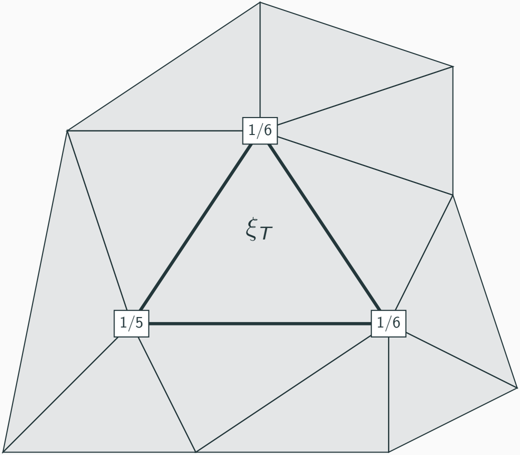

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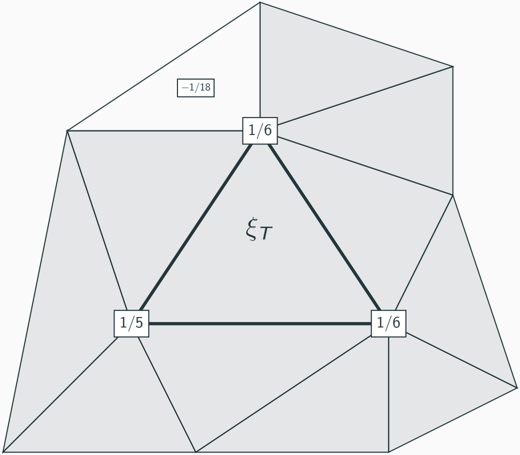

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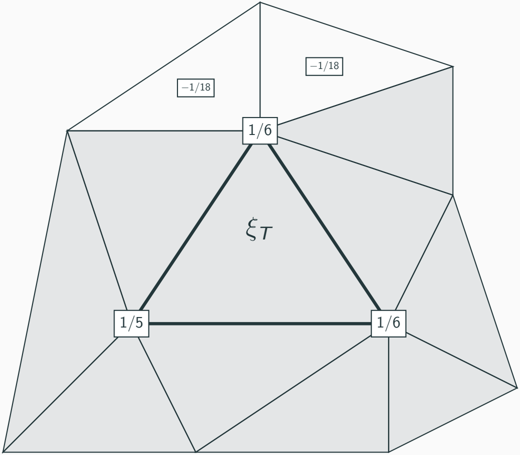

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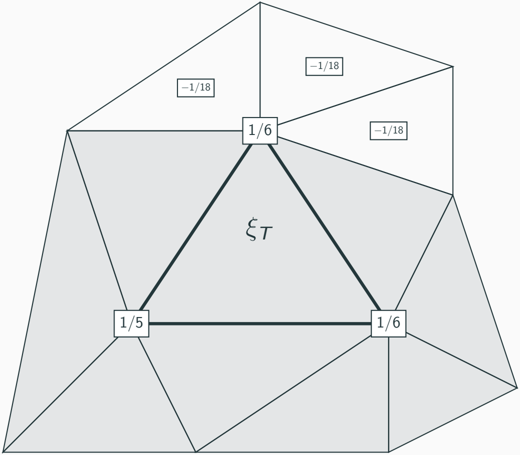
Construct $\Psi_{\mathcal{T}} \subset H^{1/2}$ such that $\boldsymbol{D}_{\mathcal{T}} = \langle \Xi_{\mathcal{T}}, \Psi_{\mathcal{T}} \rangle_{L_2}$ *diagonal*.

$$\psi_T := \sum_{v \in N_T} d_v^{-1} \phi_\nu + \theta_T - \sum_{T' \in \mathcal{T}} \Big(\frac{1}{3} \sum_{\nu \in N_T \cap N_{T'}} d_\nu^{-1}\Big) \theta_{T'}$$



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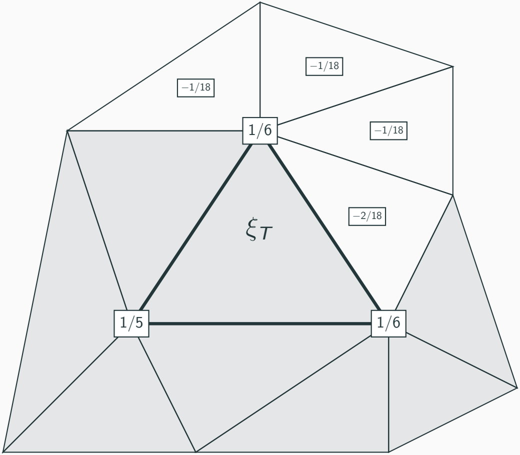

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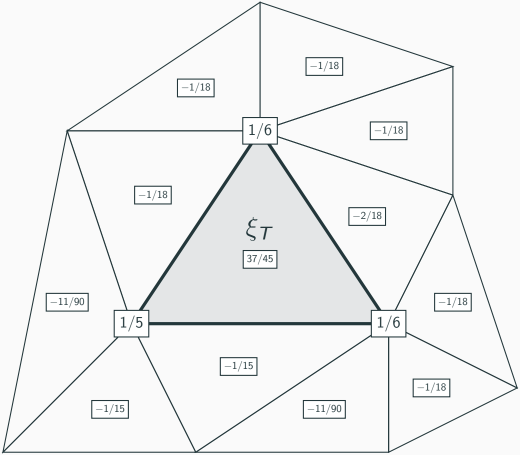

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Two important properties:

- $\boldsymbol{D}_\mathcal{T} = \langle \Xi_\mathcal{T}, \Psi_\mathcal{T} \rangle_{L_2} = \mathrm{diag}\{|T| : T \in \mathcal{T}\}$.
- $\sum_{T \in \mathcal{T}} \psi_T = \mathbb{1}$

For $\mathbb{T}$ the family of conforming *shape-regular* triangulations of Ω:

Theorem ([Sv18])

Biorthogonal proj. $P_\mathcal{T}$ *onto* $W_\mathcal{T}$*, with* $\mathrm{ran}(\mathrm{Id} - P_\mathcal{T}) \perp V_\mathcal{T}$ *is bounded in* $H^{1/2}$

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Constructing $B_\mathcal{T} : W_\mathcal{T} \to W'_\mathcal{T}$

Recall that $W_\mathcal{T} \subset S_\mathcal{T} \oplus \mathscr{B}_\mathcal{T}$ for

- Continuous piecewise linears $S_\mathcal{T} := \text{span}\{\phi_v\}$
- Bubbles $\mathscr{B}_\mathcal{T} := \text{span}\{\theta_T\}$

Matrix representation $\boldsymbol{B}_\mathcal{T} = (B\Psi_\mathcal{T})(\Psi_\mathcal{T})$ requires explicit θ_T.

Practical alternative follows from

$$\|u+v\|^2_{H^{1/2}} \eqsim \|u\|^2_{H^{1/2}} + \|v\|^2_{H^{1/2}} \quad (u \in S_\mathcal{T}, v \in \mathscr{B}_\mathcal{T}).$$

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$$(B^{\mathscr{B}}_\mathcal{T} \sum_{T \in \mathcal{T}} c_T \theta_T)(\sum_{T \in \mathcal{T}} d_T \theta_T) := \beta_0 \sum_{T \in \mathcal{T}} |T|^{1/2} c_T d_T, \quad \beta_0 > 0.$$

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Implementation preconditioner

Matrix representation is given by

$$\boldsymbol{G}_{\mathcal{T}} := \boldsymbol{D}_{\mathcal{T}}^{-1}(\boldsymbol{p}_{\mathcal{T}}^{\top}\boldsymbol{B}_{\mathcal{T}}^{S}\boldsymbol{p}_{\mathcal{T}} + \beta_0 \boldsymbol{q}_{\mathcal{T}}^{\top}\boldsymbol{D}_{\mathcal{T}}^{1/2}\boldsymbol{q}_{\mathcal{T}})\boldsymbol{D}_{\mathcal{T}}^{-1},$$

where for some $B : H^{1/2} \to H^{-1/2}$,

$$\boldsymbol{D}_{\mathcal{T}} = \operatorname{diag}\{|T| : T \in \mathcal{T}\}$$
$$\boldsymbol{B}_{\mathcal{T}}^{S} = (B\Phi_{\mathcal{T}})(\Phi_{\mathcal{T}}) \quad \text{for hat functions } \Phi_{\mathcal{T}}$$
$$\boldsymbol{p}_{\mathcal{T}}, \boldsymbol{q}_{\mathcal{T}} \quad \text{sparse.}$$

Computationally: $\operatorname{cost}(\boldsymbol{G}_{\mathcal{T}}) = \mathcal{O}(\#\mathcal{T}) + \operatorname{cost}(\boldsymbol{B}_{\mathcal{T}}^{S})$.

Numerical results: uniform refinements

$\Omega = \partial[0,1]^3$, Single Layer operator A, Hypersingular operator B.
Results for a sequence of uniformly refined meshes.

dofs	$\kappa_S(\mathrm{diag}(\boldsymbol{A}_\mathcal{T})^{-1}\boldsymbol{A}_\mathcal{T})$	$\kappa_S(\boldsymbol{G}_\mathcal{T}\boldsymbol{A}_\mathcal{T})$
12	14.56	2.50
48	29.30	2.63
192	58.25	2.77
768	116.3	2.79
3072	230.0	2.80
12288	444.8	2.86
49152	851.8	2.89
196608	1565.7	2.90

Condition numbers for preconditioned single layer system discretized by piecewise constants $V_\mathcal{T}$.
For coercivity of B we have added $\alpha\langle u, \mathbb{1}\rangle_{L_2}\langle v, \mathbb{1}\rangle_{L_2}$ for some $\alpha > 0$, here $\alpha = 0.05$, $\beta_0 = 1.25$.

Numerical results: local refinements

Sequence of locally refined triangulations.

dofs	$h_{\mathcal{T},min}$	$\kappa_S(\mathrm{diag}(\boldsymbol{A}_\mathcal{T})^{-1}\boldsymbol{A}_\mathcal{T})$	$\kappa_S(\boldsymbol{G}_\mathcal{T}\boldsymbol{A}_\mathcal{T})$
12	$7.0 \cdot 10^{-1}$	14.56	2.61
432	$2.2 \cdot 10^{-2}$	68.66	2.64
912	$6.9 \cdot 10^{-4}$	73.15	2.64
1872	$6.7 \cdot 10^{-7}$	73.70	2.64
2352	$2.1 \cdot 10^{-8}$	73.80	2.64
2976	$2.3 \cdot 10^{-10}$	73.66	2.64

Condition numbers for preconditioned single layer. Matrix $\boldsymbol{G}_\mathcal{T}$ is constructed using $\beta_0 = 1.2$. The second column is defined by $h_{\mathcal{T},min} := \min_{T\in\mathcal{T}} h_T$.

Generalizations

- Results hold for manifolds Γ, with or without boundary $\partial\Gamma$, and

$$A : H^{-s}_{0,\gamma}(\Gamma) \to H^{s}_{0,\gamma}(\Gamma) \quad s \in [0,1].$$

- Using a subspace correction method it generalizes to a preconditioner for *higher order* trial spaces $V_{\mathcal{T}} = S^{-1,\ell}_{\mathcal{T}}$
- Also works for *continuous* trial spaces $V_{\mathcal{T}} = S^{0,\ell}_{\mathcal{T}}$
- Use a cheaper operator $B : H^s \to H^{-s}$ [Sv19a]
- Similar approach (biorthogonality, bubbles) can be used to precondition the positive order operators [Sv19b]

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Higher order trial spaces

What if $V_\mathcal{T} = S_\mathcal{T}^{-1,\ell}$, piecewise polynomials for $\ell > 0$?

Lemma

For $Q_\mathcal{T}^0$ the $L_2(\Omega)$-orthogonal projector onto pw. const. $S_\mathcal{T}^{-1,0}$ we have

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$\Longrightarrow$ Splitting $V_\mathcal{T} = Q_\mathcal{T}^0 V_\mathcal{T} \oplus (\mathrm{Id} - Q_\mathcal{T}^0)V_\mathcal{T}$ stable w.r.t. $H^{-1/2}$-norm

$\Longrightarrow$ Diagonal operator on $(\mathrm{Id} - Q_\mathcal{T}^0)V_\mathcal{T}$ is bounded and coercive

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Numerical results: piecewise quadratics

Consider sequence $\{\mathcal{T}_k\}$ of uniform refined meshes, and $V_\mathcal{T} = S_\mathcal{T}^{-1,2}$ the space of discontinuous piecewise quadratics.

dofs	$\kappa_S(\mathrm{diag}(\boldsymbol{A}_\mathcal{T})^{-1}\boldsymbol{A}_\mathcal{T})$	$\kappa_S(\boldsymbol{G}_\mathcal{T}\boldsymbol{A}_\mathcal{T})$
72	167.16	9.58
288	309.12	10.4
1152	616.03	11.1
4608	1211.3	11.3
18432	2337.2	11.4

Spectral condition numbers of the preconditioned single layer system, using uniform refinements, discretized by discontinuous piecewise quadratics $S_\mathcal{T}^{-1,2}$. The matrix $\boldsymbol{G}_\mathcal{T}$ is constructed using the adapted hypersingular operator, with $\alpha = 0.05$, and $\beta_0 = \beta_1 = 1.25$.

Uniform preconditioners for positive order operators

In [Sv19b] we used a similar approach for positive order preconditioning:

- Continuous piecewise linears $S_{\mathcal{T}}$ wrt $\mathcal{T}$
- Hypersingular $B_{\mathcal{T}} \colon S_{\mathcal{T}} \to S'_{\mathcal{T}}$
- Precondition with Single Layer A

Preconditioner is given by

$$\boldsymbol{G}_{\mathcal{T}} := \boldsymbol{D}_{\mathcal{T}}^{-1}(\boldsymbol{p}_{\mathcal{T}}^{\top} \boldsymbol{A}_{\mathcal{T}}^{U} \boldsymbol{p}_{\mathcal{T}} + \beta_0 \boldsymbol{D}_{\mathcal{T}}^{3/2}) \boldsymbol{D}_{\mathcal{T}}^{-1},$$

where taking

$$U = \operatorname{span} \Sigma_{\mathcal{T}} \quad \textit{pw. cons.}\ \textbf{or}\ \textit{cont. pw. lin.}$$

$$\boldsymbol{A}_{\mathcal{T}}^{U} = (A\Sigma_{\mathcal{T}})(\Sigma_{\mathcal{T}})$$

$$\boldsymbol{D}_{\mathcal{T}} = \operatorname{diag}\{|\operatorname{supp} \phi_v| \colon \phi_v \in S_{\mathcal{T}}\}$$

$$\boldsymbol{p}_{\mathcal{T}} \quad \text{sparse.}$$

Computationally: $\operatorname{cost}(\boldsymbol{G}_{\mathcal{T}}) = \mathcal{O}(\#\mathcal{T}) + \operatorname{cost}(\boldsymbol{A}_{\mathcal{T}}^{U})$.

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Numerical results: positive order

$\Omega = \partial[0,1]^3$, B Hypersingular operator, Single Layer operator A.
Results for a sequence of uniformly refined meshes.

dofs	$\kappa_S(\boldsymbol{B}_\mathcal{T})$	$\kappa_S(\boldsymbol{G}_\mathcal{T}\boldsymbol{B}_\mathcal{T})$
12290	115.6	2.27
24578	168.7	2.24
49154	231.3	2.27
98306	336.9	2.25
196610	461.7	2.28
393218	671.9	2.28
786434	751.6	2.30

Condition numbers for preconditioned Hypersingular system discretized by continuous piecewise linears $S^{0,1}_\mathcal{T}$. Single Layer operator is discretized on piecewise constants $V_\mathcal{T}$. For coercivity of B we have added $\alpha\langle u, \mathbb{1}\rangle_{L_2(\Omega)}\langle v, \mathbb{1}\rangle_{L_2(\Omega)}$, here $\alpha = 0.05, \beta_1 = 0.34$. Results are gathered using compressed hierarchical matrices.

Conclusions

- Uniform preconditioners for operators $A\colon H^{-s} \to H^{s}$
- Requires bounded & coercive operator $B\colon H^{s} \to H^{-s}$
- Implementation of preconditioner is

$$\boldsymbol{G}_{\mathcal{T}} := \boldsymbol{D}_{\mathcal{T}}^{-1}(\boldsymbol{p}_{\mathcal{T}}^{\top}\boldsymbol{B}_{\mathcal{T}}^{S}\boldsymbol{p}_{\mathcal{T}} + \beta_0 \boldsymbol{q}_{\mathcal{T}}^{\top}\boldsymbol{D}_{\mathcal{T}}^{1-s}\boldsymbol{q}_{\mathcal{T}})\boldsymbol{D}_{\mathcal{T}}^{-1}$$

- Computationally $\text{cost}(\boldsymbol{G}_{\mathcal{T}}) = \mathcal{O}(\#\mathcal{T}) + \text{cost}(\boldsymbol{B}_{\mathcal{T}}^{S})$
- Generalizes to manifolds, and higher order (continuous) trial spaces
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