Shape and topology optimisation subject to 3D nonlinear magnetostatics - part 1: sensitivity analysis

Kevin Sturm
TU Wien

joint work with Peter Gangl (TU Graz)

Outline

- Model Problem
- Topological Derivative
- Averaged adjoint formalism
- Application to model problem

Outline

- Model Problem
- Topological Derivative
- Averaged adjoint formalism
- Application to model problem

Topology Optimization of a 3D Electric Motor

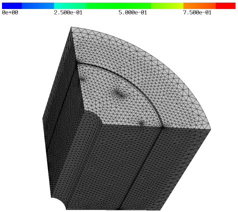
Quasilinear Model Problem

$$\min_{\Omega\in\mathcal{A}(D)} J(\Omega) = \int_{\Gamma_0} |\,\mathbf{curl}\, u\cdot \boldsymbol{n} - B_d^n|^2 \mathrm{d}S_x$$

$$\text{s.t. } u\in \mathbf{V}: \int_{\mathrm{D}} \nu_\Omega(x,|\,\mathbf{curl}\, u|)\,\mathbf{curl}\, u\cdot \mathbf{curl}\, \boldsymbol{v}\,\mathrm{d}x = \langle \boldsymbol{F},\boldsymbol{v}\rangle \quad \text{for all } \boldsymbol{v}\in\mathbf{V}.$$

Here,

- $\nu_\Omega(x,s) = \chi_\Omega(x)\nu_2(s) + \chi_{D\setminus\overline{\Omega}}(x)\nu_1(s)$,
- $\langle \boldsymbol{F},\boldsymbol{v}\rangle := \int_{\Omega_c} \boldsymbol{J}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Omega_m} \nu_m \boldsymbol{M}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad$ for $\boldsymbol{v}\in\mathbf{V}$,
- $\mathbf{V} := H_0(\mathrm{D},\mathbf{curl}) \cap H(\mathrm{D}, \mathrm{div}=0)$,
- $\nu_1,\ \nu_2 : \mathbf{R}_0^+ \to \mathbf{R}^+$ satisfy
  - $s\mapsto \nu_i(s)s$, $i=1,2$ is Lipschitz continuous and strongly monotone
  - $\nu_i\in C^2(\mathbf{R}_0^+)$, $\nu_i'(0)=0$, and that there is a constant c such that for all $s\in\mathbf{R}_0^+$, $\nu_i'(s)\le c$ and $\nu_i''(s)\le c$



Topology Optimization of a 3D Electric Motor

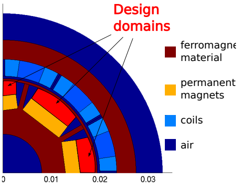
Quasilinear Model Problem

$$\min_{\Omega\in\mathcal{A}(D)} J(\Omega) = \int_{\Gamma_0} |\,\mathbf{curl}\, u\cdot \boldsymbol{n} - B_d^n|^2 \mathrm{d}S_x$$

$$\text{s.t. } u\in\mathbf{V} : \int_{\mathrm{D}} \nu_\Omega(x,|\,\mathbf{curl}\, u|)\,\mathbf{curl}\, u\cdot\mathbf{curl}\,\boldsymbol{v}\mathrm{d}x = \langle \boldsymbol{F},\boldsymbol{v}\rangle \quad \text{for all } \boldsymbol{v}\in\mathbf{V}.$$

Here,

- $\nu_\Omega(x,s) = \chi_\Omega(x)\nu_2(s) + \chi_{D\setminus\overline{\Omega}}(x)\nu_1(s)$,
- $\langle \boldsymbol{F},\boldsymbol{v}\rangle := \int_{\Omega_c} \boldsymbol{J}\cdot\boldsymbol{v}\mathrm{d}x + \int_{\Omega_m}\nu_m \boldsymbol{M}\cdot\mathbf{curl}\,\boldsymbol{v}\mathrm{d}x \quad$ for $\boldsymbol{v}\in\mathbf{V}$,
- $\mathbf{V} := H_0(\mathrm{D},\mathbf{curl})\cap H(\mathrm{D},\mathrm{div}=0)$,
- $\nu_1,\,\nu_2 : \mathbf{R}_0^+ \to \mathbf{R}^+$ satisfy
  - $s\mapsto \nu_i(s)s$, $i=1,2$ is Lipschitz continuous and strongly monotone
  - $\nu_i\in C^2(\mathbf{R}_0^+)$, $\nu_i'(0)=0$, and that there is a constant c such that for all $s\in\mathbf{R}_0^+$, $\nu_i'(s)\le c$ and $\nu_i''(s)\le c$



Topology Optimization of a 3D Electric Motor

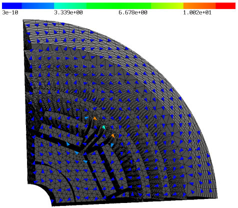
Quasilinear Model Problem

$$\min_{\Omega \in \mathcal{A}(D)} J(\Omega) = \int_{\Gamma_0} |\,\mathbf{curl}\, u \cdot \boldsymbol{n} - B_d^n|^2 \mathrm{d}S_x$$

$$\text{s.t. } u \in \mathbf{V} : \int_{\mathrm{D}} \nu_\Omega(x, |\,\mathbf{curl}\, u|)\, \mathbf{curl}\, u \cdot \mathbf{curl}\, \boldsymbol{v} \mathrm{d}x = \langle \boldsymbol{F}, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in \mathbf{V}.$$

Here,

- $\nu_\Omega(x,s) = \chi_\Omega(x)\nu_2(s) + \chi_{D\setminus\overline{\Omega}}(x)\nu_1(s)$,
- $\langle \boldsymbol{F}, \boldsymbol{v} \rangle := \int_{\Omega_c} \boldsymbol{J} \cdot \boldsymbol{v} \mathrm{d}x + \int_{\Omega_m} \nu_m \boldsymbol{M} \cdot \mathbf{curl}\, \boldsymbol{v} \mathrm{d}x \quad$ for $\boldsymbol{v} \in \mathbf{V}$,
- $\mathbf{V} := H_0(\mathrm{D}, \mathbf{curl}) \cap H(\mathrm{D}, \mathrm{div} = 0)$,
- $\nu_1,\ \nu_2 : \mathbf{R}_0^+ \to \mathbf{R}^+$ satisfy
  - $s \mapsto \nu_i(s)s$, $i = 1,2$ is Lipschitz continuous and strongly monotone
  - $\nu_i \in C^2(\mathbf{R}_0^+)$, $\nu_i'(0) = 0$, and that there is a constant c such that for all $s \in \mathbf{R}_0^+$, $\nu_i'(s) \le c$ and $\nu_i''(s) \le c$



Outline

Topological Derivative: Definition

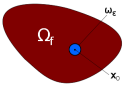
Idea:

Sensitivity of $\mathcal{J} = \mathcal{J}(\Omega) = J(\Omega, u(\Omega))$ w.r.t. insertion of hole $\omega_\varepsilon = x_0 + \varepsilon\,\omega$ (ω e.g. dots unit disk)

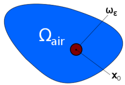
Definition (Topological derivative)

Let $\mathrm{D} \subset \mathbf{R}^3$ be an open set and $\Omega \subset \mathrm{D}$ an open subset. Let $\omega \subset \mathbf{R}^3$ be open with $0 \in \omega$. Define for $z \in \mathbf{R}^3$, $\omega_\varepsilon(z) := z + \varepsilon\omega$. Then the topological derivative of J at Ω at the point $z \in \mathbf{R}^3$ is defiend by

$$dJ(\Omega)(z) = \begin{cases} \lim_{\varepsilon \searrow 0} \frac{J(\Omega \setminus \omega_\varepsilon(z)) - J(\omega)}{|\omega_\varepsilon(z)|} & \text{if } z \in \Omega, \\ \lim_{\varepsilon \searrow 0} \frac{J(\Omega \cup \omega_\varepsilon(z)) - J(\omega)}{|\omega_\varepsilon(z)|} & \text{if } z \in \mathrm{D} \setminus \bar{\Omega} \end{cases} .$$

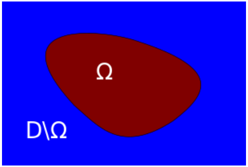


$G^{f\to air}$: air inside ferromagnetic material



$G^{air\to f}$: ferromagnetic material inside air

Topological Derivative: Example

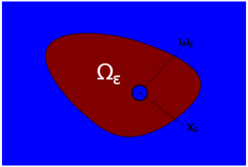


Example: $\mathcal{J}(\Omega) := |\Omega|$, $\omega_\varepsilon = x_0 + \varepsilon\omega$, $\omega \in \mathbf{R}^d$, $0 \in \omega$,

Remark

Topological derivative can be seen as a semidifferential on the space of characteristic functions.

Topological Derivative: Example



Example: $\mathcal{J}(\Omega) := |\Omega|$, $\omega_\varepsilon = x_0 + \varepsilon\omega$, $\omega \in \mathbf{R}^d$, $0 \in \omega$,

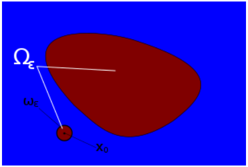
1. $x_0 \in \Omega$, $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$:

$$\frac{1}{|\omega_\varepsilon|}(\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega)) = -\frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon} f\, dx \;\to\; -f(x_0),$$

Remark

Topological derivative can be seen as a semidifferential on the space of characteristic functions.

Topological Derivative: Example



Example: $\mathcal{J}(\Omega) := |\Omega|$, $\omega_\varepsilon = x_0 + \varepsilon\omega$, $\omega \in \mathbf{R}^d$, $0 \in \omega$,

1. $x_0 \in \Omega$, $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$:

$$\frac{1}{|\omega_\varepsilon|}(\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega)) = -\frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon} f \; dx \;\to\; -f(x_0),$$

2. $x_0 \in D \setminus \overline{\Omega}$, $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon$:

$$\frac{1}{|\omega_\varepsilon|}(\mathcal{J}(\Omega_\varepsilon) - \mathcal{J}(\Omega)) = \frac{1}{|\omega_\varepsilon|}\int_{\omega_\varepsilon} f \; dx \;\to\; f(x_0),$$

Remark

Topological derivative can be seen as a semidifferential on the space of characteristic functions.

Outline

Lagrangian approach

Definition

Let X and Y be vector spaces and $\tau > 0$. A parametrised Lagrangian (or short Lagrangian) is a function

$$(\varepsilon, u, q) \mapsto L(\varepsilon, u, q) : [0, \tau] \times X \times Y \to \mathbf{R},$$

satisfying for all $(\varepsilon, u) \in [0, \tau] \times X$,

$$q \mapsto L(\varepsilon, u, q) \quad \text{is affine on } Y.$$

Example

Let $X = Y = H^1_0(\Omega)$ and

$$L(\varepsilon, u, q) := \int_\Omega u^2 \, dx + \int_\Omega \nabla u \cdot \nabla q - \varepsilon f q \, dx.$$

State and averaged adjoint state

Definition (Perturbed states)

For $\varepsilon \in [0, \tau]$ we define the (perturbed) state equation by: find $u_\varepsilon \in X$, such that

$$\partial_q L(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all } \varphi \in Y.$$

The set of perturbed states is denoted $E(\varepsilon)$.

State and averaged adjoint state

Definition (Perturbed states)

For $\varepsilon \in [0, \tau]$ we define the (perturbed) state equation by: find $u_\varepsilon \in X$, such that

$$\partial_q L(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all } \varphi \in Y.$$

The set of perturbed states is denoted $E(\varepsilon)$.

$$\text{“}\mathcal{J}(\varepsilon)\text{”} = L(\varepsilon, u_\varepsilon, \psi)$$

State and averaged adjoint state

Definition (Perturbed states)

For $\varepsilon \in [0,\tau]$ we define the (perturbed) <u>state equation</u> by: find $u_\varepsilon \in X$, such that

$$\partial_q L(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all } \varphi \in Y.$$

The set of perturbed states is denoted $E(\varepsilon)$.

$$\text{“}\mathcal{J}(\varepsilon)\text{”} = L(\varepsilon, u_\varepsilon, \psi) = L(\varepsilon, u_0, \psi) + \int_0^1 \partial_u L(\varepsilon, su_\varepsilon + (1-s)u_0, \psi)(u_\varepsilon - u_0)\, ds$$

Definition (Averaged adjoint state)

Given $\varepsilon \in [0,\tau]$ and $(u_0, u_\varepsilon) \in E(0) \times E(\varepsilon)$, the <u>averaged adjoint state equation</u> is defined as follows: find $q_\varepsilon \in X$, such that

$$\int_0^1 \partial_u L(\varepsilon, su_\varepsilon + (1-s)u_0, q_\varepsilon)(\varphi)\, ds = 0 \quad \text{for all } \varphi \in X.$$

The set of averaged adjoint states is denoted $Y(\varepsilon, u_0, u_\varepsilon)$. We set $Y(0, u_0) := Y(0, u_0, u_0)$.

Thus, “$\mathcal{J}(\varepsilon)$” $= L(\varepsilon, u_\varepsilon, \psi) = L(\varepsilon, u_\varepsilon, q_\varepsilon) = L(\varepsilon, u_0, q_\varepsilon)$.

Averaged adjoint theorem

Theorem (Delfour/Sturm)

Let $\ell : \mathbf{R} \to \mathbf{R}$ be such that $\ell(0) = 0$. Suppose the following conditions are satisfied.

(H1) *The set of perturbed states and averaged adjoint states is non-empty for all $\varepsilon \in [0, \tau]$.*

(H2) *For all $u_0 \in E(0)$ and $q_0 \in Y(0, u_0)$ the limit*

$$\partial_\ell L(0, u_0, q_0) := \lim_{\varepsilon \searrow 0} \frac{L(\varepsilon, u_0, q_0) - L(0, u_0, q_0)}{\ell(\varepsilon)} \quad \textit{exists.}$$

(H3) *The limit*

$$R := \lim_{\varepsilon \searrow 0} \frac{L(\varepsilon, u_0, q_\varepsilon) - L(\varepsilon, u_0, q_0)}{\ell(\varepsilon)} \quad \textit{exists.}$$

Then we have with $g(\varepsilon) := L(\varepsilon, u_\varepsilon, 0)$,

$$d_\ell g(0) = \partial_\ell L(0, u_0, q_0) + R.$$

Remark

For optimal control problems we usually choose $\ell(\varepsilon) = \varepsilon$.

Example

Recall the example $X = Y = H_0^1(\Omega)$, $\ell(\varepsilon) := \varepsilon$, and

$$L(\varepsilon, u, q) := \int_\Omega u^2\, dx + \int_\Omega \nabla u \cdot \nabla q - \varepsilon f q\, dx.$$

<u>State and averaged adjoint state equation:</u>

$$\begin{aligned} u_\varepsilon \in H_0^1(\Omega), \quad & \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi\, dx = \int_\Omega \varepsilon f \varphi\, dx \quad \text{for all } \varphi \in H_0^1(\Omega) \\ q_\varepsilon \in H_0^1(\Omega), \quad & \int_\Omega \nabla \psi \cdot \nabla q_\varepsilon\, dx = -\int_\Omega (u_\varepsilon + u)\psi\, dx \quad \text{for all } \psi \in H_0^1(\Omega). \end{aligned}$$

We have $(u := u_0, q := q_0)$

$$\partial_\varepsilon L(0, u, q) = \lim_{\varepsilon \searrow 0} \frac{L(\varepsilon, u, q) - L(0, u, q)}{\varepsilon} = -\int_\Omega f q\, dx$$

$$R = \lim_{\varepsilon \searrow 0} \frac{L(\varepsilon, u, q_\varepsilon) - L(\varepsilon, u, q)}{\varepsilon} = 0.$$

So $d_\varepsilon g(0) = \frac{d}{d\varepsilon} L(\varepsilon, u_\varepsilon, 0) = -\int_\Omega f q\, dx$.

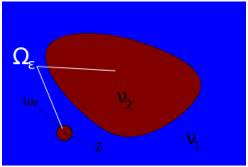
Outline

- Model Problem
- Topological Derivative
- Averaged adjoint formalism
- Application to model problem

Application to the model problem

From now on we fix:

- an open and bounded set $\omega \subset \mathbf{R}^d$ with $0 \in \omega$,
- an open set $\Omega \Subset \mathrm{D}$ and a point $z \in \mathrm{D} \setminus \overline{\Omega}$,
- the perturbation $\Omega_\varepsilon := \Omega \cup \omega_\varepsilon(z)$, where $\omega_\varepsilon(z) := z + \varepsilon\omega$ and $\varepsilon \in [0, \tau]$, $\tau > 0$.



Let $X = Y = H^1_0(\mathrm{D})$ and introduce the Lagrangian $G : [0, \tau] \times X \times Y \to \mathbf{R}$ associated with the perturbation Ω_ε by

$$L(\varepsilon, u, v) := \int_{\Gamma_0} |\,\mathbf{curl}\, u \cdot \boldsymbol{n} - B_d^n|^2 \mathrm{d}S_x + \int_{\mathrm{D}} \nu_\varepsilon(x, |\,\mathbf{curl}\, u|)\, \mathbf{curl}\, u \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x - \langle \boldsymbol{F}, \boldsymbol{v} \rangle.$$

where we use the abbreviation

$$\nu_\varepsilon(x, s) := \nu_1(s)\chi_{D\setminus\Omega_\varepsilon}(x) + \nu_2(s)\chi_{\Omega_\varepsilon}(x)$$

We will apply the averaged adjoint theorem with $\ell(\varepsilon) = |\omega_\varepsilon|$.

Bound for the perturbed state

The perturbed state equation reads: find $u_\varepsilon \in V$ such that

$$\partial_q L(\varepsilon, u_\varepsilon, 0)(v) = 0 \quad \text{for all } v \in \mathbf{V},$$

or equivalently $u_\varepsilon \in H(\mathrm{D}, \mathbf{curl})$ satisfies,

$$\int_{\mathrm{D}} \nu_\varepsilon(x, |\,\mathbf{curl}\, u_\varepsilon|)\, \mathbf{curl}\, u_\varepsilon \cdot \mathbf{curl}\, \boldsymbol{v} \mathrm{d}x = \langle \boldsymbol{F}, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in \mathbf{V}.$$

Lemma

There is a constant $C > 0$, such that for all small $\varepsilon > 0$,

$$\|u_\varepsilon - u\|_{L^2(\mathrm{D})^3} + \|\,\mathbf{curl}(u_\varepsilon - u)\|_{L^2(\mathrm{D})^3} \le C\varepsilon^{d/2}.$$

Bound for the averaged adjoint state

For $x \in \mathrm{D}$ and $v \in \mathbf{R}^3$, let $F_\varepsilon(x,v) := \nu_\varepsilon(x,|v|)v$. We introduce for $\varepsilon \in [0,\tau]$ and $v, w \in \mathbf{R}^3$, $v \neq 0$,

$$b_\varepsilon(x,v,w) := \partial_v F_\varepsilon(x,v)w = \left(\nu_\varepsilon(x,|v|)I + \frac{\nu_\varepsilon'(x,|v|)}{|v|} v \otimes v\right) w.$$

The averaged adjoint $q_\varepsilon \in \mathbf{V}$ is defined by

$$\int_0^1 \partial_u L(\varepsilon, su_\varepsilon + (1-s)u_0, q_\varepsilon)(v)\, ds = 0 \quad \text{for all } v \in \mathbf{V}.$$

This is equivalent to

$$\int_{\mathrm{D}} \int_0^1 b_\varepsilon(x, \mathbf{curl}(su_\varepsilon + (1-s)u_0), \mathbf{curl}(v))\, ds \cdot \mathbf{curl}(q_\varepsilon)\, dx$$
$$= -\int_{\Gamma_0} (\mathbf{curl}(u_\varepsilon + u_0)\cdot \mathbf{n} - 2B_d^n)\, \mathbf{curl}(v) \cdot \mathbf{n}\, dS_x$$

for all $v \in \mathbf{V}$.

Lemma

There is a constant $C > 0$, such that for all small $\varepsilon > 0$,

$$\|q_\varepsilon - q\|_{H(\mathrm{D},\mathbf{curl})} \leq C\left(\|\mathbf{curl}(u_\varepsilon - u)\|_{L_2(\mathrm{D})^3} + \varepsilon^{d/2}\right).$$

Topological Derivative

By the previous theorem, we get that the topological derivative is given by

$$G^{1\to 2}(z) = d_\ell g(0) = \partial_\ell L(0,u,q) + R$$

if both terms exist.

Lemma

We have

$$\partial_\ell L(0,u,q) = \lim_{\varepsilon\searrow 0} \frac{L(\varepsilon,u,q) - L(0,u,q)}{|\omega_\varepsilon|}$$
$$= (\nu_2(|\operatorname{\mathbf{curl}} u(z)|) - \nu_1(|\operatorname{\mathbf{curl}} u(z)|))\operatorname{\mathbf{curl}} u(z)\cdot \operatorname{\mathbf{curl}} q(z)$$

Proof.

Change of variables and using that all functions are continuous at z. □

Topological Derivative

Let us now consider the second term

$$R = \lim_{\varepsilon \searrow 0} \frac{L(\varepsilon, u, q_\varepsilon) - L(\varepsilon, u, q)}{\ell(\varepsilon)}$$

where $\ell(\varepsilon) = |\omega_\varepsilon|$. First note that testing the state equation for $\varepsilon = 0$ with $\boldsymbol{v} := q_\varepsilon - q$ yields

$$\int_{\mathrm{D}} \nu_0(x, |\,\mathbf{curl}\, u|)\,\mathbf{curl}\, u \cdot \mathbf{curl}(q_\varepsilon - q)\; dx = \langle \boldsymbol{F}, q_\varepsilon - q\rangle. \tag{1}$$

Therefore

$$\begin{aligned} L(\varepsilon, u, q_\varepsilon) - L(\varepsilon, u, q) &= \int_{\mathrm{D}} \nu_\varepsilon(x, |\,\mathbf{curl}\, u|)\,\mathbf{curl}\, u \cdot \mathbf{curl}(q_\varepsilon - q)\; dx - \langle \boldsymbol{F}, q_\varepsilon - q\rangle \\ &\overset{(1)}{=} \int_{\mathrm{D}} (\nu_\varepsilon(x, |\,\mathbf{curl}\, u|) - \nu_0(x, |\,\mathbf{curl}\, u|))\,\mathbf{curl}\, u \cdot \mathbf{curl}(q_\varepsilon - q)\; dx \\ &= \int_{\omega_\varepsilon} (\nu_2(|\,\mathbf{curl}\, u|) - \nu_1(|\,\mathbf{curl}\, u|))\,\mathbf{curl}\, u \cdot \mathbf{curl}(q_\varepsilon - q)\; dx. \end{aligned}$$

Helmholtz decomposition

Definition

We use the following Helmholtz decomposition for u_ε:

$$u_\varepsilon = \nabla\phi_\varepsilon + w_\varepsilon, \quad \phi_\varepsilon \in H_0^1(\mathrm{D}),\ w_\varepsilon \in H_0^1(\mathrm{D})^3.$$

Similarly we decompose q_ε:

$$q_\varepsilon = \nabla\psi_\varepsilon + z_\varepsilon, \quad \psi_\varepsilon \in H_0^1(\mathrm{D}),\ z_\varepsilon \in H_0^1(\mathrm{D})^3.$$

Topological Derivative

Definition

The variation of the averaged adjoint state q_ε is defined pointwise a.e. in $\mathbf{R}^d$ by

$$Q_\varepsilon(x) := \frac{\tilde{z}_\varepsilon(T_\varepsilon(x)) - \tilde{z}(T_\varepsilon(x))}{\varepsilon},$$

and the variation of the direct state u_ε is defined pointwise a.e. in $\mathbf{R}^d$ by

$$W_\varepsilon(x) := \frac{\tilde{w}_\varepsilon(T_\varepsilon(x)) - \tilde{w}(T_\varepsilon(x))}{\varepsilon}.$$

Here, $\tilde{q}_\varepsilon$, $\tilde{q}$, $\tilde{u}_\varepsilon$, $\tilde{u}$ are extensions of q_ε, q, u_ε, u by zero.

Invoking the change of variables $y = T_\varepsilon(x) = z + \varepsilon x$, we get

$$\frac{L(\varepsilon, u, q_\varepsilon) - L(\varepsilon, u, q)}{|\omega_\varepsilon|}$$
$$= \frac{1}{|\omega|} \int_\omega (\nu_2(|\,\mathbf{curl}\, u(T_\varepsilon(x))|) - \nu_1(|\,\mathbf{curl}\, u(T_\varepsilon(x))|))\,\mathbf{curl}\, u(T_\varepsilon(x)) \cdot \mathbf{curl}\, Q_\varepsilon \; dx.$$

Topological Derivative

Definition

The variation of the averaged adjoint state q_ε is defined pointwise a.e. in $\mathbf{R}^d$ by

$$Q_\varepsilon(x) := \frac{\tilde{z}_\varepsilon(T_\varepsilon(x)) - \tilde{z}(T_\varepsilon(x))}{\varepsilon},$$

and the variation of the direct state u_ε is defined pointwise a.e. in $\mathbf{R}^d$ by

$$W_\varepsilon(x) := \frac{\tilde{w}_\varepsilon(T_\varepsilon(x)) - \tilde{w}(T_\varepsilon(x))}{\varepsilon}.$$

Here, $\tilde{q}_\varepsilon$, $\tilde{q}$, $\tilde{u}_\varepsilon$, $\tilde{u}$ are extensions of q_ε, q, u_ε, u by zero.

Invoking the change of variables $y = T_\varepsilon(x) = z + \varepsilon x$, we get

$$\frac{L(\varepsilon, u, q_\varepsilon) - L(\varepsilon, u, q)}{|\omega_\varepsilon|}$$
$$= \frac{1}{|\omega|}\int_\omega (\nu_2(|\,\mathbf{curl}\, u(T_\varepsilon(x))|) - \nu_1(|\,\mathbf{curl}\, u(T_\varepsilon(x))|))\,\mathbf{curl}\, u(T_\varepsilon(x)) \cdot \mathbf{curl}\, Q_\varepsilon \; dx.$$
$$\xrightarrow{\varepsilon \to 0} \mathbf{???}$$

Conjecture

There exists $W \in BL(\mathbf{R}^3)$, such that

$$\partial W_\varepsilon \rightharpoonup \partial W \qquad \text{in } L_2(\mathbf{R}^3)^{3\times 3} \quad \text{as } \varepsilon \searrow 0$$
$$\mathbf{curl}(W_\varepsilon) \to \mathbf{curl}(W) \qquad \text{in } L_2(\mathbf{R}^3)^3 \quad \text{as } \varepsilon \searrow 0.$$

Moreover, W satisfies

$$\int_{\mathbf{R}^3} (F_\omega(x, \mathbf{curl}\, W + \mathbf{curl}(w_0)(z)) - F_\omega(x, \mathbf{curl}\, w_0(z))) \cdot \mathbf{curl}\, v \, dx = -\int_\omega \big(\nu_1(|\,\mathbf{curl}\, w_0(z)|) - \nu_2(|\,\mathbf{curl}\, w_0(z)|)\big)\, \mathbf{curl}\, w_0(z)\big) \cdot \mathbf{curl}\, v \, dx$$

for all $v \in BL(\mathbf{R}^3)$. Here $F_\omega(x,y) := \nu_1(y)\chi_{\mathbf{R}^3\setminus\omega}(x) + \nu_2(y)\chi_\omega(x)$.

Corollary of Conjecture

The weak limit $\mathbf{Q}$ of $\mathbf{Q}_\varepsilon = (\mathbf{q}_\varepsilon - \mathbf{q}_0)/\varepsilon \circ T_\varepsilon$ satisfies

$$\int_{\mathbf{R}^3} \left(\int_0^1 \partial_y \tilde{F}_\omega(x, s\,\mathbf{curl}\,W + \mathbf{curl}\,\mathbf{u}_0(z))\,\mathbf{curl}\,\bar{\mathbf{v}}\;ds \right) \cdot \mathbf{curl}\,\mathbf{Q}\;dx$$

$$= -\int_{\mathbf{R}^3} \Big[\int_0^1 \partial_y \tilde{F}_\omega(x, s\,\mathbf{curl}\,W + \mathbf{curl}\,u_0(z))\,\mathbf{curl}(\bar{\mathbf{v}})\;ds$$

$$- \partial_y \tilde{F}_\omega(x, \mathbf{curl}\,\mathbf{u}_0(z))\,\mathbf{curl}(\bar{\mathbf{v}}) \Big] \cdot \mathbf{curl}\,\mathbf{q}_0(z)\;dx$$

$$- \int_\omega [\partial_y F_2(\mathbf{curl}\,\mathbf{u}_0(z)) - \partial_y F_1(\mathbf{curl}\,\mathbf{u}_0(z))]\,\mathbf{curl}(\bar{\mathbf{v}}) \cdot \mathbf{curl}\,\mathbf{q}_0(z)\;dx.$$

Topological Derivative

Suppose that Conjecture holds. Then

$$\frac{L(\varepsilon, u, q_\varepsilon) - L(\varepsilon, u, q)}{|\omega_\varepsilon|}$$

$$= \frac{1}{|\omega|}\int_\omega (\nu_2(|\,\mathbf{curl}\, u(T_\varepsilon(x))|) - \nu_1(|\,\mathbf{curl}\, u(T_\varepsilon(x))|))\,\mathbf{curl}\, u(T_\varepsilon(x)) \cdot \mathbf{curl}\, Q_\varepsilon \; dx.$$

$$\stackrel{\varepsilon\to 0}{\longrightarrow} (\nu_2(|\,\mathbf{curl}\, u(z)|) - \nu_1(|\,\mathbf{curl}\, u(z)|))\,\mathbf{curl}\, u(z) \cdot \frac{1}{|\omega|}\int_\omega \mathbf{curl}\, Q \; dx$$

For the topological derivative, we would get

$$\partial J(z) = d_\ell g(0) = \partial_\ell L(0, u_0, q_0) + R$$

$$= (\nu_2(|\,\mathbf{curl}\, u(z)|) - \nu_1(|\,\mathbf{curl}\, u(z)|))\,\mathbf{curl}\, u(z) \cdot \left(\mathbf{curl}\, q(z) + \frac{1}{|\omega|}\int_\omega \mathbf{curl}\, Q \; dx\right)$$

It's not the end of the story

- computation of Q is numerically infeasible
- BUT: we can eliminate it by testing the equation for W by Q and using the fundamental theorem of calculus.

$$
\begin{aligned}
&R(u,p) = (\nu_1(|\,\mathbf{curl}\,u(z)|) - \nu_2(|\,\mathbf{curl}\,u(z)|))\,\mathbf{curl}\,u(z)\cdot\frac{1}{|\omega|}\int_\omega \mathbf{curl}\,Q\;dx\\
&=\frac{1}{|\omega|}\int_{\mathbf{R}^3}\Big[\tilde{F}_\omega(x,\mathbf{curl}\,W+\mathbf{curl}\,w_0(z)) - \tilde{F}_\omega(x,\mathbf{curl}\,w_0(z))\\
&\quad - \partial_y\tilde{F}_\omega(x,\mathbf{curl}\,w_0(z))\,\mathbf{curl}(W)\Big]\cdot\mathbf{curl}\,q_0\;dx\\
&\quad + \frac{1}{|\omega|}\int_\omega [\partial_y F_2(\mathbf{curl}\,w_0(z)) - \partial_y F_1(\mathbf{curl}\,w_0(z))]\,\mathbf{curl}(W)\cdot\mathbf{curl}\,q_0(z)\;dx.
\end{aligned}
$$