Martyna Soszyńska

# Temporal Multiscale Simulations for Multiphysics Problems

Strobl, July 2, 2019

**Institut für Analysis und Numerik**
Otto-von-Guericke-Universität Magdeburg

DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

# Outline

# Motivation

- Multiphysics problems model multiple physical phenomena interacting with each other.
- One of the classes of multiphysics problems is fluid-structure interactions (FSI).
- Each of the subproblems might change in time with different speeds.

**Goal**

The goal of this project is deriving robust numerical schemes for multiphysics problems composed of subproblems operating in different temporal scales.

# Motivation

Figure: Water floating around a ship

Figure: Fluid in a pipe

# Model Problem

$$
\begin{cases}
\partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\
-\Delta u_f = 0 & \text{in } I \times \Omega_f, \\
\partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\
\partial_t u_s = v_s & \text{in } I \times \Omega_s, \\
u_f = u_s & \text{on } I \times \Gamma, \\
v_f = v_s & \text{on } I \times \Gamma, \\
\partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\
u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\
u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\
u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \\
u_s(0) = v_s(0) = 0 & \text{in } \Omega_s,
\end{cases}
$$

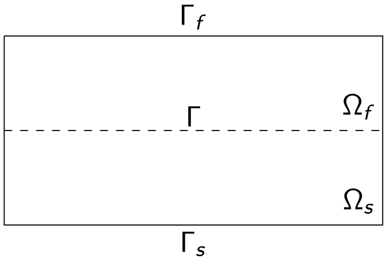


Figure: View of the domain

# Model Problem
## Heat Equation

$$
\begin{cases}
\partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\
-\Delta u_f = 0 & \text{in } I \times \Omega_f, \\
\partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\
\partial_t u_s = v_s & \text{in } I \times \Omega_s, \\
u_f = u_s & \text{on } I \times \Gamma, \\
v_f = v_s & \text{on } I \times \Gamma, \\
\partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\
u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\
u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\
u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \\
u_s(0) = v_s(0) = 0 & \text{in } \Omega_s,
\end{cases}
$$

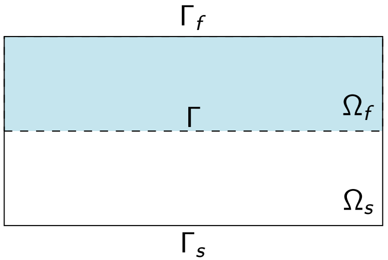


Figure: View of the domain

# Model Problem
## Wave Equation

$$
\begin{cases}
\partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\
-\Delta u_f = 0 & \text{in } I \times \Omega_f, \\
\partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\
\partial_t u_s = v_s & \text{in } I \times \Omega_s, \\
u_f = u_s & \text{on } I \times \Gamma, \\
v_f = v_s & \text{on } I \times \Gamma, \\
\partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\
u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\
u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\
u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \\
u_s(0) = v_s(0) = 0 & \text{in } \Omega_s,
\end{cases}
$$

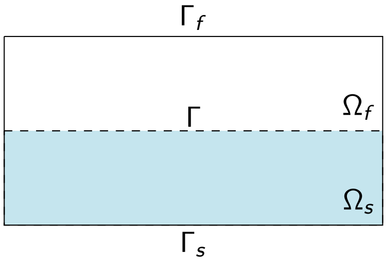


Figure: View of the domain

# Model Problem

## Poisson Equation

$$
\begin{cases}
\partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\
-\Delta u_f = 0 & \text{in } I \times \Omega_f, \\
\partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\
\partial_t u_s = v_s & \text{in } I \times \Omega_s, \\
u_f = u_s & \text{on } I \times \Gamma, \\
v_f = v_s & \text{on } I \times \Gamma, \\
\partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\
u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\
u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\
u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \\
u_s(0) = v_s(0) = 0 & \text{in } \Omega_s,
\end{cases}
$$

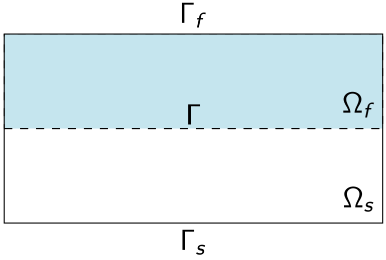


Figure: View of the domain

# Model Problem
## Coupling Conditions

$$
\begin{cases}
\partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\
-\Delta u_f = 0 & \text{in } I \times \Omega_f, \\
\partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\
\partial_t u_s = v_s & \text{in } I \times \Omega_s, \\
u_f = u_s & \text{on } I \times \Gamma, \\
v_f = v_s & \text{on } I \times \Gamma, \\
\partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\
u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\
u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\
u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \\
u_s(0) = v_s(0) = 0 & \text{in } \Omega_s,
\end{cases}
$$

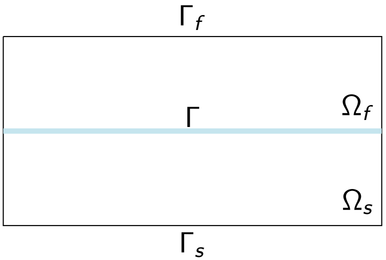


Figure: View of the domain

# Model Problem
## Boundary Conditions

$$
\begin{cases}
\partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\
-\Delta u_f = 0 & \text{in } I \times \Omega_f, \\
\partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\
\partial_t u_s = v_s & \text{in } I \times \Omega_s, \\
u_f = u_s & \text{on } I \times \Gamma, \\
v_f = v_s & \text{on } I \times \Gamma, \\
\partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\
u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\
u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\
u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \\
u_s(0) = v_s(0) = 0 & \text{in } \Omega_s,
\end{cases}
$$

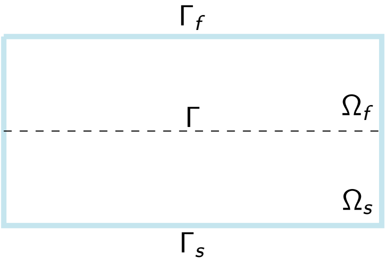


Figure: View of the domain

# Model Problem
## Initial Values

$$
\begin{cases}
\partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\
-\Delta u_f = 0 & \text{in } I \times \Omega_f, \\
\partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\
\partial_t u_s = v_s & \text{in } I \times \Omega_s, \\
u_f = u_s & \text{on } I \times \Gamma, \\
v_f = v_s & \text{on } I \times \Gamma, \\
\partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\
u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\
u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\
u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \\
u_s(0) = v_s(0) = 0 & \text{in } \Omega_s,
\end{cases}
$$

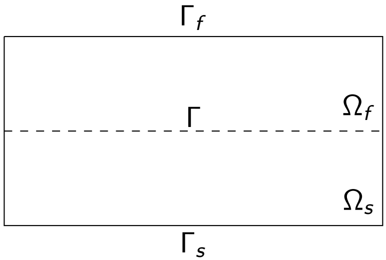


Figure: View of the domain

# Model Problem

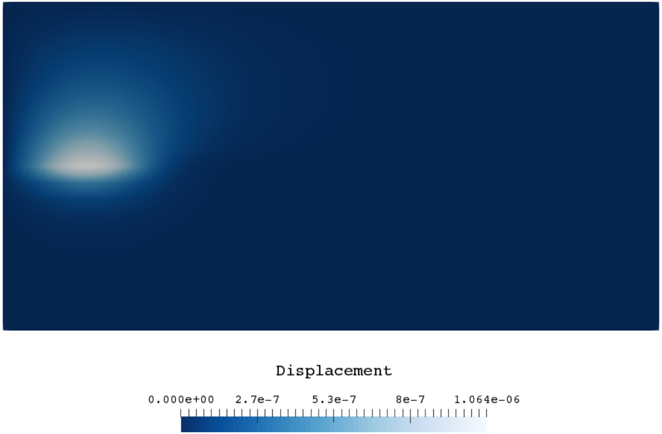


Figure: An example of a simple FSI problem

# Model Problem

$$\begin{cases} \partial_t v_f - \Delta v_f = f & \text{in } I \times \Omega_f, \\ -\Delta u_f = 0 & \text{in } I \times \Omega_f, \\ u_f = u_s & \text{on } I \times \Gamma, \\ v_f = v_s & \text{on } I \times \Gamma, \\ u_f = v_f = 0 & \text{on } I \times \Gamma_f, \\ u_f(0) = v_f(0) = 0 & \text{in } \Omega_f, \end{cases}$$

$$\begin{cases} \partial_t v_s - \Delta u_s = 0 & \text{in } I \times \Omega_s, \\ \partial_t u_s = v_s & \text{in } I \times \Omega_s, \\ \partial_{\mathbf{n}_s} u_s = \partial_{\mathbf{n}_f} v_f & \text{on } I \times \Gamma, \\ u_s = v_s = 0 & \text{on } I \times \Gamma_s, \\ u_s(0) = v_s(0) = 0 & \text{in } \Omega_s, \end{cases}$$

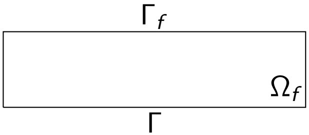


Figure: View of the fluid domain

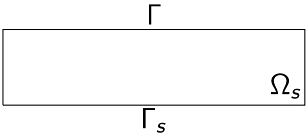


Figure: View of the solid domain

# Model Problem

## Heat Equation

- parabolic
- smooth

$\Rightarrow$ slow dynamics

## Wave Equation

- hyperbolic
- oscillatory

$\Rightarrow$ fast dynamics

**Goal**

Consider different time-step sizes for each of the subproblems.

**Method**

Apply partitioned approach using effective decoupling strategies.

# Fluid Subproblem
## Variational Formulation

Find $u_f, v_f \in X_f$ such that:

$$
\begin{cases}
\int_I (\partial_t v_f, \varphi_f)_f \,\mathrm{d}t + \int_I (\nabla v_f, \nabla \varphi_f)_f \,\mathrm{d}t \\
- \int_I \langle \partial_{\mathbf{n}_f} v_f, \varphi_f \rangle_\Gamma \,\mathrm{d}t + \frac{\gamma}{h} \int_I \langle v_f, \varphi_f \rangle_\Gamma \,\mathrm{d}t & = \int_I (f, \varphi_f)_f \,\mathrm{d}t + \frac{\gamma}{h} \int_I \langle v_s, \varphi_f \rangle_\Gamma \,\mathrm{d}t, \\
\int_I (\nabla u_f, \nabla \psi_f)_f \,\mathrm{d}t - \int_I \langle \partial_{\mathbf{n}_f} u_f, \psi_f \rangle_\Gamma \,\mathrm{d}t \\
+ \frac{\gamma}{h} \int_I \langle u_f, \psi_f \rangle_\Gamma \,\mathrm{d}t & = \frac{\gamma}{h} \int_I \langle u_s, \psi_f \rangle_\Gamma \,\mathrm{d}t
\end{cases}
$$

for all $\varphi_f, \psi_f \in X_f$.

# Solid Subproblem
## Variational Formulation

Find $u_s, v_s \in X_s$ such that:

$$\begin{cases} \int_I (\partial_t v_s, \varphi_s)_s \,\mathrm{d}t + \int_I (\nabla u_s, \nabla \varphi_s)_s \,\mathrm{d}t & = \int_I \langle \partial_{\mathbf{n}_f} v_f, \varphi_s \rangle_\Gamma \,\mathrm{d}t, \\ \int_I (\partial_t u_s, \psi_s)_s \,\mathrm{d}t - \int_I (v_s, \psi_s)_s \,\mathrm{d}t & = 0 \end{cases}$$

for all $\varphi_s, \psi_s \in X_s$.

# Notations

We introduce notations

$$\mathbf{U}_f := \begin{pmatrix} u_f \\ v_f \end{pmatrix}, \mathbf{U}_s := \begin{pmatrix} u_s \\ v_s \end{pmatrix}, \mathbf{\Phi}_f := \begin{pmatrix} \varphi_f \\ \psi_f \end{pmatrix}, \mathbf{\Phi}_s := \begin{pmatrix} \varphi_s \\ \psi_s \end{pmatrix}.$$

We define

$$\begin{aligned} A_f(\mathbf{U}_f, \mathbf{U}_s)(\mathbf{\Phi}_f) &:= (\nabla v_f, \nabla \varphi_f)_f - \langle \partial_{\mathbf{n}_f} v_f, \varphi_f \rangle_\Gamma + \frac{\gamma}{h} \langle v_f, \varphi_f \rangle_\Gamma \\ &\quad - (f, \varphi_f)_f - \frac{\gamma}{h} \langle v_s, \varphi_f \rangle_\Gamma + (\nabla u_f, \nabla \psi_f)_f \\ &\quad - \langle \partial_{\mathbf{n}_f} u_f, \psi_f \rangle_\Gamma + \frac{\gamma}{h} \langle u_f, \psi_f \rangle_\Gamma - \frac{\gamma}{h} \langle u_s, \psi_f \rangle_\Gamma, \\ A_s(\mathbf{U}_f, \mathbf{U}_s)(\mathbf{\Phi}_s) &:= (\nabla u_s, \nabla \varphi_s)_s - \langle \partial_{\mathbf{n}_f} v_f, \varphi_s \rangle_\Gamma - (v_s, \psi_s)_s. \end{aligned}$$

# Variational Formulation

Find $\mathbf{U}_f \in X_f \times X_f$ and $\mathbf{U}_s \in X_s \times X_s$ such that:

$$\begin{cases} \int_I (\partial_t v_f, \varphi_f)_f \, \mathrm{d}t + \int_I A_f(\mathbf{U}_f, \mathbf{U}_s)(\mathbf{\Phi}_f) \, \mathrm{d}t = 0 \\ \int_I (\partial_t v_s, \varphi_s)_s \, \mathrm{d}t + \int_I (\partial_t u_s, \psi_s)_s \, \mathrm{d}t + \int_I A_s(\mathbf{U}_f, \mathbf{U}_s)(\mathbf{\Phi}_s) \, \mathrm{d}t = 0 \end{cases}$$

for all $\mathbf{\Phi}_f \in X_f \times X_f$ and $\mathbf{\Phi}_s \in X_s \times X_s$.

# Time Discretization
## Crank-Nicolson Method

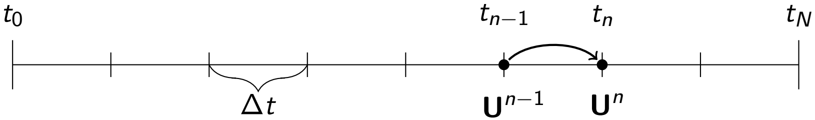


Figure: Time interval

# Time Discretization
## Crank-Nicolson Method

We define

$$\begin{aligned}
B_f(\mathbf{U}_f^n, \mathbf{U}_s^n)(\mathbf{\Phi}_f) :=\ & \left(v_f^n - v_f^{n-1}, \varphi_f\right)_f \\
& + \frac{\Delta t}{2}\left(A_f(\mathbf{U}_f^n, \mathbf{U}_s^n)(\mathbf{\Phi}_f) + A_f(\mathbf{U}_f^{n-1}, \mathbf{U}_s^{n-1})(\mathbf{\Phi}_f)\right), \\
B_s(\mathbf{U}_f^n, \mathbf{U}_s^n)(\mathbf{\Phi}_s) :=\ & \left(v_s^n - v_s^{n-1}, \varphi_s\right)_s + \left(u_s^n - u_s^{n-1}, \psi_s\right)_s \\
& + \frac{\Delta t}{2}\left(A_s(\mathbf{U}_f^n, \mathbf{U}_s^n)(\mathbf{\Phi}_s) + A_s(\mathbf{U}_f^{n-1}, \mathbf{U}_s^{n-1})(\mathbf{\Phi}_s)\right)
\end{aligned}$$

# Space Discretization
## Finite Element Method

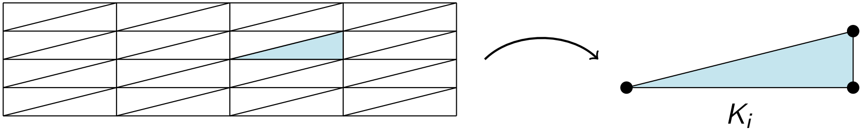


Figure: Rectangle regular mesh

$$X_f^h := \left\{ v \in C^0(\Omega_f) : v_{|\Gamma_f} = 0, v_{|K_i} \in \mathcal{P}_1 \text{ for all } i \right\},$$
$$X_s^h := \left\{ v \in C^0(\Omega_s) : v_{|\Gamma_s} = 0, v_{|K_j} \in \mathcal{P}_1 \text{ for all } j \right\}.$$

# Variational Formulation

Find $\mathbf{U}_f^n \in X_f^h \times X_f^h$ and $\mathbf{U}_s^n \in X_s^h \times X_s^h$ such that:

$$\begin{cases} B_f(\mathbf{U}_f^n, \mathbf{U}_s^n)(\mathbf{\Phi}_f) = 0, \\ B_s(\mathbf{U}_f^n, \mathbf{U}_s^n)(\mathbf{\Phi}_s) = 0 \end{cases}$$

for all $\mathbf{\Phi}_f \in X_f^h \times X_f^h$ and $\mathbf{\Phi}_s \in X_s^h \times X_s^h$.

# Variational Formulation
## Decoupled Problem

For a given $\mathbf{U}_s^{n,(m-1)} \in X_s^h \times X_s^h$ find $\mathbf{U}_f^{n,(m)} \in X_f^h \times X_f^h$ and $\widetilde{\mathbf{U}}_s^{n,(m)} \in X_s^h \times X_s^h$ such that:

$$\begin{cases} B_f(\mathbf{U}_f^{n,(m)}, \mathbf{U}_s^{n,(m-1)})(\mathbf{\Phi}_f) = 0, \\ B_s(\mathbf{U}_f^{n,(m)}, \widetilde{\mathbf{U}}_s^{n,(m)})(\mathbf{\Phi}_s) = 0 \end{cases}$$

for all $\mathbf{\Phi}_f \in X_f^h \times X_f^h$ and $\mathbf{\Phi}_s \in X_s^h \times X_s^h$.

# Decoupling Algorithm

1. Setting boundary conditions on fluid subproblem.
2. Solving the fluid subproblem.
3. Setting boundary conditions on solid subproblem.
4. Solving the solid subproblem.

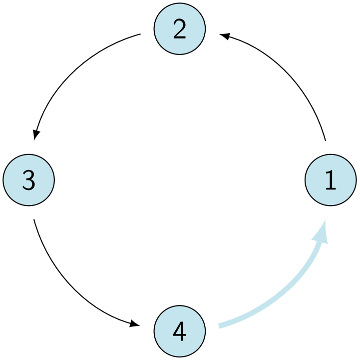


Figure: Illustration of one decoupling iteration

# Fractional Time-stepping

| Number of fractional time-steps | Fractional time-step size | Extrapolation formula |
|---|---|---|
| $K$ | $\frac{\Delta t}{K}$ | $\mathbf{U}_f^{n+\frac{k}{n}} := \frac{K-k}{K}\mathbf{U}_f^n + \frac{k}{K}\mathbf{U}_f^{n+1}$ for $k \in \{0, ..., K\}$ |
| $M$ | $\frac{\Delta t}{M}$ | $\mathbf{U}_s^{n+\frac{m}{n}} := \frac{M-m}{M}\mathbf{U}_s^n + \frac{m}{M}\mathbf{U}_s^{n+1}$ for $m \in \{0, ..., M\}$ |

# Relaxation Method

Let $R : X_s^h \times X_s^h \to X_s^h \times X_s^h$ be defined as [1]

$$R(\mathbf{U}_s^{n,(m-1)}) := \tau \widetilde{\mathbf{U}}_s^{n,(m)} + (1-\tau)\mathbf{U}_s^{n,(m-1)},$$

where $\tau \in [0,1]$. We set

$$\begin{cases} \mathbf{U}_s^{n,(0)} := \mathbf{U}_s^{n-1}, \\ \mathbf{U}_s^{n,(m)} := R(\mathbf{u}_s^{n,(m-1)}). \end{cases}$$

# Shooting Method

Let $S : X_s^h \times X_s^h \to X_s^h \times X_s^h$ be defined as [2]

$$S(\mathbf{U}_s^{n,(m-1)}) := \mathbf{U}_s^{n,(m-1)} - \widetilde{\mathbf{U}}_s^{n,(m)}.$$

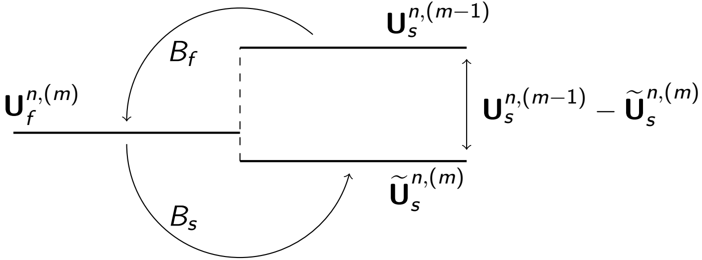


Figure: Illustration of the shooting function $S$

# Shooting Method

We employ Newton's method for finding the root of function $S$

$$\begin{cases} S'(\mathbf{U}_s^{n,(m-1)})\mathbf{d} & = -S(\mathbf{U}_s^{n,(m-1)}) \\ \mathbf{U}_s^{n,(m)} & := \mathbf{U}_s^{n,(m-1)} + \mathbf{d}. \end{cases}$$

Jacobian matrix vector product is approximated using directional derivative

$$S'(\mathbf{U}_s^{n,(m-1)})\mathbf{d} \approx \frac{S(\mathbf{U}_s^{n,(m-1)} + \varepsilon\mathbf{d}) - S(\mathbf{U}_s^{n,(m-1)})}{\varepsilon}$$

and incorporated as a part of an iterative linear solver. For our problem we chose generalized minimal residual method (GMRES).

# Comparison

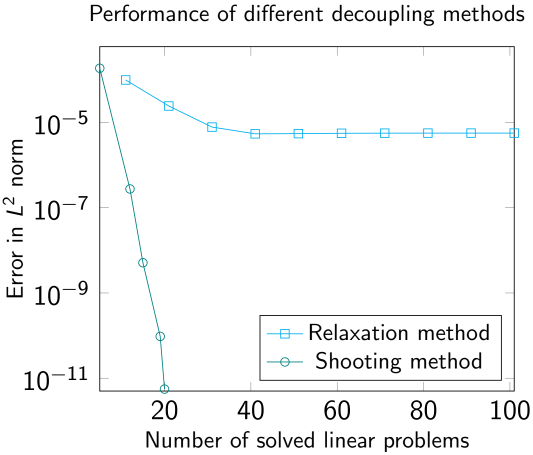


Figure: Comparison between relaxation and shooting methods

# Future Goals

- We would like to develop efficient ways of transferring normal derivatives across the interface between two meshes.
- As the next step, we would like to focus on suitable preconditioning of the shooting method.

# Bibliography


J. Stoer and R. Bulirsch.
*Introduction to numerical analysis*, volume 12 of *Texts in Applied Mathematics*.
Springer-Verlag, New York, third edition, 2002.
Translated from the German by R. Bartels, W. Gautschi and C. Witzgall.

Thomas Carraro, Michael Geiger, and Stefan Körkel, editors.
*Multiple shooting and time domain decomposition methods*, volume 9 of *Contributions in Mathematical and Computational Sciences*.
Springer, Cham, 2015.
Papers based on the International Workshop (MuSTDD 2013) held at Heidelberg University, Heidelberg, May 6–8, 2013.

Thomas Richter.
*Fluid-structure interactions*, volume 118 of *Lecture Notes in Computational Science and Engineering*.
Springer, Cham, 2017.