Initial source recovery of the wave equation given internal boundary measurements

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Geometry + Simulation
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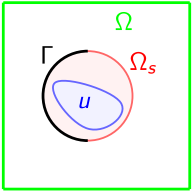
We study the inverse problem



Assuming that the given data $z_d \in L^2(\Gamma_T)$, where $\Gamma_T = \Gamma \times [0, T]$, we aim at identifiying the initial condition $u \in H_0^1(\Omega)$ which minimizes

$$\min_{u \in H_0^1(\Omega)} f(u) = \frac{1}{2}\|y - z_d\|^2_{L^2(\Gamma_T)} + \frac{\alpha}{2}\|u\|^2_{H_0^1(\Omega)},$$

where y solves the hyperbolic equation

$$\begin{cases} y'' - \Delta y = 0, & (x,t) \in \Omega \times (0,T) \\ y = 0, & (x,t) \in \partial\Omega \times [0,T] \\ y(0) = u,\ y'(0) = 0, & x \in \Omega. \end{cases}$$

Outline

- Solution theory
- Saddle point problem
- Robust preconditioning
- Stable discretization
- Numerical result
- Discussion

The wave equation: Solution theory

For any $(f, y_0, y_1) \in L^2(\Omega \times (0,T)) \times H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique

$$y \in \mathcal{W} = \{y \in L^2(0,T; H_0^1(\Omega)) \,|\, y' \in L^2(0,T; L^2(\Omega)),\, y'' \in L^2(0,T; H^{-1}(\Omega))\}$$

which satisfies (a variational form of) the wave equation with $y(0) = y_0$, $y'(0) = y_1$.

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which satisfies (a variational form of) the wave equation with $y(0) = y_0$, $y'(0) = y_1$.

Additional regularity: $(y, y') \in C([0,T]; H_0^1(\Omega)) \times C([0,T]; L^2(\Omega))$,

$$\begin{aligned} \|y\|^2_{C([0,T];H_0^1(\Omega))} + \|y'\|^2_{C([0,T];L^2(\Omega))} \\ &\leqslant \text{const}\left(\|f\|^2_{L^2(\Omega\times(0,T))} + \|y_0\|^2_{H_0^1(\Omega)} + \|y_1\|^2_{L^2(\Omega)} \right) \\ &= \text{const}\left(\|Ly\|^2_{L^2(0,T;L^2(\Omega))} + \|y(0)\|^2_{H_0^1(\Omega)} + \|y'(0)\|^2_{L^2(\Omega)} \right) \end{aligned}$$

where

$$Ly = y'' - \Delta y.$$

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$$\begin{aligned} \|y\|^2_{C([0,T];H_0^1(\Omega))} + \|y'\|^2_{C([0,T];L^2(\Omega))} + \|Ly\|^2_{L^2(0,T;L^2(\Omega))} \\ \leqslant \text{const} \left(\|f\|^2_{L^2(\Omega\times(0,T))} + \|y_0\|^2_{H_0^1(\Omega)} + \|y_1\|^2_{L^2(\Omega)}\right) \\ = \text{const} \left(\|Ly\|^2_{L^2(0,T;L^2(\Omega))} + \|y(0)\|^2_{H_0^1(\Omega)} + \|y'(0)\|^2_{L^2(\Omega)}\right) \end{aligned}$$

where

$$Ly = y'' - \Delta y.$$

Definition

Let $Y \subset \mathcal{W}$ denote the space spanned by y as (f, y_0, y_1) ranges over $L^2(\Omega \times (0,T)) \times H_0^1(\Omega) \times L^2(\Omega)$,

$$Y = \{y \in \mathcal{W} \mid (y, y') \in C([0,T]; H_0^1(\Omega)) \times C([0,T]; L^2(\Omega)),\ Ly \in L^2(0,T; L^2(\Omega))\}.$$

Endowed with the norm

$$\|y\|_Y^2 = \|Ly\|^2_{L^2(0,T;L^2(\Omega))} + \|y(0)\|^2_{H_0^1(\Omega)} + \|y'(0)\|^2_{L^2(\Omega)},$$

this is a Hilbert space.

Lemma

There exists a positive constant $C = C(\Omega, \Omega_s, T)$ such that $\|y\|_{L^2(\Gamma_T)} \leqslant C\|y\|_Y$ for all $y \in Y$.

Corollary

For every $\alpha > 0$, the norm

$$\|y\|_{Y_\alpha}^2 = \|y\|^2_{L^2(\Gamma_T)} + \|Ly\|^2_{L^2(0,T;L^2(\Omega))} + \alpha\|y(0)\|^2_{H_0^1(\Omega)} + \|y'(0)\|^2_{L^2(\Omega)}$$

is equivalent to $\|\cdot\|_Y$ on Y.

The saddle point problem

Set $\Lambda = L^2(0,T;L^2(\Omega)) \times L^2(\Omega)$. The Lagrangian reads

$$\mathcal{L} : Y \times \Lambda \to \mathbb{R}, \quad \mathcal{L}(y,\lambda) = \frac{1}{2} a(y,y) + b(y,\lambda) - l(y),$$

where the (continuous) bilinear forms $a : Y \times Y \to \mathbb{R}$ and $b : Y \times \Lambda \to \mathbb{R}$ are given by

$$a(y,\overline{y}) = (y,\overline{y})_{L^2(\Gamma_T)} + \alpha\,(y(0),\overline{y}(0))_{H_0^1(\Omega)}\,,$$
$$b(y,\lambda) = (Ly,w)_{L^2(0,T;L^2(\Omega))} + \left(y'(0),\phi\right)_{L^2(\Omega)}\,, \quad \lambda = (w,\phi),$$

together with the linear form $l : Y \to \mathbb{R}$, $l(y) = (y,z_d)_{L^2(\Gamma_T)}$. Find $(y,\lambda) \in Y \times \Lambda$ such that

$$\begin{cases} a(y,\overline{y}) + b(\overline{y},\lambda) = l(\overline{y}) & \text{for all } \overline{y} \in Y, \\ b(y,\overline{\lambda}) = 0 & \text{for all } \overline{\lambda} \in \Lambda. \end{cases}$$

Well-posedness of the saddle point problem

Existence and uniqueness under the Brezzi conditions:

1. The bilinear forms $a : Y_\alpha \times Y_\alpha \to \mathbb{R}$ and $b : Y_\alpha \times \Lambda \to \mathbb{R}$ are bounded,
$$|a(y, \bar{y})| \leqslant C_a \|y\|_{Y_\alpha} \|\bar{y}\|_{Y_\alpha}, \quad |b(y, \lambda)| \leqslant C_b \|y\|_{Y_\alpha} \|\lambda\|_\Lambda$$
for all $y, \bar{y} \in Y_\alpha, \lambda \in \Lambda$.
2. The bilinear form a is coercive on $\mathcal{N}(b) = \{y \in Y_\alpha \mid b(y, \lambda) = 0 \text{ for all } \lambda \in \Lambda\}$. There exists a constant $k_0 > 0$ such that
$$a(y, y) \geqslant k_0 \|y\|^2_{Y_\alpha}$$
for all $y \in \mathcal{N}(b)$.
3. The bilinear form b satisfies the inf-sup condition. There exists a constant $\beta_0 > 0$ such that
$$\sup_{0 \neq y \in Y_\alpha} \frac{b(y, \lambda)}{\|y\|_{Y_\alpha}} \geqslant \beta_0 \|\lambda\|_\Lambda$$
for all $\lambda \in \Lambda$.

Theorem

The bilinear forms a and b satisfy the Brezzi conditions. Moreover, C_a, C_b, k_0, β_0 can be chosen independent of α for
$$a : Y_\alpha \times Y_\alpha \to \mathbb{R} \quad \text{and} \quad b : Y_\alpha \times \Lambda \to \mathbb{R}.$$

The saddle point problem in operator notation

Let $A : Y_\alpha \to Y_\alpha'$ and $B : Y_\alpha \to \Lambda'$ be given by

$$\langle Ay, \bar{y} \rangle_{Y_\alpha' \times Y_\alpha} = a(y, \bar{y}) \quad \text{and} \quad \langle By, \lambda \rangle_{\Lambda' \times \Lambda} = b(y, \lambda) \quad \text{for all} \quad y, \bar{y} \in Y_\alpha,\ \lambda \in \Lambda.$$

Using this operator notation, the saddle point problem can be written as

$$\mathcal{A}_\alpha : Y_\alpha \times \Lambda \to Y_\alpha' \times \Lambda', \qquad \mathcal{A}_\alpha \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} A & B' \\ B & 0 \end{pmatrix} \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} l \\ 0 \end{pmatrix}.$$

Corollary

For every $\alpha > 0$ the linear self-adjoint operator $\mathcal{A}_\alpha$ is bounded and continuously invertible.
Moreover, there exist positive constants $\overline{c}$, $\underline{c}$, both independent of α, such that

$$\|\mathcal{A}_\alpha\|_{\mathcal{L}(Y_\alpha \times \Lambda, (Y_\alpha \times \Lambda)')} \leqslant \overline{c} \quad \text{and} \quad \|\mathcal{A}_\alpha^{-1}\|_{\mathcal{L}((Y_\alpha \times \Lambda)', Y_\alpha \times \Lambda)} \leqslant \underline{c}^{-1}.$$

Robust preconditioning

For the operator $\mathcal{A}_\alpha$ we define the *preconditioner*

$$\mathcal{B}_\alpha : Y_\alpha \times \Lambda \to Y_\alpha' \times \Lambda', \qquad \mathcal{B}_\alpha = \begin{pmatrix} P_{Y_\alpha} & 0 \\ 0 & P_\Lambda \end{pmatrix}.$$

Thus, $\mathcal{B}_\alpha^{-1}\mathcal{A}_\alpha : Y_\alpha \times \Lambda \to Y_\alpha \times \Lambda$ is an isomorphism and self-adjoint with respect to the inner product on $Y_\alpha \times \Lambda$. Moreover,

$$\|\mathcal{B}_\alpha^{-1}\mathcal{A}_\alpha\|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)} = \|\mathcal{A}_\alpha\|_{\mathcal{L}(Y_\alpha \times \Lambda, (Y_\alpha \times \Lambda)')} \quad \text{and}$$
$$\|(\mathcal{B}_\alpha^{-1}\mathcal{A}_\alpha)^{-1}\|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)} = \|\mathcal{A}_\alpha^{-1}\|_{\mathcal{L}((Y_\alpha \times \Lambda)', Y_\alpha \times \Lambda)}.$$

Corollary

The condition number of $\mathcal{B}_\alpha^{-1}\mathcal{A}_\alpha$ is uniformly bounded with respect to α. In other words

$$\kappa(\mathcal{B}_\alpha^{-1}\mathcal{A}_\alpha) := \|\mathcal{B}_\alpha^{-1}\mathcal{A}_\alpha\|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)}\|(\mathcal{B}_\alpha^{-1}\mathcal{A}_\alpha)^{-1}\|_{\mathcal{L}(Y_\alpha \times \Lambda, Y_\alpha \times \Lambda)} \leqslant \frac{\overline{c}}{\underline{c}}.$$

Discretized problem

Given conforming discretization spaces $Y_h \subset Y$ and $\Lambda_h \subset \Lambda$:

$$\underbrace{\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}}_{\mathcal{A}_h} \underbrace{\begin{pmatrix} \underline{y}_h \\ \underline{\lambda}_h \end{pmatrix}}_{\mathbf{x}_h} = \underbrace{\begin{pmatrix} \underline{z}_{d,h} \\ 0 \end{pmatrix}}_{\mathbf{f}_h}$$

$$A_h : \quad \langle A\, y_h, \overline{y_h} \rangle = (y_h, \overline{y}_h)_{L^2(\Gamma_T)} + \alpha\, (y_h(0), \overline{y_h}(0))_{H_0^1(\Omega)}$$

$$B_h : \quad \langle B y_h, \overline{\lambda}_h \rangle = (L y_h, \overline{w}_h)_{L^2(0,T;L^2(\Omega))} + \left(y_h'(0), \overline{\phi}_h\right)_{L^2(\Omega)}$$

with $\lambda_h = (w_h, \phi_h)$

$$(L y_h, \overline{w}_h)_{L^2(0,T;L^2(\Omega))} = \int_0^T \int_\Omega y_h'' \, \overline{w}_h \, dx\, dt - \int_0^T \int_\Omega \Delta y_h \, \overline{w}_h \, dx\, dt$$

Stable discretization

We need to satisfy the discrete Brezzi conditions:

- **Coercivity:** We need to choose Y_h s.t.,

$$\langle Ay_h, y_h \rangle \geqslant c_1 \|y_h\|^2_{Y_{\alpha,h}} \quad \text{for all} \quad y_h \in \ker B.$$

- **Inf-sup:** We need to choose Y_h and Λ_h s.t.

$$\inf_{0 \neq \lambda_h \in \Lambda_h} \sup_{0 \neq y_h \in Y_h} \frac{\langle By_h, \lambda_h \rangle}{\|y_h\|_{Y_{\alpha,h}} \|\lambda_h\|_{\Lambda_h}} \geqslant \delta_0 > 0.$$

Stable discretization: Coercivity

We circumvent the coercivity by replacing A with

$$\langle A_\rho y_h, \overline{y_h} \rangle = \langle A y_h, \overline{y_h} \rangle + \rho \left(L y_h, L \overline{y}_h \right)_{L^2(0,T;L^2(\Omega))} + \rho (y_h'(0), \overline{y}_h'(0))_{L^2(\Omega)}$$

for $\rho > 0$.

On the continuous level this is consistence since

$$Ly = 0 \quad \text{and} \quad y'(0) = 0.$$

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for $\rho > 0$.

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$$Ly = 0 \quad \text{and} \quad y'(0) = 0.$$

Note that:

$$\langle A_\rho y_h, y_h\rangle \geqslant c_\rho \|y_h\|^2_{Y_{\alpha,h}} \quad \text{for all} \quad y_h \in Y_h.$$

This stabilization method is sometimes called augmented Lagrangian.

Stable discretization: Inf-sup

Assume we have chosen Y_h such that $Y_h \subset Y$. We then choose Λ_h to be

$$\Lambda_h := \left\{ \lambda_h = (Ly_h, y_h'(0)) \,|\, y_h(0) = 0,\ y_h \in Y_h \right\}.$$

Note that $\Lambda_h \subset \Lambda := L^2(0, T; L^2(\Omega)) \times L^2(\Omega)$.

A basis of Λ_h is found by using a basis of Y_h.

The **inf-sup** condition holds with the same constant as the continuous case!

Preconditioning

We precondition the system:

$$\mathcal{B}_{\rho,h}^{-1}\mathcal{A}_{\rho,h} = \begin{pmatrix} P_{Y_h} & 0 \\ 0 & P_{\Lambda_h} \end{pmatrix}^{-1} \begin{pmatrix} A_{\rho,h} & B_h^T \\ B_h & 0 \end{pmatrix}$$

where

$$P_{Y_h} = A_{\rho,h} \quad \text{and} \quad P_{\Lambda_h} = \|\lambda_h\|_\Lambda^2 = (\lambda_h, \lambda_h)_{L^2}.$$

With this preconditioner the condition number κ is independed of α and h, but **depend** on ρ!

Discretization spaces

Our domain is a rectangle.
We consider tensor product B-splines as discretization space

$$S_{p,\ell} = S_{p_t,\ell_t}(0,T) \otimes S_{p_x,\ell_x}((0,1)^d).$$

Our discretization spaces are

$$Y_h := S_{p,\ell} \cap H_0^1(0,1) \quad \text{and} \quad \Lambda_h := \left\{ \lambda_h = (L y_h, y_h'(0)) \,|\, y_h \in Y_{h,0} \right\}.$$

Our observation domain is the boundary of $(\frac{1}{4}, \frac{3}{4})^d$.

Condition numbers

$\ell \backslash \alpha$	10^0	10^{-2}	10^{-5}	10^{-7}	DoFs
2	2.65163	2.64811	2.64779	2.64770	176
3	2.66313	2.65083	2.64879	2.64879	1216

Table: Condition numbers : $\mathcal{B}_{\rho,h}^{-1}\mathcal{A}_{\rho=1,h}$, $d = 2$ and $p = 2$

$\ell \backslash \alpha$	10^0	10^{-2}	10^{-5}	10^{-7}	DoFs
2	2.65112	2.649	2.64887	2.64587	704

Table: Iteration numbers: $\mathcal{B}_{\rho,h}^{-1}\mathcal{A}_{\rho=1,h}$, $d = 3$ and $p = 2$

Iteration numbers

$\ell \backslash \alpha$	10^0	10^{-2}	10^{-5}	10^{-7}	DoFs
2	9	9	9	9	176
3	9	9	9	9	1216
4	9	7	7	7	8960
5	7	7	7	7	68608

Table: Iteration numbers: $\mathcal{B}_{\rho,h}^{-1}\mathcal{A}_{\rho=1,h}$, $d = 2$ and $p = 2$

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Iteration numbers

$\rho\backslash\alpha$	10^0	10^{-2}	10^{-5}	10^{-7}
10^0	7	7	7	7
10^{-2}	23	21	19	19
10^{-5}	351	343	163	153
10^{-7}	3039	2940	1145	769

Table: Iteration numbers: $\mathcal{B}_h^{-1}\mathcal{A}_{\rho,h}$, $d = 2$, $p = 2$ and $\ell = 5$

Initial source recovery in 2D

1. Choose an initial image $y(0, x)$ on $(0, 1)^2$
2. Calculate $y(t, x)$ on $(0, T) \times (0, 1)^2$ by solving

$$\begin{aligned} \partial_{tt} y - \Delta y &= 0 \quad \text{in} \quad (0, T) \times (0, 1)^2 \\ \partial_t y(0) &= 0 \quad \text{on} \quad (0, 1)^2 \end{aligned}$$

3. Set $z_d = y|_{\Gamma_T}$
4. Use z_d in the optimal control problem and calculate $\tilde{y}$

Initial image

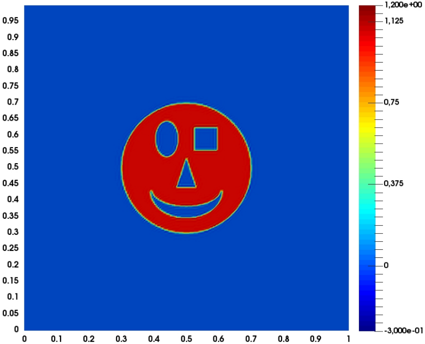

Figure: Initial image

Initial $y(0)$, projection

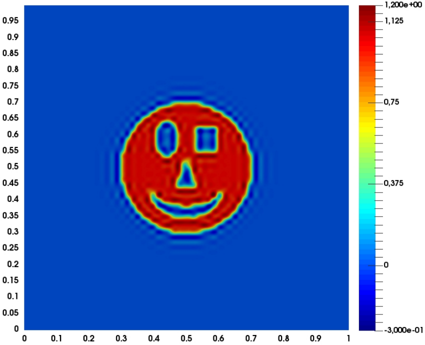

Figure: $p = 2$ and $\ell = 6$

Recovery $\tilde{y}(0)$ with full observation

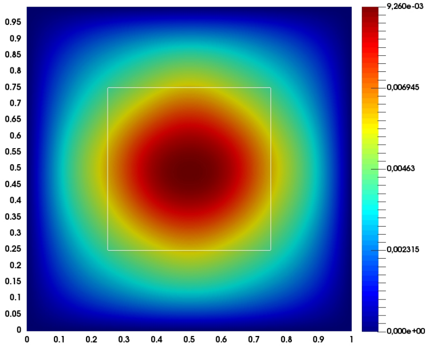

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 1.0 \quad \rho = 1.0$

Recovery $\tilde{y}(0)$ with full observation

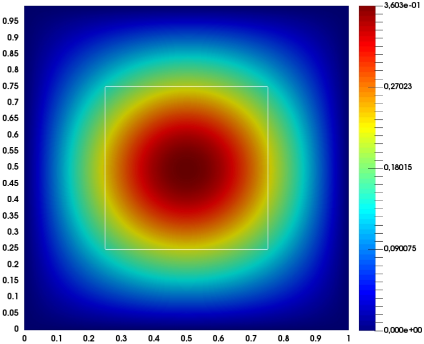

Figure: $p = 2 \quad \ell = \quad \alpha = 10^{-7} \quad \rho = 1.0$

Recovery $\tilde{y}(0)$ with full observation

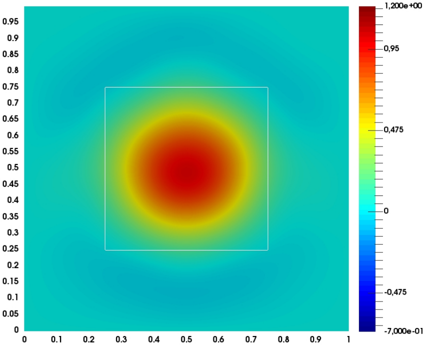

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 10^{-7} \quad \rho = 10^{-2}$

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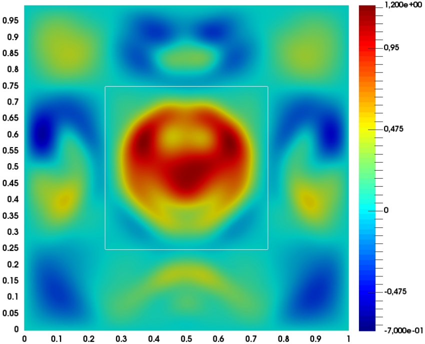

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 10^{-7} \quad \rho = 10^{-5}$

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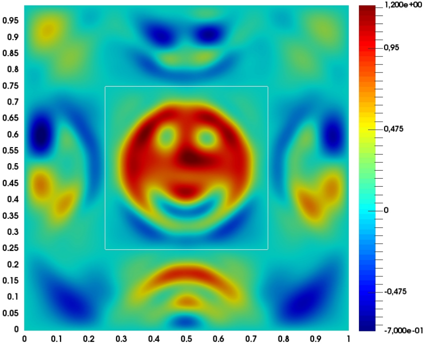

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 10^{-7} \quad \rho = 10^{-6}$

Recovery $\tilde{y}(0)$ with full observation

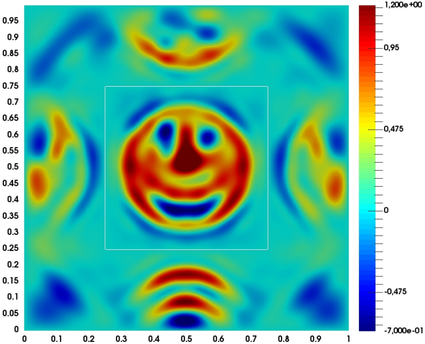

Figure: $p = 2 \quad \ell = 6 \quad \alpha = 10^{-7} \quad \rho = 10^{-7}$

Discussion

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  - Need low ρ to recover image, but iterative methods does not converge for very low ρ
  - Λ_h might not have optimal approximation properties
- Need efficient "inversion" of the preconditioners
- System matrix is very large (3 space dimension + time) + "outer" domain. Possible to exploit tensorization!

Thank you for your attention!

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