Stable schemes for weakly compressible flows

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AANMPDE 12

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Joint work with ...

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- Sebastian Noelle (Aachen)

Outline

- Introduction: Isentropic Euler at low Mach number

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- RS-IMEX: Isentropic Euler

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- RS-IMEX: Isentropic Euler
- RS-IMEX: Full Euler

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- Asymptotic consistency

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Low Mach

- Isentropic Euler equations

$$\rho_t + \nabla \cdot (\rho \boldsymbol{u}) = 0, \qquad (\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{\varepsilon^2} \nabla p(\rho) = 0$$

- Singular perturbed system of conservation laws

$$w_t + \nabla \cdot f_\varepsilon(w) = 0, \qquad \varepsilon \to 0$$

- Eigenvalues of $f_\varepsilon{}'(w) \cdot \boldsymbol{n}$

$$\lambda_0 = \boldsymbol{u} \cdot \boldsymbol{n}, \qquad \lambda_\pm = \boldsymbol{u} \cdot \boldsymbol{n} \pm \frac{\sqrt{p'(\rho)}}{\varepsilon}$$

Asymptotic Expansion

- Formal expansion

$$w = w_{(0)} + \varepsilon w_{(1)} + \varepsilon^2 w_{(2)} + \dots$$

- Asymptotic solution: *Incompressible* Euler [Klainerman / Majda 1981]

$$\rho_{(0)} \equiv \text{const}, \quad \nabla \cdot \boldsymbol{u}_{(0)} = 0,$$
$$(\boldsymbol{u}_{(0)})_t + \nabla \cdot (\boldsymbol{u}_{(0)} \otimes \boldsymbol{u}_{(0)}) + \frac{\nabla p_{(2)}}{\rho_{(0)}} = 0.$$

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Algorithmic challenges for $\varepsilon \to 0$

- CFL condition:

$$\Delta t \lesssim \varepsilon \Delta x, \qquad \varepsilon \to 0.$$

- Discrete convergence as $\varepsilon \to 0$ for *numerics*?
- Highly stiff linear algebra equations.
- Order degradation.
- ...

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IMEX schemes

$$w_t + \nabla \cdot f_\varepsilon(w) = 0, \qquad \varepsilon \to 0.$$

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$$w_t + \nabla \cdot \left(\widetilde{f_\varepsilon}(w) + \widehat{f_\varepsilon}(w) \right) = 0, \qquad \varepsilon \to 0.$$

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$$w_t + \nabla \cdot \left(\widetilde{f_\varepsilon}(w) + \widehat{f_\varepsilon}(w) \right) = 0, \qquad \varepsilon \to 0.$$

- $\widetilde{f_\varepsilon}(w)$ *stiff* contribution, $\widehat{f_\varepsilon}(w)$ *non-stiff* contribution.
- Treat implicitly and explicitly :

$$\frac{w^{n+1} - w^n}{\Delta t} + \nabla \cdot \left(\widetilde{f_\varepsilon}(w^{n+1}) + \widehat{f_\varepsilon}(w^n) \right) = 0$$

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- ... or with some higher-order extension (IMEX-BDF, IMEX-RK, ...)
- [Klein 1994], [Degond, Tang 2008], [Haack, Jin, Liu 2012], [Noelle et al. 2014], [Kaiser et al. 2017] ...

RS-IMEX: Isentropic Euler

$$\rho_t + \nabla \cdot (\rho \boldsymbol{u}) = 0, \qquad (\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{\varepsilon^2} \nabla p(\rho) = 0.$$

- Asymptotic solution $w_{(0)}$:

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- Splitting:

$$\widetilde{f}_\varepsilon(w) = f_\varepsilon(w_{(0)}) + f'_\varepsilon(w_{(0)})(w - w_{(0)}).$$
$$\widehat{f}_\varepsilon(w) = f_\varepsilon(w) - \widetilde{f}_\varepsilon(w).$$

[Giraldo et al. 2010], [Bispen et al. 2014], [Schütz and Kaiser 2016]

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Splitting: Formal requirements

- ☑ Hyperbolicity and Consistency:

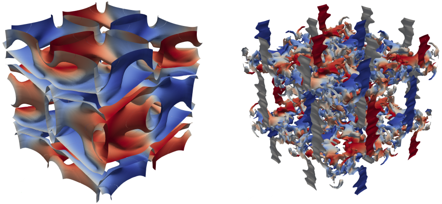
$$f_\varepsilon(w) = \widetilde{f_\varepsilon}(w) + \widehat{f_\varepsilon}(w).$$

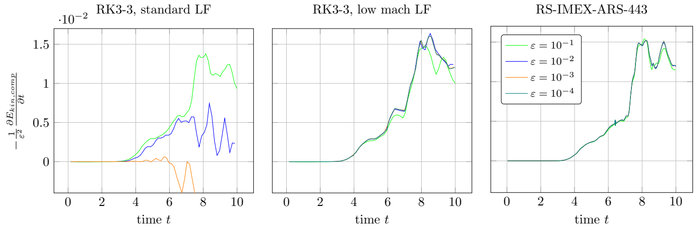
- ☑ Non-stiffness:

$$\varrho\left(\widehat{f'_\varepsilon}(w)\cdot \boldsymbol{n}\right) = 2\left|(\boldsymbol{u} - \boldsymbol{u}_{(0)})\cdot \boldsymbol{n}\right| = \mathcal{O}(1).$$

- ☑ Linearity of $\widetilde{f_\varepsilon}(w)$
- ☑ Time-step restriction *independent* of ε.

Taylor-Green





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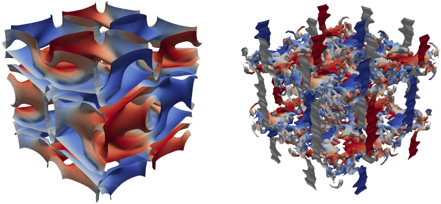

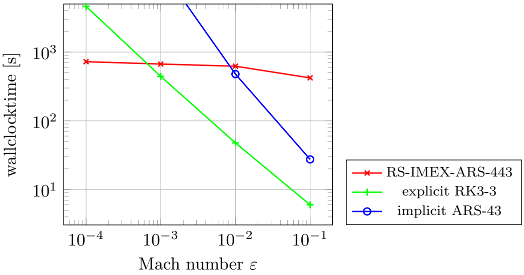

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Full Euler equations

- Trivial extension?

$$\widetilde{f}_\varepsilon(w) = f_\varepsilon(w_{(0)}) + f'_\varepsilon(w_{(0)})(w - w_{(0)}).$$
$$\widehat{f}_\varepsilon(w) = f_\varepsilon(w) - \widetilde{f}_\varepsilon(w).$$

Full Euler equations

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$$\widehat{f}_\varepsilon(w) = f_\varepsilon(w) - \widetilde{f}_\varepsilon(w).$$

- Pitfalls?

$$\sigma(\widehat{f}'_\varepsilon(w)\cdot \boldsymbol{n}) = \begin{pmatrix} 0 \\ \gamma(\boldsymbol{u}-\boldsymbol{u}_{(0)})\cdot\boldsymbol{n} \\ \left(2-\frac{1}{2}\gamma\right)\left(\boldsymbol{u}-\boldsymbol{u}_{(0)}\right)\cdot\boldsymbol{n} + \frac{1}{2}\sqrt{(4-4\gamma)\|\boldsymbol{u}-\boldsymbol{u}_{(0)}\|^2 + \gamma^2\left((\boldsymbol{u}-\boldsymbol{u}_{(0)})\cdot\boldsymbol{n}\right)^2} \\ \left(2-\frac{1}{2}\gamma\right)\left(\boldsymbol{u}-\boldsymbol{u}_{(0)}\right)\cdot\boldsymbol{n} - \frac{1}{2}\sqrt{(4-4\gamma)\|\boldsymbol{u}-\boldsymbol{u}_{(0)}\|^2 + \gamma^2\left((\boldsymbol{u}-\boldsymbol{u}_{(0)})\cdot\boldsymbol{n}\right)^2} \end{pmatrix}$$

Full Euler equations

- Trivial extension?

$$\widetilde{f_\varepsilon}(w) = f_\varepsilon(w_{(0)}) + f_\varepsilon'(w_{(0)})(w - w_{(0)}).$$
$$\widehat{f_\varepsilon}(w) = f_\varepsilon(w) - \widetilde{f_\varepsilon}(w).$$

- Pitfalls?

$$\sigma(\widehat{f_\varepsilon}'(w) \cdot \boldsymbol{n}) = \begin{pmatrix} 0 \\ \gamma(\boldsymbol{u} - \boldsymbol{u}_{(0)}) \cdot \boldsymbol{n} \\ \left(2 - \frac{1}{2}\gamma\right)\left(\boldsymbol{u} - \boldsymbol{u}_{(0)}\right) \cdot \boldsymbol{n} + \frac{1}{2}\sqrt{(4 - 4\gamma)\|\boldsymbol{u} - \boldsymbol{u}_{(0)}\|^2 + \gamma^2\left((\boldsymbol{u} - \boldsymbol{u}_{(0)}) \cdot \boldsymbol{n}\right)^2} \\ \left(2 - \frac{1}{2}\gamma\right)\left(\boldsymbol{u} - \boldsymbol{u}_{(0)}\right) \cdot \boldsymbol{n} - \frac{1}{2}\sqrt{(4 - 4\gamma)\|\boldsymbol{u} - \boldsymbol{u}_{(0)}\|^2 + \gamma^2\left((\boldsymbol{u} - \boldsymbol{u}_{(0)}) \cdot \boldsymbol{n}\right)^2} \end{pmatrix}$$

- Non-hyperbolic subproblems!

A modified flux

- Explicit computation:

$$\widehat{f_\varepsilon}(w) = \begin{pmatrix} 0 \\ \rho(\boldsymbol{u}-\boldsymbol{u}_{(0)})\otimes(\boldsymbol{u}-\boldsymbol{u}_{(0)}) + \frac{1}{\varepsilon^2}\widehat{p}Id \\ (\boldsymbol{u}-\boldsymbol{u}_{(0)})\frac{(\rho_{(0)}E-E_{(0)}\rho)}{\rho_{(0)}} + \widehat{\boldsymbol{u}p} \end{pmatrix},$$

$$\widehat{p} = -(\gamma-1)\frac{\varepsilon^2}{2}\rho\|\boldsymbol{u}_{(0)}-\boldsymbol{u}\|^2,$$

$$\widehat{\boldsymbol{u}p} = (\gamma-1)(\boldsymbol{u}-\boldsymbol{u}_{(0)})\left(E-\frac{\rho}{\rho_{(0)}}E_{(0)}\right) + \mathcal{O}(\varepsilon^2)$$

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- Note that

$$\widehat{p} = \mathcal{O}(\varepsilon^4), \quad \widehat{\boldsymbol{u}p} = \mathcal{O}(\varepsilon^2).$$

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- Neglect :

$$\widehat{f}_{\varepsilon,\mathbf{mod}}(w) := \begin{pmatrix} 0 \\ \rho(\boldsymbol{u} - \boldsymbol{u}_{(0)}) \otimes (\boldsymbol{u} - \boldsymbol{u}_{(0)}) \\ \gamma(\boldsymbol{u} - \boldsymbol{u}_{(0)}) \frac{(\rho_{(0)}E - E_{(0)}\rho)}{\rho_{(0)}} \end{pmatrix}.$$

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A modified flux (ctd.)

- Modified flux

$$\widehat{f}_{\varepsilon,\mathbf{mod}}(w) := \begin{pmatrix} 0 \\ \rho(\boldsymbol{u}-\boldsymbol{u}_{(\mathbf{0})})\otimes(\boldsymbol{u}-\boldsymbol{u}_{(\mathbf{0})}) \\ \gamma(\boldsymbol{u}-\boldsymbol{u}_{(\mathbf{0})})\frac{(\rho_{(\mathbf{0})}E-E_{(\mathbf{0})}\rho)}{\rho_{(\mathbf{0})}} \end{pmatrix}.$$

A modified flux (ctd.)

- Modified flux

$$\widehat{f}_{\varepsilon,\mathbf{mod}}(w) := \begin{pmatrix} 0 \\ \rho(\boldsymbol{u} - \boldsymbol{u}_{(\mathbf{0})}) \otimes (\boldsymbol{u} - \boldsymbol{u}_{(\mathbf{0})}) \\ \gamma(\boldsymbol{u} - \boldsymbol{u}_{(\mathbf{0})}) \frac{(\rho_{(\mathbf{0})} E - E_{(\mathbf{0})} \rho)}{\rho_{(\mathbf{0})}} \end{pmatrix} .$$

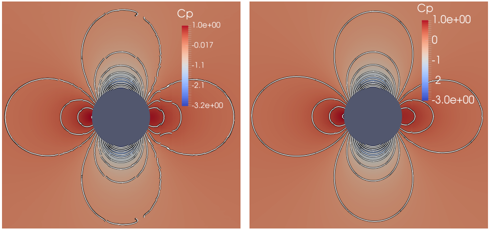
- Voila:

$$\sigma(\widehat{f}'_{\varepsilon,\mathbf{mod}}(w) \cdot \boldsymbol{n}) = \begin{pmatrix} 0 \\ 1 \\ \gamma \\ 2 \end{pmatrix} (\boldsymbol{u} - \boldsymbol{u}_{(\mathbf{0})}) \cdot \boldsymbol{n}$$

- Hence, a hyperbolic sub-system.

Flow over a cylinder

- DGSEM with quadratics
- Time integration: IMEX-ARS-443
- (convective) CFL = 0.5



$\varepsilon = 10^{-1}$ and $\varepsilon = 10^{-3}$

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Asymptotic preserving property [Jin 1999]

- Semi-discrete scheme:

$$\frac{w_{\Delta t}^{n+1} - w_{\Delta t}^{n}}{\Delta t} + \nabla \cdot \left(\widetilde{f}_{\varepsilon}(w_{\Delta t}^{n+1}) + \widehat{f}_{\varepsilon}(w_{\Delta t}^{n}) \right) = 0$$

Asymptotic preserving property [Jin 1999]

- Semi-discrete scheme:

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- *Assuming* Hilbert expansion: Method is AP!

$$\begin{array}{ccc} w_{\Delta t}(\varepsilon) & \xrightarrow{\varepsilon \to 0} & w_{\Delta t,(0)} \\ \Big\downarrow {\scriptstyle \Delta t \to 0} & & \Big\downarrow {\scriptstyle \Delta t \to 0\text{: AP}} \\ w(\varepsilon) & \xrightarrow{\varepsilon \to 0} & w_{(0)} \end{array}$$

Hilbert expansion?

$$w_{\Delta t} \overset{?}{=} w_{(0),\Delta t} + \varepsilon w_{(1),\Delta t} + \mathcal{O}(\varepsilon^2)$$

- Rarely shown in literature: difficult! [Bispen, Lukacova, Yelash 2017]
- Can be shown under restrictive assumptions
  - Show that $\widetilde{p}$ fulfills fourth-order PDE.
  - $\widetilde{p} = \text{const} + \mathcal{O}(\varepsilon^2)$.
  - Done ...

Overzicht

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Conclusion and outlook

- Presented a uniformly stable, efficient low-Mach splitting.
- Future work:
  - Hilbert expansion in more general settings?
  - Stability analysis?
  - Error convergence?
  - ...

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Thank you for your attention!

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