On some compact embeddings in various Hilbert complexes

Michael Schomburg
joined work with Dirk Pauly and Walter Zulehner

AANMPDE12

Strobl, July 5, 2019

Overview

Investigate compact embeddings and related topics in various Hilbert space complexes:

- the classical de Rham complex for vector fields
- the classical de Rham complex for differential forms
- the elasticity complex
- and the biharmonic complex

Main tools and related topics:

- Helmholtz (type) decompositions
- regular potentials
- regular decompositions
- Rellich's selection theorem

We consider bounded strong Lipschitz domains and mixed boundary conditions.

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot\cup \Gamma_n}$

(electro-magnetics, Maxwell's equations)

$$\{0\} \;\underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}}{\overset{\mathring{\nabla}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\operatorname{rot}}{\overset{\mathring{\operatorname{rot}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla}{\overset{\mathring{\operatorname{div}}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}}\; \mathbb{R}$$

mixed boundary conditions and inhomogeneous and anisotropic media

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_{\varepsilon} \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\mu^{-1}\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_{\mu} \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}\mu}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

classical de Rham complex in 3D (∇-rot-div-complex)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain, $\partial\Omega = \Gamma = \overline{\Gamma_t \dot{\cup} \Gamma_n}$

(electro-magneto dynamics, Maxwell's equations with mixed boundary conditions)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_\varepsilon \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{llllllll} \nabla_{\Gamma_t} u = A & \text{in } \Omega & | \quad \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & | \quad \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & | \quad \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | \quad \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & | \quad -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{llllll} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & | \quad \varepsilon^{-1}\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & | \quad -\nabla_{\Gamma_n}\operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | \quad \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n}\varepsilon) = \mathsf{H}_{\Gamma_t}(\operatorname{rot},\Omega) \cap \mathsf{H}_{\Gamma_n}(\operatorname{div},\Omega) \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}) = \mathsf{H}_{\Gamma_t}(\operatorname{div},\Omega) \cap \mathsf{H}_{\Gamma_n}(\operatorname{rot},\Omega) \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem, '74)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/Pauly/Schomburg ('16)

Weck's selection theorem (Weck '74, (Habil. '72) stimulated by Rolf Leis)
(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Fernandes/Gilardi '97, Kuhn '99, Picard/Weck/Witsch '01, Pauly '96, '03, '06, '07, '08)

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations **the mother of all complexes**)

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^{2,0} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \cdots \boxed{\mathsf{L}^{2,q} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,q+1}} \cdots \mathsf{L}^{2,N-1} \underset{-\delta}{\overset{\mathring{\mathrm{d}}}{\rightleftarrows}} \mathsf{L}^{2,N} \underset{\iota_{\mathbb{R}}}{\overset{\pi_{\mathbb{R}}}{\rightleftarrows}} \mathbb{R}$$

de Rham complex in ND or on Riemannian manifolds (d-complex)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

(generalized Maxwell equations)

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^{2,0} \;\underset{-\delta^1_{\Gamma_n}}{\overset{\mathrm{d}^0_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^{2,1} \;\underset{-\delta^2_{\Gamma_n}}{\overset{\mathrm{d}^1_{\Gamma_t}}{\rightleftarrows}}\; \cdots \; \mathsf{L}^{2,q} \;\underset{-\delta^{q+1}_{\Gamma_n}}{\overset{\mathrm{d}^q_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^{2,q+1} \cdots \mathsf{L}^{2,N-1} \;\underset{-\delta^N_{\Gamma_n}}{\overset{\mathrm{d}^{N-1}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^{2,N} \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$
$$-\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

related sos

$$-\delta^{q+1}_{\Gamma_n} \mathrm{d}^q_{\Gamma_t} E = F \qquad \text{in } \Omega$$
$$-\delta^q_{\Gamma_n} E = G \qquad \text{in } \Omega$$

includes: EMS rot / div, Laplacian, rot rot, and more...
corresponding compact embeddings:

$$D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) = \mathsf{H}_{\Gamma_t}(\mathrm{d}^q, \Omega) \cap \mathsf{H}_{\Gamma_n}(\delta^q, \Omega) \hookrightarrow \mathsf{L}^{2,q} \qquad \text{(Weck's selection theorems, '74)}$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/Pauly/Schomburg ('17)

elasticity complex in 3D (sym ∇-Rot $\mathrm{Rot}_{\mathbb{S}}^{\top}$-$\mathrm{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain, full boundary condition

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\,\mathrm{Div}_{\mathbb{S}}}{\overset{\mathring{\mathrm{sym}}\,\nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S}}^{\top}}{\overset{\mathrm{Rot}\,\mathring{\mathrm{Rot}}_{\mathbb{S}}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\,\mathrm{sym}\,\nabla}{\overset{\mathring{\mathrm{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RM}}}{\overset{\pi_{\mathsf{RM}}}{\rightleftarrows}} \mathsf{RM}$$

... and with mixed boundary conditions

$$\{0\} \text{ or } \mathsf{RM} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{-\,\mathrm{Div}_{\mathbb{S},\Gamma_n}}{\overset{\mathrm{sym}\,\nabla_{\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S},\Gamma_n}^{\top}}{\overset{\mathrm{Rot}\,\mathrm{Rot}_{\mathbb{S},\Gamma_t}^{\top}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\,\mathrm{sym}\,\nabla_{\Gamma_n}}{\overset{\mathrm{Div}_{\mathbb{S},\Gamma_t}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \{0\} \text{ or } \mathsf{RM}$$

elasticity complex in 3D (sym ∇-Rot $\text{Rot}_{\mathbb{S}}^{\top}$-$\text{Div}_{\mathbb{S}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\operatorname{sym}} \nabla}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}}}{\overset{\mathring{\operatorname{Rot}}\operatorname{Rot}^{\top}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathrm{RM}}}{\overset{\pi_{\mathrm{RM}}}{\rightleftarrows}} \mathrm{RM}$$

related fos ($\mathring{\operatorname{Rot}}\operatorname{Rot}^{\top}_{\mathbb{S}}$, $\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}}$ first order operators!)

$\mathring{\operatorname{sym}}\nabla v = M$ in Ω	$\mathring{\operatorname{Rot}}\operatorname{Rot}^{\top}_{\mathbb{S}} M = F$ in Ω	$\mathring{\operatorname{Div}}_{\mathbb{S}} N = g$ in Ω	$\pi v = r$ in Ω
$\pi v = 0$ in Ω	$-\operatorname{Div}_{\mathbb{S}} M = f$ in Ω	$\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}} N = G$ in Ω	$-\operatorname{sym}\nabla v = M$ in Ω

related sos ($\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}}\mathring{\operatorname{Rot}}\operatorname{Rot}^{\top}_{\mathbb{S}}$ second order operator!)

$-\operatorname{Div}_{\mathbb{S}}\mathring{\operatorname{sym}}\nabla v = f$ in Ω	$\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}}\mathring{\operatorname{Rot}}\operatorname{Rot}^{\top}_{\mathbb{S}} M = G$ in Ω	$-\operatorname{sym}\nabla\mathring{\operatorname{Div}}_{\mathbb{S}} N = M$ in Ω
$\pi v = 0$ in Ω	$-\operatorname{Div}_{\mathbb{S}} M = f$ in Ω	$\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}} N = G$ in Ω

corresponding compact embeddings:

$$D(\mathring{\operatorname{sym}}\nabla) \cap D(\pi) = D(\mathring{\nabla}) = \mathring{\mathsf{H}}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

$$D(\mathring{\operatorname{Rot}}\operatorname{Rot}^{\top}_{\mathbb{S}}) \cap D(\operatorname{Div}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\mathring{\operatorname{Div}}_{\mathbb{S}}) \cap D(\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S}}) \hookrightarrow \mathsf{L}^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{sym}\nabla) = D(\nabla) = \mathsf{H}^1 \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

two new selection theorems for strong Lip. dom.: Pauly/Schomburg/Zulehner ('18)

main tool: regular potentials in $3D$

Let (Ω, Γ_t) be topologically trivial and strong Lipschitz, $m \in \mathbb{N}_0$. Then

$$\mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\mathrm{rot}_0, \Omega) = \nabla \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$

$$\mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{H}_{\Gamma_t}(\mathrm{div}_0, \Omega) = \mathrm{rot}\, \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$

$$\mathsf{H}^k_{\Gamma_t}(\Omega) = \mathrm{div}\, \mathsf{H}^{k+1}_{\Gamma_t}(\Omega), \; \mathsf{H}^k_{\Gamma}(\Omega) \cap \mathsf{L}^2_{\perp}(\Omega) = \mathrm{div}\, \mathsf{H}^{k+1}_{\Gamma}(\Omega) \text{ resp.}$$

hold with linear and continuous potential operators. Furthermore

$$\mathsf{H}^{-1}(\mathrm{rot}_0, \Omega) = \nabla \mathsf{L}^2(\Omega) = \nabla \mathsf{L}^2_{\perp}(\Omega)$$

$$\mathsf{H}^{-1}(\mathrm{div}_0, \Omega) = \mathrm{rot}\, \mathsf{L}^2(\Omega)$$

$$\mathsf{H}^{-1}(\Omega) = \mathrm{div}\, \mathsf{L}^2(\Omega)$$

with linear and continuous potential operator.

Realization of the operators

Realize operators as linear, densely defined, closed unbounded operators. Define operators on smooth test vector/tensor fields

$$\widetilde{\operatorname{sym}\nabla}_{\Gamma_t} : \mathsf{C}^\infty_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), \qquad E \mapsto \operatorname{sym}\nabla E,$$

$$\widetilde{\operatorname{Rot}\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_t} : \mathsf{C}^\infty_{\Gamma_t}(\Omega) \cap \mathsf{L}^2_{\mathbb{S}}(\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), \qquad M \mapsto \operatorname{Rot}\operatorname{Rot}^\top M,$$

$$\widetilde{\operatorname{Div}}_{\mathbb{S},\Gamma_t} : \mathsf{C}^\infty_{\Gamma_t}(\Omega) \cap \mathsf{L}^2_{\mathbb{S}}(\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega), \qquad H \mapsto \operatorname{Div} H,$$

and close them

$$\operatorname{sym}\nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t} \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), \qquad E \mapsto \operatorname{sym}\nabla E,$$

$$\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), \qquad M \mapsto \operatorname{Rot}\operatorname{Rot}^\top M,$$

$$\operatorname{Div}_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega), \qquad H \mapsto \operatorname{Div} H.$$

Realization of the operators

Realize operators as linear, densely defined, closed unbounded operators. Define operators on smooth test vector/tensor fields

$$\begin{aligned}
\widetilde{\operatorname{sym}\nabla}_{\Gamma_t} &: \mathrm{C}^\infty_{\Gamma_t}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2_{\mathbb{S}}(\Omega), && E \mapsto \operatorname{sym}\nabla E,\\
\widetilde{\operatorname{Rot}\operatorname{Rot}}^\top_{\mathbb{S},\Gamma_t} &: \mathrm{C}^\infty_{\Gamma_t}(\Omega) \cap \mathrm{L}^2_{\mathbb{S}}(\Omega) \subset \mathrm{L}^2_{\mathbb{S}}(\Omega) \to \mathrm{L}^2_{\mathbb{S}}(\Omega), && M \mapsto \operatorname{Rot}\operatorname{Rot}^\top M,\\
\widetilde{\operatorname{Div}}_{\mathbb{S},\Gamma_t} &: \mathrm{C}^\infty_{\Gamma_t}(\Omega) \cap \mathrm{L}^2_{\mathbb{S}}(\Omega) \subset \mathrm{L}^2_{\mathbb{S}}(\Omega) \to \mathrm{L}^2(\Omega), && H \mapsto \operatorname{Div} H,
\end{aligned}$$

and close them

$$\begin{aligned}
\operatorname{sym}\nabla_{\Gamma_t} &: \mathrm{H}^1_{\Gamma_t} \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2_{\mathbb{S}}(\Omega), && E \mapsto \operatorname{sym}\nabla E,\\
\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t} &: \mathrm{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top, \Omega) \subset \mathrm{L}^2_{\mathbb{S}}(\Omega) \to \mathrm{L}^2_{\mathbb{S}}(\Omega), && M \mapsto \operatorname{Rot}\operatorname{Rot}^\top M,\\
\operatorname{Div}_{\mathbb{S},\Gamma_t} &: \mathrm{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) \subset \mathrm{L}^2_{\mathbb{S}}(\Omega) \to \mathrm{L}^2(\Omega), && H \mapsto \operatorname{Div} H.
\end{aligned}$$

adjoints

The adjoints are

$$(\operatorname{sym}\nabla_{\Gamma_t})^* = -\operatorname{Div}_{\mathbb{S}} : \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega), \qquad H \mapsto -\operatorname{Div} H,$$

$$(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t})^* = \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}} : \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), \qquad M \mapsto \operatorname{Rot}\operatorname{Rot}^\top M,$$

$$(\operatorname{Div}_{\mathbb{S},\Gamma_t})^* = -\operatorname{sym}\nabla_{\Gamma_n} : \mathcal{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), \qquad E \mapsto -\operatorname{sym}\nabla E.$$

Here

$$\mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) = \{H \in \mathsf{H}_{\mathbb{S}}(\operatorname{Div},\Omega) : \langle H, \operatorname{sym}\nabla E\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = -\langle \operatorname{Div} H, E\rangle_{\mathsf{L}^2(\Omega)},$$
$$\text{for all } E \in \mathsf{H}^1_{\Gamma_t}\},$$

$$\mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) = \{E \in \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) : \langle E, \operatorname{Rot}\operatorname{Rot}^\top M\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = \langle \operatorname{Rot}\operatorname{Rot}^\top E, M\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)},$$
$$\text{for all } M \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega)\},$$

$$\mathcal{H}^1_{\Gamma_n}(\Omega) = \{E \in \mathsf{H}^1(\Omega) : \langle E, \operatorname{Div} H\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{sym}\nabla E, H\rangle_{\mathsf{L}^2_{\mathbb{S}(\Omega)}},$$
$$\text{for all } H \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)\}.$$

adjoints

The adjoints are

$$\begin{aligned}(\operatorname{sym}\nabla_{\Gamma_t})^* &= -\operatorname{Div}_{\mathbb{S}} : \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2(\Omega), && H \mapsto -\operatorname{Div} H,\\ (\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t})^* &= \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}} : \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \subset \mathsf{L}^2_{\mathbb{S}}(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), && M \mapsto \operatorname{Rot}\operatorname{Rot}^\top M,\\ (\operatorname{Div}_{\mathbb{S},\Gamma_t})^* &= -\operatorname{sym}\nabla_{\Gamma_n} : \mathcal{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2_{\mathbb{S}}(\Omega), && E \mapsto -\operatorname{sym}\nabla E.\end{aligned}$$

Here

$$\begin{aligned}\mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) &= \{H \in \mathsf{H}_{\mathbb{S}}(\operatorname{Div},\Omega) : \langle H, \operatorname{sym}\nabla E\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = -\langle \operatorname{Div} H, E\rangle_{\mathsf{L}^2(\Omega)},\\ &\qquad \text{for all } E \in \mathsf{H}^1_{\Gamma_t}\},\\ \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) &= \{E \in \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) : \langle E, \operatorname{Rot}\operatorname{Rot}^\top M\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = \langle \operatorname{Rot}\operatorname{Rot}^\top E, M\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)},\\ &\qquad \text{for all } M \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega)\},\\ \mathcal{H}^1_{\Gamma_n}(\Omega) &= \{E \in \mathsf{H}^1(\Omega) : \langle E, \operatorname{Div} H\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{sym}\nabla E, H\rangle_{\mathsf{L}^2_{\mathbb{S}(\Omega)}},\\ &\qquad \text{for all } H \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega)\}.\end{aligned}$$

weak and strong boundary conditions

Clearly

$$\mathrm{H}^1_{\Gamma_n}(\Omega) \subset \mathcal{H}^1_{\Gamma_n}(\Omega),$$
$$\mathrm{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \subset \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega),$$
$$\mathrm{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \subset \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega).$$

but one can proof

$$\mathrm{H}^1_{\Gamma_n}(\Omega) = \mathcal{H}^1_{\Gamma_n}(\Omega),$$
$$\mathrm{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega),$$
$$\mathrm{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) = \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega).$$

weak and strong boundary conditions

Clearly

$$
\begin{aligned}
\mathsf{H}^1_{\Gamma_n}(\Omega) &\subset \mathcal{H}^1_{\Gamma_n}(\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) &\subset \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) &\subset \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega).
\end{aligned}
$$

but one can proof

$$
\begin{aligned}
\mathsf{H}^1_{\Gamma_n}(\Omega) &= \mathcal{H}^1_{\Gamma_n}(\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) &= \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega),\\
\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) &= \mathcal{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega).
\end{aligned}
$$

Theorem (regular potentials)

Let $\Omega \subset \mathbb{R}^3$ and let (Ω, Γ_t) be a bounded, topologically trivial strong Lipschitz pair and $\Gamma \neq \Gamma_t \neq \emptyset$. Then

$$\mathcal{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}_0^\top, \Omega) = N(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S}\Gamma_t}) = R(\operatorname{sym}\nabla_{\Gamma_t}) = \operatorname{sym}\nabla \mathsf{H}^1_{\Gamma_t}(\Omega),$$
$$\mathcal{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0^\top, \Omega) = N(\operatorname{Div}_{\mathbb{S},\Gamma_t}) = R(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t}) = \operatorname{Rot}\operatorname{Rot}^\top \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\mathsf{L}^2(\Omega) = N(0) = R(\operatorname{Div}_{\mathbb{S},\Gamma_t}) = \operatorname{Div}\mathsf{H}^1_{\mathbb{S},\Gamma_t}(\Omega).$$

with linear and continuous potential operators. Especially, all ranges are closed.

In case of full boundary conditions (i.e. $\Gamma_t = \Gamma$) it holds

$$\mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} = N(\pi_{\mathsf{RM}}) = R(\operatorname{Div}_{\mathbb{S},\Gamma}) = \operatorname{Div}\mathsf{H}^1_{\mathbb{S},\Gamma}(\Omega).$$

Theorem (regular potentials)

Let $\Omega \subset \mathbb{R}^3$ *and let* (Ω, Γ_t) *be a bounded, topologically trivial strong Lipschitz pair and* $\Gamma \neq \Gamma_t \neq \emptyset$. *Then*

$$\mathcal{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}_0^\top, \Omega) = N(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}\Gamma_t}^\top) = R(\operatorname{sym}\nabla_{\Gamma_t}) = \operatorname{sym}\nabla \mathrm{H}_{\Gamma_t}^1(\Omega),$$
$$\mathcal{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0^\top, \Omega) = N(\operatorname{Div}_{\mathbb{S},\Gamma_t}) = R(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma_t}^\top) = \operatorname{Rot}\operatorname{Rot}^\top \mathrm{H}_{\mathbb{S},\Gamma_t}^2(\Omega),$$
$$\mathrm{L}^2(\Omega) = N(0) = R(\operatorname{Div}_{\mathbb{S},\Gamma_t}) = \operatorname{Div}\mathrm{H}_{\mathbb{S},\Gamma_t}^1(\Omega).$$

with linear and continuous potential operators. Especially, all ranges are closed.

In case of full boundary conditions (i.e. $\Gamma_t = \Gamma$) it holds

$$\mathrm{RM}^{\perp_{\mathrm{L}^2(\Omega)}} = N(\pi_{\mathrm{RM}}) = R(\operatorname{Div}_{\mathbb{S},\Gamma}) = \operatorname{Div}\mathrm{H}_{\mathbb{S},\Gamma}^1(\Omega).$$

Theorem (higher order regular potentials)

Let $\Omega \subset \mathbb{R}^3$ and let (Ω, Γ_t) be a bounded, topologically trivial strong Lipschitz pair and $\Gamma \neq \Gamma_t \neq \emptyset$. Then

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) = \operatorname{sym}\nabla \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}^\top_0, \Omega) = \operatorname{Rot}\operatorname{Rot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\mathsf{H}^k_{\Gamma_t}(\Omega) = \operatorname{Div} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega).$$

with linear and continuous potential operators. Especially, all ranges are closed.

In case of full boundary conditions (i.e. $\Gamma_t = \Gamma$) it holds

$$\mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} = \operatorname{Div} \mathsf{H}^{k+1}_{\mathbb{S},\Gamma}(\Omega).$$

Theorem (higher order regular potentials)

Let $\Omega \subset \mathbb{R}^3$ and let (Ω, Γ_t) be a bounded, topologically trivial strong Lipschitz pair and $\Gamma \neq \Gamma_t \neq \emptyset$. Then

$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) = \operatorname{sym}\nabla \mathsf{H}^{k+1}_{\Gamma_t}(\Omega),$$
$$\mathsf{H}^k_{\mathbb{S},\Gamma_t}(\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}^\top_0, \Omega) = \operatorname{Rot}\operatorname{Rot}^\top \mathsf{H}^{k+2}_{\mathbb{S},\Gamma_t}(\Omega),$$
$$\mathsf{H}^k_{\Gamma_t}(\Omega) = \operatorname{Div} \mathsf{H}^{k+1}_{\Gamma_t}(\Omega).$$

with linear and continuous potential operators. Especially, all ranges are closed.

In case of full boundary conditions (i.e. $\Gamma_t = \Gamma$) it holds

$$\mathsf{H}^k_{\Gamma_t}(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} = \operatorname{Div} \mathsf{H}^{k+1}_{\mathbb{S},\Gamma}(\Omega).$$

construction of the potentials

look at $E \in N(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^{\top})$, i.e. $E \in \mathsf{L}^2_{\mathbb{S}}(\Omega) \wedge \operatorname{Rot}\operatorname{Rot}^{\top} E = 0$.

$$\Rightarrow \qquad \operatorname{Rot}^{\top} E \in \mathsf{H}^{-1}(\operatorname{Rot}_0, \Omega)$$

$$\Rightarrow \qquad \operatorname{Rot}^{\top} E = \nabla\phi, \qquad \phi \in \mathsf{L}^2(\Omega)$$

$$\Rightarrow \qquad \operatorname{Rot} E = \nabla^{\top}\phi = (\operatorname{div}\phi) I - \operatorname{Rot}(\operatorname{spn}\phi)$$

as $\operatorname{tr}\operatorname{Rot} E = 0$ it holds

$$0 = \operatorname{tr}\nabla^{\top}\phi = \operatorname{div}\phi$$

and hence $\operatorname{Rot}(E + \operatorname{spn}\phi) = 0$, so

$$E + \operatorname{spn}\phi = \nabla\psi, \qquad \psi \in \mathsf{H}^1(\Omega)$$

and as E is symmetric

$$E = \operatorname{sym}\nabla\psi$$

cohomology groups

... are trivial! $\Gamma \neq \Gamma_t \neq \emptyset$:

$$\begin{aligned}
N(\operatorname{sym}\nabla_{\Gamma_t}) \cap N(0) &= \{0\} \cap \mathsf{L}^2(\Omega) = \{0\},\\
N(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t}) \cap N(-\operatorname{Div}_{\mathbb{S},\Gamma_n}) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega)\\
&= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega) \cap \operatorname{Rot}\operatorname{Rot}^\top \mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega)\\
&= N(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t}) \cap R(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_n}) = \{0\},\\
N(\operatorname{Div}_{\mathbb{S},\Gamma_t}) \cap N(\operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_n}) &= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0)\\
&= \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega) \cap \operatorname{sym}\nabla \mathsf{H}^1_{\Gamma_n}(\Omega)\\
&= N(\operatorname{Div}_{\mathbb{S},\Gamma_t}) \cap R(\operatorname{sym}\nabla_{\Gamma_n}) = \{0\},\\
N(0) \cap N(-\operatorname{sym}\nabla_{\Gamma_n}) &= \mathsf{L}^2(\Omega) \cap \{0\}.
\end{aligned}$$

$\Gamma_t = \Gamma$:

$$N(\pi_{\mathsf{RM}}) \cap N(-\operatorname{sym}\nabla) = \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} \cap \mathsf{RM} = \{0\}.$$

reduced operators

$\Gamma \neq \Gamma_t \neq \emptyset$:

$$\mathcal{A}_0 = \operatorname{sym}\nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0 \Omega),$$

$$\mathcal{A}_1 = \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0, \Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0, \Omega),$$

$$\mathcal{A}_2 = \operatorname{Div}_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \to \mathsf{L}^2(\Omega),$$

$$\mathcal{A}_0^* = -\operatorname{Div}_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \to \mathsf{L}^2(\Omega),$$

$$\mathcal{A}_1^* = \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0, \Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0, \Omega),$$

$$\mathcal{A}_2^* = -\operatorname{sym}\nabla_{\Gamma_n} : \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0 \Omega).$$

$\Gamma_t = \Gamma$:

$$\mathcal{A}_2 = \operatorname{Div}_{\mathbb{S},\Gamma} : \mathsf{H}_{\mathbb{S},\Gamma}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \subset \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \to \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}},$$

$$\mathcal{A}_2^* = -\operatorname{sym}\nabla : \mathsf{H}^1(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} \subset \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}_{\mathbb{S},\Gamma}(\operatorname{Rot}\operatorname{Rot}^\top_0 \Omega).$$

reduced operators

$\Gamma \neq \Gamma_t \neq \emptyset$:

$$\begin{aligned}
\mathcal{A}_0 &= \operatorname{sym}\nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}_0^\top \Omega),\\
\mathcal{A}_1 &= \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega),\\
\mathcal{A}_2 &= \operatorname{Div}_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}_0^\top,\Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}_0^\top,\Omega) \to \mathsf{L}^2(\Omega),\\
\mathcal{A}_0^* &= -\operatorname{Div}_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}_0^\top,\Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}_0^\top,\Omega) \to \mathsf{L}^2(\Omega),\\
\mathcal{A}_1^* &= \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega),\\
\mathcal{A}_2^* &= -\operatorname{sym}\nabla_{\Gamma_n} : \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}_0^\top \Omega).
\end{aligned}$$

$\Gamma_t = \Gamma$:

$$\begin{aligned}
\mathcal{A}_2 &= \operatorname{Div}_{\mathbb{S},\Gamma} : \mathsf{H}_{\mathbb{S},\Gamma}(\operatorname{Div},\Omega) \cap \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}_0^\top,\Omega) \subset \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}_0^\top,\Omega) \to \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}},\\
\mathcal{A}_2^* &= -\operatorname{sym}\nabla : \mathsf{H}^1(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} \subset \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}_{\mathbb{S},\Gamma}(\operatorname{Rot}\operatorname{Rot}_0^\top \Omega).
\end{aligned}$$

reduced operators

$\Gamma \neq \Gamma_t \neq \emptyset$:

$$\mathcal{A}_0 = \operatorname{sym}\nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0 \Omega),$$
$$\mathcal{A}_1 = \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0, \Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0, \Omega),$$
$$\mathcal{A}_2 = \operatorname{Div}_{\mathbb{S},\Gamma_t} : \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \to \mathsf{L}^2(\Omega),$$
$$\mathcal{A}_0^* = -\operatorname{Div}_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \to \mathsf{L}^2(\Omega),$$
$$\mathcal{A}_1^* = \operatorname{Rot}\operatorname{Rot}^\top_{\mathbb{S},\Gamma_n} : \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0, \Omega) \subset \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0, \Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0, \Omega),$$
$$\mathcal{A}_2^* = -\operatorname{sym}\nabla_{\Gamma_n} : \mathsf{H}^1_{\Gamma_n}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0 \Omega).$$

$\Gamma_t = \Gamma$:

$$\mathcal{A}_2 = \operatorname{Div}_{\mathbb{S},\Gamma} : \mathsf{H}_{\mathbb{S},\Gamma}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \subset \mathsf{H}_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top_0, \Omega) \to \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}},$$
$$\mathcal{A}_2^* = -\operatorname{sym}\nabla : \mathsf{H}^1(\Omega) \cap \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} \subset \mathsf{RM}^{\perp_{\mathsf{L}^2(\Omega)}} \to \mathsf{H}_{\mathbb{S},\Gamma}(\operatorname{Rot}\operatorname{Rot}^\top_0 \Omega).$$

Helmholtz type decompositions

$$\begin{aligned} L^2_{\mathbb{S}}(\Omega) &= H_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega) \oplus_{L^2_{\mathbb{S}}(\Omega)} H_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega) \\ &= H_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega) \oplus_{L^2_{\mathbb{S}}(\Omega)} H_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0). \end{aligned}$$

represent kernels as

$$\begin{aligned} H_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega) &= \operatorname{sym}\nabla H^1_{\Gamma_t}(\Omega), \\ H_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega) &= \operatorname{sym}\nabla H^1(\Omega) = \operatorname{sym}\nabla(H^1 \cap RM^{\perp_{L^2(\Omega)}}) \\ H_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega) &= \operatorname{Rot}\operatorname{Rot}^\top H^2_{\mathbb{S},\Gamma_n}(\Omega) \\ &= \operatorname{Rot}\operatorname{Rot}^\top H_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \\ &= \operatorname{Rot}\operatorname{Rot}^\top(H_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \cap H_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega)) \end{aligned}$$

Furthermore

$$\begin{aligned} L^2(\Omega) &= \operatorname{Div} H^1_{\mathbb{S},\Gamma_t}(\Omega) = \operatorname{Div} H_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) = \operatorname{Div}(H_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) \cap H_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega)), \\ RM^{\perp_{L^2(\Omega)}} &= \operatorname{Div} H^1_{\mathbb{S},\Gamma}(\Omega) = N(\pi_{RM}) = \operatorname{Div} H_{\mathbb{S},\Gamma}(\operatorname{Div},\Omega) = \operatorname{Div}(H_{\mathbb{S},\Gamma}(\operatorname{Div},\Omega) \cap H_{\mathbb{S}}(\operatorname{Rot}\operatorname{Rot}^\top_0)). \end{aligned}$$

Friedrichs/Poincaré-type estimates

$$\forall E \in \mathsf{H}^1_{\Gamma_t}(\Omega) \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_{s\nabla} |\operatorname{sym} \nabla E|_{\mathsf{L}^2(\Omega)},$$

$$\forall M \in \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}_0^\top, \Omega) \qquad |M|_{\mathsf{L}^2(\Omega)} \leq c_{s\nabla} |\operatorname{Div} M|_{\mathsf{L}^2(\Omega)},$$

$$\forall M \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}_0^\top, \Omega) \qquad |M|_{\mathsf{L}^2(\Omega)} \leq c_D |\operatorname{Div} M|_{\mathsf{L}^2(\Omega)},$$

$$\forall E \in \mathsf{H}^1_{\Gamma_n}(\Omega) \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_D |\operatorname{sym} \nabla E|_{\mathsf{L}^2(\Omega)},$$

$$\forall M \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}, \Omega) \qquad |M|_{\mathsf{L}^2(\Omega)} \leq c_R |\operatorname{Rot}\operatorname{Rot}^\top M|_{\mathsf{L}^2(\Omega)},$$

$$\forall M \in \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Rot}\operatorname{Rot}^\top, \Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}, \Omega) \qquad |M|_{\mathsf{L}^2(\Omega)} \leq c_R |\operatorname{Rot}\operatorname{Rot}^\top M|_{\mathsf{L}^2(\Omega)},$$

compact embedding on topological trivial domains

want to show

$$D(\operatorname{Rot}\operatorname{Rot}^{\top}_{\mathbb{S},\Gamma_t}) \cap D(\operatorname{Div}_{\mathbb{S},\Gamma_n}) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^{\top},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \hookrightarrow\!\!\!\rightarrow \mathsf{L}^2_{\mathbb{S}}(\Omega).$$

pick (M_n) bounded in $\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^{\top},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)$.
using the decomposition

$$\mathsf{L}^2_{\mathbb{S}}(\Omega) = \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^{\top}_0,\Omega) \oplus \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega)$$

we get

$$\begin{aligned}&\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^{\top},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)\\ &\quad = (\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^{\top}_0,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega)) \oplus (\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^{\top},\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega)).\end{aligned}$$

use kernel representations

$$\begin{aligned}\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^{\top}_0,\Omega) &= \operatorname{sym}\nabla\mathsf{H}^1_{\Gamma_t}(\Omega),\\ \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega) &= \operatorname{Rot}\operatorname{Rot}^{\top}\mathsf{H}^2_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega).\end{aligned}$$

compact embedding on topological trivial domains

decompose

$$M_n = M_{n,r} + M_{n,d} \in (\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}_0^\top,\Omega)\cap\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega))\oplus(\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega)\cap\mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div}_0,\Omega))$$

we have

$$M_{n,r} \in \operatorname{sym}\nabla\mathsf{H}^1_{\Gamma_t} \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega),$$
$$M_{n,d} \in \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top) \cap \operatorname{Rot}\operatorname{Rot}^\top \mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega)$$

and it holds

$$\operatorname{Rot}\operatorname{Rot}^\top M_{n,d} = \operatorname{Rot}\operatorname{Rot}^\top M_n, \qquad M_{n,r} = \operatorname{sym}\nabla E_n,\ E_n \in \mathsf{H}^1_{\Gamma_t}(\Omega),$$
$$\operatorname{Div} M_{n,r} = \operatorname{Div} M_n, \qquad M_{n,d} = \operatorname{Rot}\operatorname{Rot}^\top H_n,\ H_n \in \mathsf{H}^2_{\mathbb{S},\Gamma_n}(\Omega).$$

and

$$|E_n|_{\mathsf{H}^1(\Omega)} \leq c|M_{n,r}|_{\mathsf{L}^2(\Omega)} \leq c|M_n|_{\mathsf{L}^2(\Omega)},$$
$$|H_n|_{\mathsf{H}^2(\Omega)} \leq c|M_{n,d}|_{\mathsf{L}^2(\Omega)} \leq c|M_n|_{\mathsf{L}^2(\Omega)}.$$

compact embedding on topological trivial domains

Rellich $\Rightarrow E_n \overset{L^2(\Omega)}{\longrightarrow} E,\ H_n \overset{H^1(\Omega)}{\longrightarrow} H$
define

$$M_{n,m} := M_n - M_m,\ M_{n,m,r} := M_{n,r} - M_{m,r},$$
$$M_{n,m,d} := M_{n,d} - M_{m,d},\ E_{n,m} := E_n - E_m,\ H_{n,m} := H_n - H_m.$$

remember

$$M_n = M_{n,r} + M_{n,d},$$
$$M_{n,r} = \operatorname{sym} \nabla E_n,\ E_n \in H^1_{\Gamma_t}(\Omega),$$
$$M_{n,d} = \operatorname{Rot} \operatorname{Rot}^\top H_n,\ H_n \in H^2_{\mathbb{S},\Gamma_n}(\Omega).$$

compute

$$|M_{n,m,r}|^2_{L^2(\Omega)} = \langle M_{n,m,r}, \operatorname{sym} \nabla E_{n,m}\rangle_{L^2_{\mathbb{S}}(\Omega)} = \langle \operatorname{Div} M_{n,m,r}, E_{n,m}\rangle_{L^2(\Omega)}$$
$$= \langle \operatorname{Div} M_{n,m}, E_{n,m}\rangle_{L^2(\Omega)} \le c|E_{n,m}|_{L^2(\Omega)}.$$

compact embedding on topological trivial domains

do the same for $M_{n,m,d}$

$$\begin{aligned}|M_{n,m,d}|^2_{\mathsf{L}^2(\Omega)} &= \langle M_{n,m,d}, \operatorname{Rot}\operatorname{Rot}^\top H_{n,m}\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} = \langle \operatorname{Rot}\operatorname{Rot}^\top M_{n,m,d}, H_{n,m}\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}\\ &= \langle \operatorname{Rot}\operatorname{Rot}^\top M_{n,m}, H_{n,m}\rangle_{\mathsf{L}^2_{\mathbb{S}}(\Omega)} \leq c|H_{n,m}|_{\mathsf{L}^2_{\mathbb{S}}(\Omega)}.\end{aligned}$$

alltogether (M_n) is a Cauchy series and hence

$$\mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) \cap \mathsf{H}_{\mathbb{S},\Gamma_n}(\operatorname{Div},\Omega) \hookrightarrow\!\!\!\rightarrow \mathsf{L}^2_{\mathbb{S}}(\Omega)$$

is compact.

regular decompositions

The regular decompositions

$$\begin{aligned} \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top,\Omega) &= \mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega) + \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega), \\ \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div},\Omega) &= \mathsf{H}^1_{\mathbb{S},\Gamma_t} + \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega), \end{aligned}$$

hold with linear and continuous (regular) decomposition and potential operators. The kernels can be represented by

$$\begin{aligned} \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Rot}\operatorname{Rot}^\top_0,\Omega) &= \operatorname{sym}\nabla\mathsf{H}^1_{\Gamma_t}(\Omega) \\ \mathsf{H}_{\mathbb{S},\Gamma_t}(\operatorname{Div}_0,\Omega) &= \operatorname{Rot}\operatorname{Rot}^\top\mathsf{H}^2_{\mathbb{S},\Gamma_t}(\Omega) \end{aligned}$$

biharmonic / general relativity complex in 3D ($\nabla\nabla$-$\text{Rot}_{\mathbb{S}}$-$\text{Div}_{\mathbb{T}}$-complex)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

$$\{0\} \underset{\pi_{\{0\}}}{\overset{\iota_{\{0\}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\mathring{\nabla\nabla}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\mathring{\operatorname{Rot}}_{\mathbb{S}}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\mathring{\operatorname{Div}}_{\mathbb{T}}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota_{\mathsf{RT}}}{\overset{\pi_{\mathsf{RT}}}{\rightleftarrows}} \mathsf{RT}$$