Space-Time Finite Element Methods for Parabolic Initial-Boundary Value Problems with Distributional Data

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Computational Mathematics
Numerical Analysis and Symbolic Computation

Der Wissenschaftsfonds.

Outline

A parabolic model problem

Where we aim

Let $Q = Q_T := \Omega \times (0,T)$ be the space-time cylinder and $\Sigma := \partial\Omega \times (0,T)$, $\Sigma_0 := \overline{\Omega} \times \{0\}$ and $\Sigma_T := \overline{\Omega} \times \{T\}$.
Then: Given f, $\mathbf{f}$, g, σ, ν and u_0, find u such that

$$
\begin{aligned}
\sigma(x,t)\partial_t u - \operatorname{div}_x(\nu(x,t,|\nabla_x u|)\nabla_x u) =& f + \operatorname{div}_x(\mathbf{f}) && \text{in } Q,\\
u(x,t) =& g(x,t), && (x,t) \in \Sigma,\\
u(x,0) =& u_0(x), && x \in \overline{\Omega},
\end{aligned}
$$

where σ, ν are uniformly positive and bounded coefficients.

A parabolic model problem

Where we are

Let $Q = Q_T := \Omega \times (0,T)$ be the space-time cylinder and $\Sigma := \partial\Omega \times (0,T)$, $\Sigma_0 := \overline{\Omega} \times \{0\}$ and $\Sigma_T := \overline{\Omega} \times \{T\}$.
Then: Given f, g, ν and u_0, find u such that

$$
\begin{aligned}
\partial_t u - \operatorname{div}_x(\nu(x,t)\nabla_x u) &= f + \operatorname{div}_x(\mathbf{f}) && \text{in } Q,\\
u(x,t) &= 0, && (x,t) \in \Sigma,\\
u(x,0) &= 0, && x \in \overline{\Omega},
\end{aligned}
$$

where ν is a uniformly positive and bounded coefficient.

Examples: diffusion, heat-conduction and 2D eddy current.

References

Overview papers:

Gander ('15): Historical overview on 50 years time-parallel meth.
Steinbach, Yang ('19): Overview on space-time methods

Recent unstructured FE space-time methods

Steinbach ('15): theory for the limit case of our method
Bank, Vassilevski, Zikatanov ('16): non-localized version
Devaud, Schwab ('18): mesh-grading in time

Space-time Variational Formulations

(1) Line Variational Formulation and Vertical Method of Lines
- ▷ discretize first in space, then in time

(2) Line Variational Formulation and Horizontal Method of Lines
- ▷ discretize first in time, then in space

(3) Space-time Variational Formulation
- ▷ discretize **all at once** on unstructured decompositions of the space-time cylinder

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Pros? Cons?

A (very) weak space-time variational formulation

Find $u \in H_0^{1,0}(Q)$ s.t.

$$a(u,v) = l(v) \qquad \forall v \in H_{0,\overline{0}}^{1,1}(Q), \qquad \text{(VI)}$$

with

$$a(u,v) := -\int_Q u(x,t)\ \partial_t v(x,t) + \nu(x,t)\nabla_x u(x,t) \cdot \nabla_x v(x,t)\ \mathrm{d}(x,t),$$

$$l(v) := \int_Q f(x,t)\, v(x,t)\ \mathrm{d}(x,t) + \int_Q \mathbf{f}(x,t) \cdot \nabla_x v(x,t)\ \mathrm{d}(x,t) + \int_\Omega u_0(x)\, v(x,0)\ \mathrm{d}x.$$

Regularity of solutions

Using a test function $\varphi \in \dot{C}^\infty(Q_i)$, we obtain from (VI)

$$-\int_{Q_i} u\, \partial_t \varphi + \nu \nabla_x u \cdot \nabla_x \varphi \, \mathrm{d}(x,t) = \int_{Q_i} f\, \varphi \, \mathrm{d}(x,t) + \int_{Q_i} \mathbf{f} \cdot \nabla_x \varphi \, \mathrm{d}(x,t),$$

where $\overline{Q} = \bigcup_{i=1}^N \overline{Q_i}$. Integration by parts yields

$$\int_{Q_i} \begin{pmatrix} \nu \nabla_x u \\ -u \end{pmatrix} \cdot \nabla \varphi \, \mathrm{d}(x,t) = \int_{Q_i} (f - \mathrm{div}_x \mathbf{f}) \varphi \, \mathrm{d}(x,t),$$

i.e, the definition of the weak (space-time) divergence

$$\mathrm{div} \begin{pmatrix} \nu \nabla_x u \\ -u \end{pmatrix} = f - \mathrm{div}_x \mathbf{f} \quad \text{in } L_2(Q_i).$$

Using this on (VI), we obtain for the sum of fluxes

$$\sum_{i=1}^N \int_{\partial Q_i} \begin{pmatrix} -\nu \nabla_x u \\ u \end{pmatrix} \cdot \begin{pmatrix} \vec{n}_x \\ n_t \end{pmatrix} v \, \mathrm{d}s_{(x,t)} = \sum_{i=1}^N \int_{\partial Q_i} \mathbf{f} \cdot \vec{n}_x \, v \, \mathrm{d}(x,t).$$

Space-time FEM with time-upwind stabilization

The main idea:

- decompose Q into shape-regular finite elements $K \in \mathcal{T}_h$,
- define conform finite element space

$$V_{0h} := \{v \in C(\overline{Q}) : v(x_K(\cdot)) \in \mathbb{P}_p(\widehat{K}),\ v|_{\overline{\Sigma} \cup \overline{\Sigma}_0} = 0\}$$

- for each element K, define *individual* upwind test function

$$v_h(x,t) + \boldsymbol{\theta_K h_K} \partial_t v_h(x,t), \text{ for all } (x,t) \in K, \qquad (*)$$

with $\boldsymbol{\theta_K}$ positive parameter, and $\boldsymbol{h_K} := \mathrm{diam}(K)$,

The generalized space-time FE scheme

Find $u_h \in V_{0h} := \{v \in C(\overline{Q}) : v(x_K(\cdot)) \in \mathbb{P}_p(K),\ v|_{\overline{\Sigma} \cup \overline{\Sigma}_0} = 0\}$:

$$\tilde{a}_h(u_h, v_h) = \tilde{l}_h(v_h), \quad \forall v_h \in V_{0h}, \tag{1}$$

where

$$\begin{aligned}
\tilde{a}_h(u_h, v_h) &:= \sum_{K \in \mathcal{T}_h} \int_K \left[\begin{pmatrix} \nu \nabla_x u_h \\ -u_h \end{pmatrix} \cdot \nabla(v_h + \theta_K h_K \partial_t v_h) \right] \mathrm{d}(x,t) \\
&\quad - \theta_K h_K \int_{\partial K} \left[\begin{pmatrix} \nu \nabla_x u_h \\ -u_h \end{pmatrix} \cdot \begin{pmatrix} \vec{n}_x \\ n_t \end{pmatrix} \right] \partial_t v_h \, \mathrm{d}s_{(x,t)} \\
\tilde{l}_h(v_h) &:= \sum_{K \in \mathcal{T}_h} \int_K f(v_h + \theta_K h_K \partial_t v_h) + \mathbf{f} \cdot \nabla_x (v_h + \theta_K h_K \partial_t v_h) \, \mathrm{d}(x,t) \\
&\quad - \theta_K h_K \int_{\partial K} \mathbf{f} \cdot \vec{n}_x \, \partial_t v_h \, \mathrm{d}(x,t)
\end{aligned}$$

Ellipticity = Stability

Definition (Mesh dependent norm).

$$\|v_h\|_h^2 := \sum_{K\in\mathcal{T}_h} \left[\|\nu^{1/2}\nabla_x v_h\|_{L_2(K)}^2 + \theta_K h_K \|\partial_t v_h\|_{L_2(K)}^2 \right] + \frac{1}{2}\|v_h\|_{L_2(\Sigma_T)}^2.$$

Lemma (Coercivity on the FE space [1]).

There exits a constant μ_c such that

$$a_h(v_h, v_h) \geq \mu_c \|v_h\|_h^2, \quad \forall v_h \in V_{0h},$$

with $\mu_c = \min_{K\in\mathcal{T}_h} \left\{1 - c_{Idiv}\sqrt{\frac{\overline{\nu}_K \theta_K}{4h_K}}\right\} \geq \frac{1}{2}$ *for* $\theta_K \leq \frac{h_K}{c_{Idiv}^2 \overline{\nu}_K}$,
i.e., $\mu_c = \frac{1}{2}$ *for* $\theta_K = \frac{h_K}{c_{Idiv}^2 \overline{\nu}_K}$.

Existence & Uniqueness

For the *finite dimensional* problem (1):

Coercivity $\Rightarrow$ **Uniqueness** $\Rightarrow$ **Existence**

The linear system

usual FEM procedure yields

$$\mathbf{K}_h \mathbf{u}_h = \mathbf{f}_h$$

with

- $\mathbf{K}_h$ **non-symmetric**, but **positive definite** system matrix,
- $\mathbf{u}_h$ the coefficient vector wrt the ansatz functions,
- $\mathbf{f}_h$ the load vector stemming from the rhs

Boundedness

Definition.

$$\|v\|_{h,*}^2 := \|v\|_h^2 + \sum_{K\in\mathcal{T}_h} \left[(\theta_K h_K)^{-1}\|v\|_{L_2(K)}^2 + \theta_K h_K \|\mathrm{div}_x(\nu\nabla_x v)\|_{L_2(K)}^2\right]$$

Lemma.

The bilinear form $\tilde{a}_h(\,.\,,\,.\,)$ is uniformly bounded on $V_{0h,*}\times V_{0h}$, i.e.,

$$|\tilde{a}_h(u,v_h)| \le \mu_b \|u\|_{h,*}\,\|v_h\|_h,$$

where $V_{0h,*} := V_{0h} + H^{\mathcal{L},1}(\mathcal{T}_h)$.

A Cèa-like estimate

Lemma ([1]).

Let $u \in H^{\mathcal{L},1}_{0,\underline{0}}(\mathcal{T}_h)$ *be the exact solution, and* $u_h \in V_{0h}$ *the solution of the finite element scheme* (1). *Then there holds the Cèa-like estimate*

$$\|u - u_h\|_h \leq \left(1 + \frac{\mu_b}{\mu_c}\right) \inf_{v_h \in V_{0h}} \|u - v_h\|_{h,*}.$$

An a priori error estimate

Theorem ([2]).

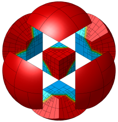
Let $1 < l \leq p+1$, $u \in V_0 \cap H^l(Q)$ *be the exact solution, and* $u_h \in V_{0h}$ *be the solution of the finite element scheme (1). Furthermore, let* $a_h(\cdot,\cdot)$ *be coercive and bounded. Then the a priori error estimate*

$$\|u-u_h\|_h \leq C\left(\sum_{K\in\mathcal{T}_h} h_K^{2(s-1)}\left(|u|^2_{H^s(S_K)}+\|\mathrm{div}_x(\nu\nabla_x u)\|^2_{L_2(K)}\right)\right)^{1/2}$$

holds with $s = \min\{l, p+1\}$, S_K *a neighborhood of* K, *and some generic positive constant* C.

Numerical Experiments

Key information



- Space-time FEM was implemented with MFEM[1],
- Linear systems were solved by means of *hypre*,
- Tests were performed on `quartz`[2] (uniform) or on my workstation (adaptive),

[1] http://mfem.org
[2] 2604 nodes, 93 744 cores, LLNL

Linear solvers

hypre[3] *boomerAMG* and GMRES

boomerAMG

	$d = 2$	$d = 3$
Relaxation	(ℓ_1-scaled) hybrid Gauß-Seidel	
Coarsening	Falgout	HMIS
Interpolation	Standard	Extended+i
AMG strength threshold	0.75	0.8

Parallel flexible GMRES

- stopped after initial residual is reduced by 10^{-8}
- no restarts

[3] https://www.llnl.gov/casc/hypre/

The variable-step nonlinear AMLI cycle MG [3]

Algorithm (Recursive Definition)

For any fiven $\mathbf{v}$ *at level* $k < \ell$ *to compute* $B_k[\mathbf{v}]$*, we perform:*

- *Pre-smooth, i.e., solve* $M_k \mathbf{y} = \mathbf{v}$ *and compute residual* $\mathbf{r} = \mathbf{v} - A_k \mathbf{y}$.
- *Restrict residual, i.e., compute* $\mathbf{r}_c = P_k^T \mathbf{r}$.
- *If* $k + 1 = \ell$*, solve exactly, i.e.,* $\mathbf{x}_c = A_\ell^{-1} \mathbf{r}_c$*. Otherwise, apply recursion to solve* $A_{k+1} \mathbf{y}_c = \mathbf{r}_c$*, i.e., use* ν_{k+1} *iterations of flexible GMRES with* $B_{k+1}[.]$ *(recursively defined) as a preconditioner, staring with zero initial iterate* $\mathbf{y}_c^{(0)} = 0$*. The result, is* $\mathbf{x}_c = B_{k+1}^{(\nu_{k+1})}[\mathbf{r}_c] := \mathbf{y}_c^{(\nu_{k+1})}$.
- *Update fine-level iterate, i.e., compute*

$$\mathbf{y} = \mathbf{y} + P_k \mathbf{x}_c.$$

- *Post-smooth, i.e., solve for* $\mathbf{z}$*,* $M_k^{'}(\mathbf{z} - \mathbf{y}) = \mathbf{v} - A_k \mathbf{y}$.
- *This gives* $B_k[\mathbf{v}] = \mathbf{z}$.

Example 1 (Smooth solution, constant coefficient).

We consider $Q = (0,1)^{d+1}$, $\nu \equiv 1$, the exact solution

$$u(x,t) = \sum_{i=1}^{d} \sin(x_i\,\pi)\,\sin(t\,\pi).$$

Example 1: Performance for $d = 2$.

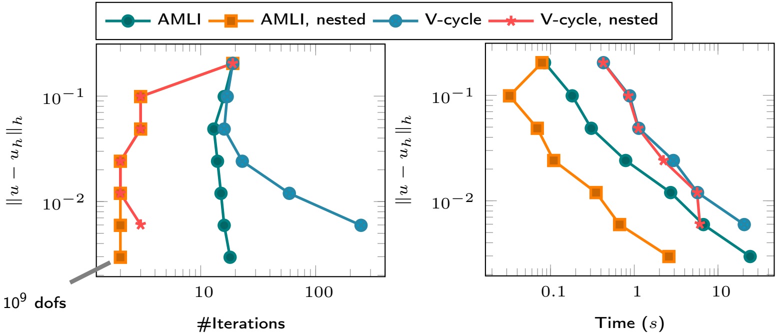
(a) $p = 1$, 4096 cores

dofs	AMLI		V-cycle	
4913	19	(0.08 s)	19	(0.43 s)
35 937	16	(0.18 s)	17	(0.88 s)
274 625	13	(0.31 s)	16	(1.13 s)
2 146 689	14	(0.79 s)	23	(2.93 s)
16 974 593	15	(2.72 s)	59	(5.70 s)
135 005 697	16	(6.70 s)	248	(20.61 s)
1 076 890 625	18	(24.24 s)	$>$300	(246.46 s)

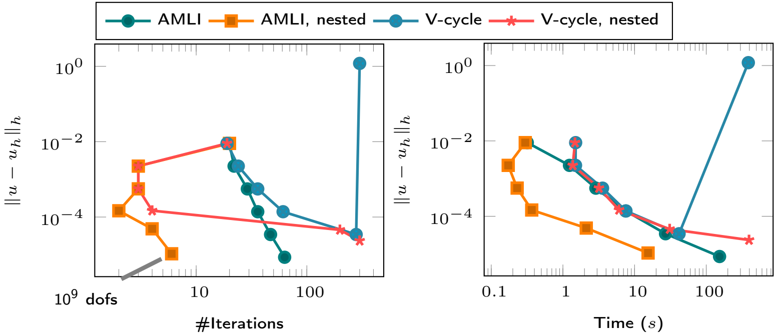
(b) $p = 2$, 4096 cores, 2 sweeps

dofs	AMLI		V-cycle	
35 937	20	(0.32 s)	19	(1.51 s)
274 625	22	(1.24 s)	24	(1.50 s)
2 146 689	29	(2.88 s)	36	(3.58 s)
16 974 593	36	(7.54 s)	61	(7.58 s)
135 005 697	47	(27.05 s)	279	(41.81 s)
1 076 890 625	63	(153.48 s)	$>$300	(386.86 s)

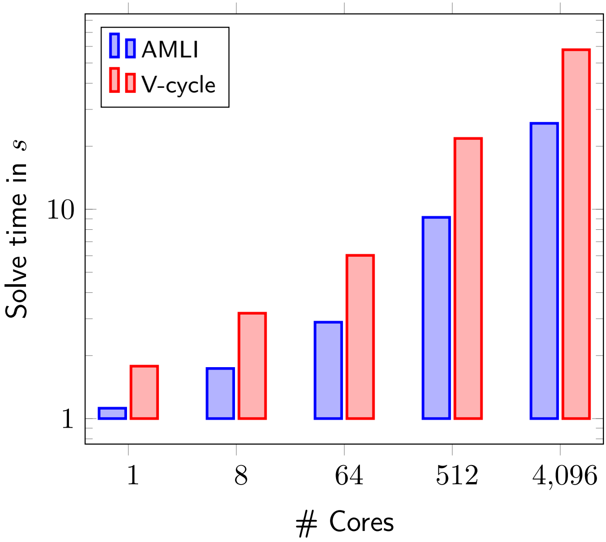
Example 1: Iteration and time to solution, with $p = 1$ and $d = 2$.



Example 1: Iteration and time to solution, with $p = 2$ and $d = 2$.



Example 1: Weak scaling with ∼274 625 dofs per core.



Example 1: Performance for $d = 3$.

(a) $p = 1$, 256 cores

dofs	**AMLI**		**V-cycle**	
625	10	(0.01 s)	10	(0.02 s)
6561	16	(0.05 s)	16	(0.09 s)
83 521	22	(0.24 s)	23	(0.17 s)
1 185 921	33	(1.01 s)	35	(2.46 s)
17 850 625	58	(13.35 s)	58	(9.30 s)

(b) $p = 2$, 256 cores

dofs	**AMLI**		**V-cycle**	
6561	32	(0.08 s)	27	(0.10 s)
83 521	51	(0.43 s)	44	(0.23 s)
1 185 921	83	(2.76 s)	75	(2.06 s)
17 850 625	162	(45.68 s)	148	(31.35 s)

$$(\eta^*)^2 = \sum_{K \in \mathcal{T}_h} (\eta_K^*)^2$$

An error indicator

Residual-based error indicator by Steinbach & Yang ('17)

$$(\eta_K^{\mathcal{R}})^2 := h_K^2 R_h(u_h) + h_K J_h(u_h),$$

where

$$R_h(u_h) := \|f + \mathrm{div}_x(\nu \nabla_x u_h) - \partial_t u_h\|_{L_2(K)}^2,$$
$$J_h(u_h) := \|[\nu \nabla_x u_h]_e\|_{L_2(\partial K)}^2.$$

Functional error indicator

Guaranteed error bound, see e.g. Repin ('08)

$$\|\nabla_x(u-u_h)\|_{L_2(Q)}+\frac{1}{2}\|u(\cdot,T)-u_h(\cdot,T)\|_{L_2(\Omega)} \leq \overline{\mathfrak{M}}^2_{EV1}(\beta,\delta,v,\mathbf{y}_h),$$

where $\mathbf{y}_h = \arg\min_{\mathbf{y}} \overline{\mathfrak{M}}^2_{EV1}(\beta,\delta,u_h,\mathbf{y})$, with $\delta = 1/2$, and the first form of the error majorant

$$\overline{\mathfrak{M}}^2_{EV1}(\beta,\delta,v,\mathbf{y}) := \int_Q \Big[(1+\beta(t))|\mathbf{y}-\nu\nabla_x v|^2 + (1+\frac{1}{\beta(t)})|f-\partial_t v+\mathrm{div}_x\mathbf{y}|^2\Big]\,\mathrm{d}(x,t).$$

Error indicator

$$(\eta_K^{\mathfrak{M}})^2 := \|\mathbf{y}_h-\nu\nabla_x u_h\|^2_{L_2(K)}$$

Functional error indicator (generalized)

Guaranteed error bound, see e.g. Repin ('08)

$$\|\nabla_x(u-u_h)\|_{L_2(Q)}+\frac{1}{2}\|u(\cdot,T)-u_h(\cdot,T)\|_{L_2(\Omega)} \leq \overline{\mathfrak{M}}^2_{EV1}(\beta,\delta,v,\mathbf{y}_h),$$

where $\mathbf{y}_h = \arg\min_{\mathbf{y}} \overline{\mathfrak{M}}^2_{EV1}(\beta,\delta,u_h,\mathbf{y})$, with $\delta = 1/2$, and the first form of the error majorant

$$\overline{\mathfrak{M}}^2_{EV1}(\beta,\delta,v,\mathbf{y}) := \int_Q \Big[(1+\beta(t))|\mathbf{y}-\nu\nabla_x v+\mathbf{f}|^2 + (1+\frac{1}{\beta(t)})|f-\partial_t v+\operatorname{div}_x\mathbf{y}|^2\Big]\,\mathrm{d}(x,t).$$

Error indicator

$$(\eta_K^{\mathfrak{M}})^2 := \|\mathbf{y}_h-\nu\nabla_x u_h+\mathbf{f}\|^2_{L_2(K)}$$

Dörfler's Marking

Determine set $\mathcal{M}_h \subset \mathcal{T}_h$ of minimal cardinality s.t.

$$\Xi(\eta^*)^2 \leq \sum_{K \in \mathcal{M}_h} (\eta_K^*)^2,$$

and **MARK** all $K \in \mathcal{M}_h$

Example 2 (NIST Benchmark "Moving Wavefront").

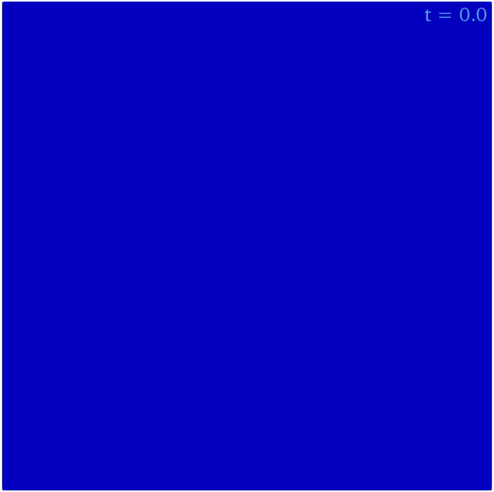
Choose space-time cylinder $Q = \prod_{i=1}^{d}(x_{i,0}, x_{i,1}) \times (0,T)$, $\nu \equiv 1$, and exact solution

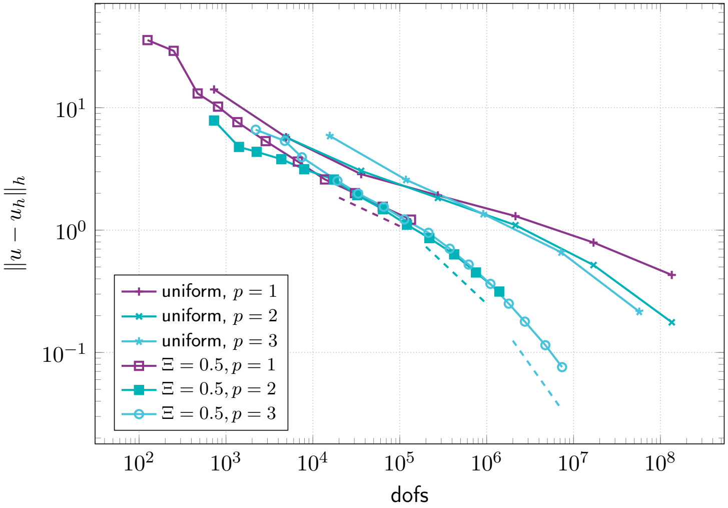
$$u(x,t) = B(x)\arctan(t)\left(\frac{\pi}{2} - \arctan(\alpha(r-t))\right)$$

with $r^2 := \sum_{i=1}^{d}(x_i - x_{i,c})^2$ and

$$B(x) := \prod_{i=1}^{d} \frac{(x_i - x_{i,0})(x_i - x_{i,1})}{(-1)^{d+1}C}.$$

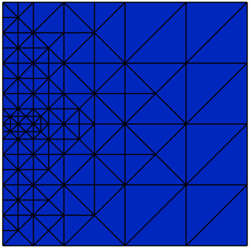
$$Q = (0,10) \times (-5,5) \times (0,10),\ \alpha = 20,\ C = 10\,000,\ x_c = (0,0)$$



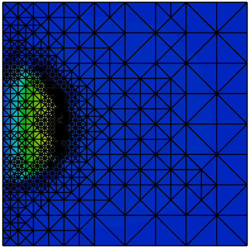


Example 2 Error rates for *functional estimator* $\eta^{\mathfrak{M}}$ and Dörfler's marking.

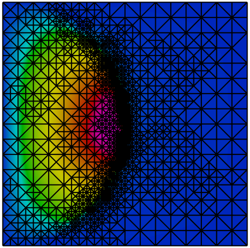
$t = 0$



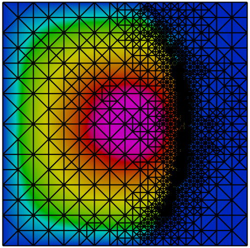
$t = 0.25$



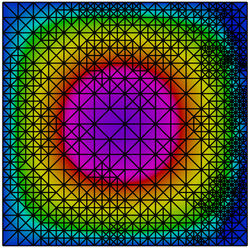
$t = 0.5$

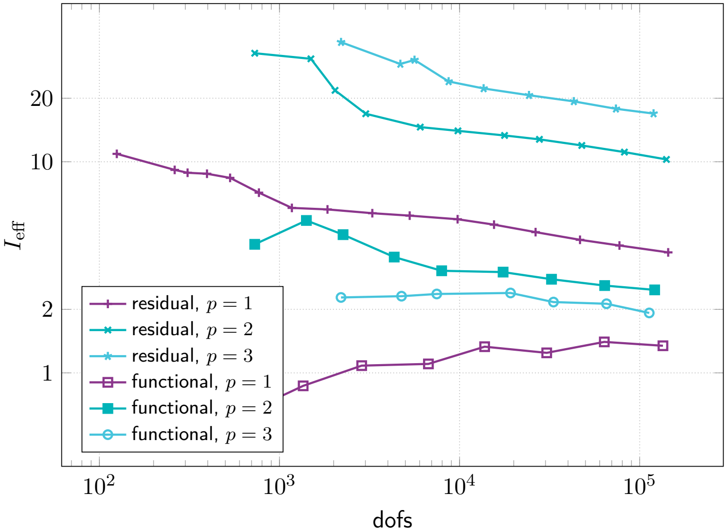


$t = 0.75$



$t = 1$

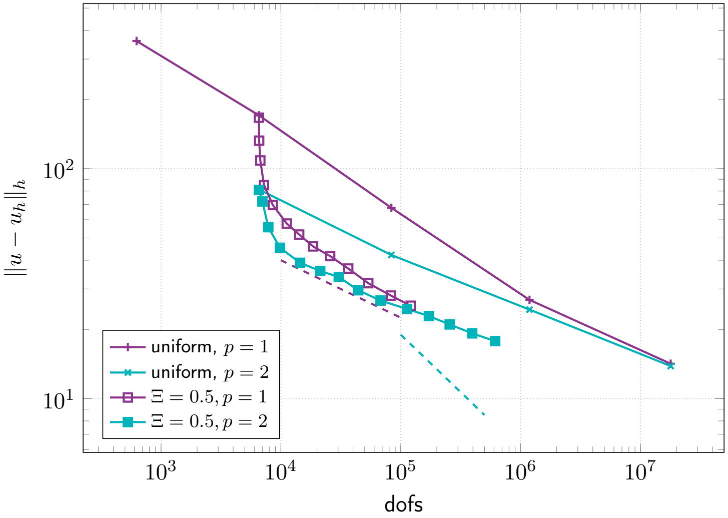




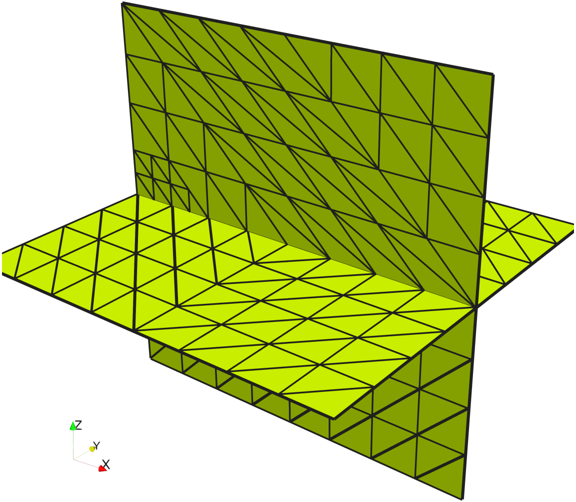
Example 2 Efficiency indices for Dörfler's marking.

$$Q = (0,10) \times (-5,5)^2 \times (0,10),\ \alpha = 20,\ C = 100\,000,\ x_c = (0,0,0)$$

t = 0.00



Example 2 Error rates for *functional estimator* $\eta^{\mathfrak{M}}$ and Dörfler's marking.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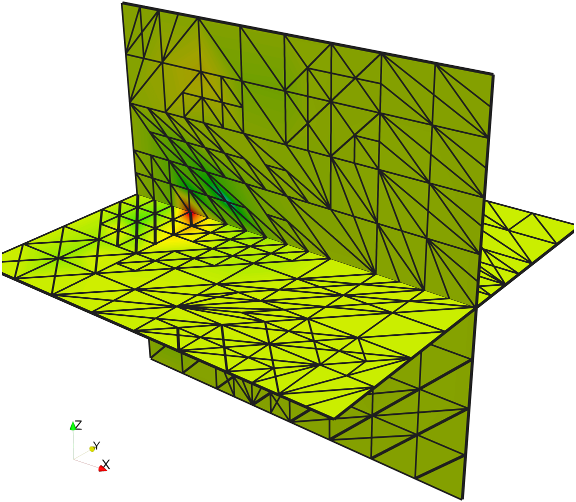
Y
X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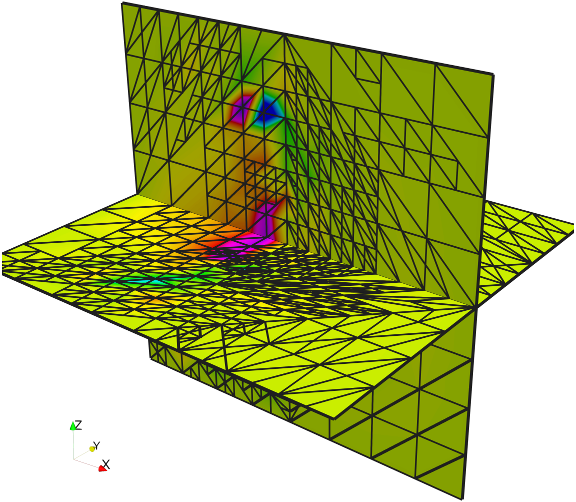
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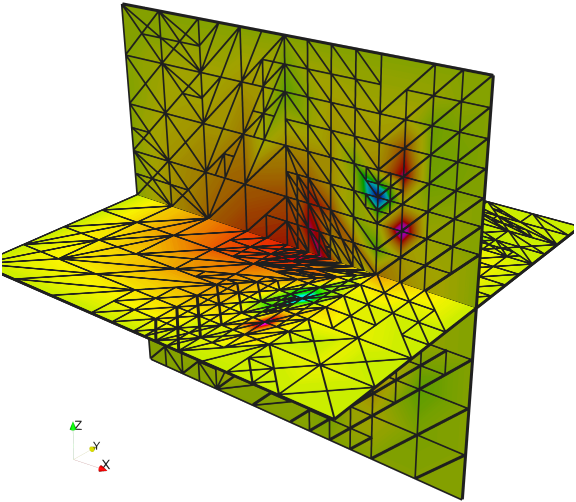
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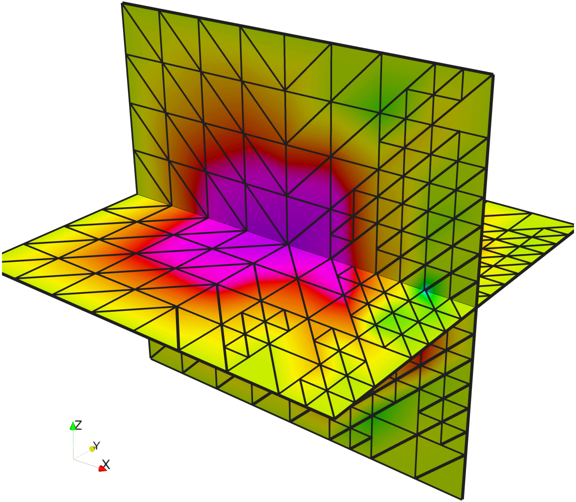
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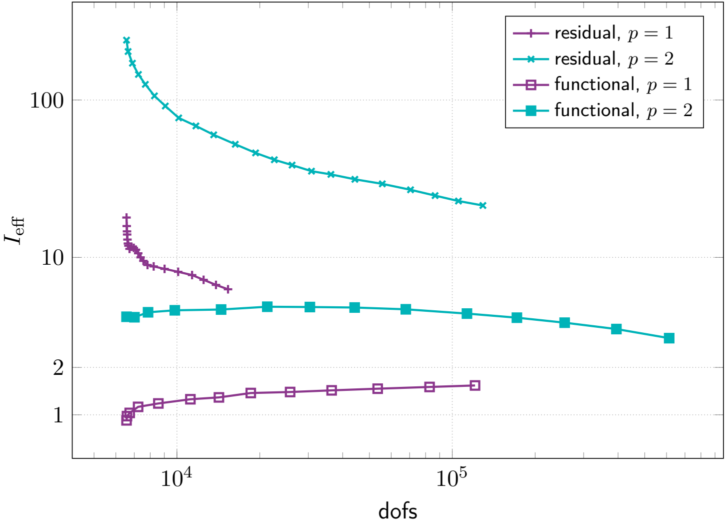

Example 2 Efficiency indices for Dörfler's marking.

Conclusions

- **stability** of the FE scheme provided that $\theta_K = \mathcal{O}(h_K)$,
- **a priori estimates** for the discretization error,
- results can be generalized to a space-time scheme with **distributional rhs** $= f + \mathrm{div}_x \mathbf{f}$
- the numerical experiments are in accordance with the theory,
- fully *parallel* space-time FEM
- adaptivity simultaneously in space and time for $d = 1, 2, 3$ for residual and functional error indicators

Outlook & Future Work

- Analysis
  - reliable and efficient a posterior error estimators
- Computational
  - application to more complex problems,
  - preconditioners for $p > 1$, and $d = 3$,
  - parallelization of the AMR (load balancing)

Main future goal

non-linear parabolic problems and *eddy-current problems*

[1] Langer, U., Neumüller, M., and Schafelner, A.
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[2] Langer, U., and Schafelner, A.
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[3] Schafelner, A., and Vassilevski, P. S.
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In preparation.

Thank you!

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The constant c_{I,div_x}

- the constant c_{I,div_x} comes from the *inverse inequality*

$$\|\mathrm{div}_x(\nu w_h)\|_{L_2(K)} \leq c_{I,\mathrm{div}_x} h_K^{-1} \|\nu w_h\|_{L_2(K)}, \forall w_h \in W_h|_K,$$

  with $W_h|_K := \nabla_x (V_{0h}|_K)$ and $\nu \in W^{1,\infty}(\mathcal{T}_h)$,
- c_{I,div_x} is independent of h_K, but depends on polynomial degree p and the dimension d,
- c_{I,div_x} is **computable!**
  - to be precise, we can compute a *sharp* lower bound for $c_{I,\mathrm{div}_x} h_K^{-1}$

Ellipticity = Stability (generalized)

Definition (A different norm).

$$\|v_h\|_{x,T}^2 := \sum_{K\in\mathcal{T}_h}\left[\|\nu^{1/2}\nabla_x v_h\|_{L_2(K)}^2\right] + \frac{1}{2}\|v_h\|_{L_2(\Sigma_T)}^2.$$

Lemma (Coercivity on the FE space?).

There exits a constant $\tilde{\mu}_c$ *such that*

$$\tilde{a}_h(v_h, v_h) \geq \tilde{\mu}_c\|v_h\|_{x,T}^2, \quad \forall v_h \in V_{0h},$$

with $\tilde{\mu}_c = 1/2$ *for* $\theta_K = \frac{h_K}{c_{Idiv}^2 \overline{\nu}_K}$.

Boundedness (generalized)

Definition.

$$\|v\|_{x,T,\mathrm{div}}{}^2 := \|v\|_{x,T}^2 + \sum_{K\in\mathcal{T}_h} (\theta_K h_K)^{-1}\|u\|_{L_2(K)}^2 + \theta_K h_K \|\mathrm{div}\begin{pmatrix}\nu\nabla_x u\\ -u\end{pmatrix}\|_{L_2(K)}^2$$

Conjecture.

The bilinear form $\tilde{a}_h(\,.\,,\,.\,)$ is uniformly bounded on $V_{0h,\mathrm{div}} \times V_{0h}$, i.e.,

$$|\tilde{a}_h(u, v_h)| \leq \tilde{\mu}_b \; \|u\|_{x,T,\mathrm{div}} \, \|v_h\|_{x,T},$$

where $V_{0h,\mathrm{div}} := V_{0h} + H^{1,0}(Q)$.

Example 3 (Kellogg's problem ('74)).

Consider space-time cylinder $Q = (-1,1)^2 \times (0,1)$,

$$\nu(x,t) = \begin{cases} 161.4476387975885, & \text{if } (x,t) \in Q_1 \cup Q_3 \\ 1, & \text{if } (x,t) \in Q_2 \cup Q_4 \end{cases},$$

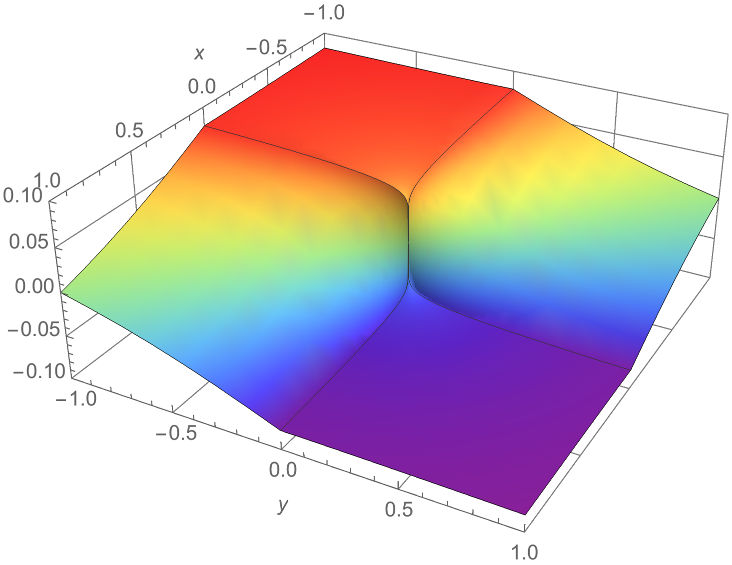
and choose

$$u(r,\phi,t) = r^\gamma \mu(\phi)\, t$$

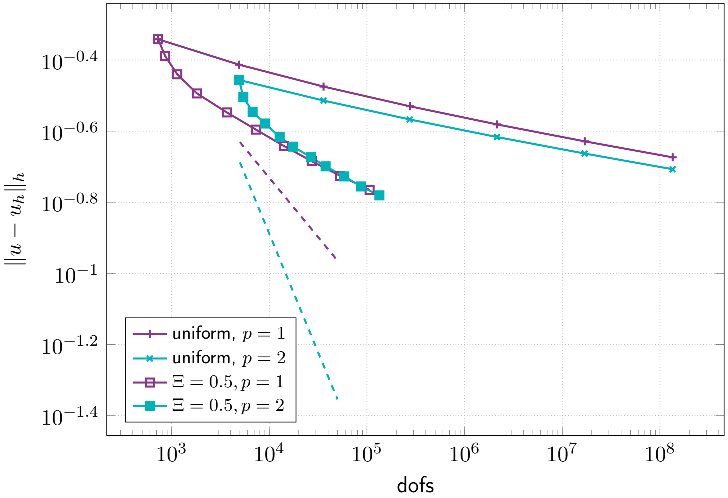
with

$$\mu(\phi) := \begin{cases} \cos(\gamma(\frac{\pi}{2} - \sigma))\cos(\gamma(\phi - \frac{\pi}{2} + \rho)), & \text{for } \phi \in [0, \frac{\pi}{2}], \\ \cos(\gamma\rho)\cos(\gamma(\phi - \pi + \sigma)), & \text{for } \phi \in [\frac{\pi}{2}, \pi], \\ \cos(\gamma\sigma)\cos(\gamma(\phi + \pi - \rho)), & \text{for } \phi \in [-\pi, -\frac{\pi}{2}], \\ \cos(\gamma(\frac{\pi}{2} - \rho))\cos(\gamma(\phi + \frac{\pi}{2} - \sigma)), & \text{for } \phi \in [-\frac{\pi}{2}, 0], \end{cases}$$

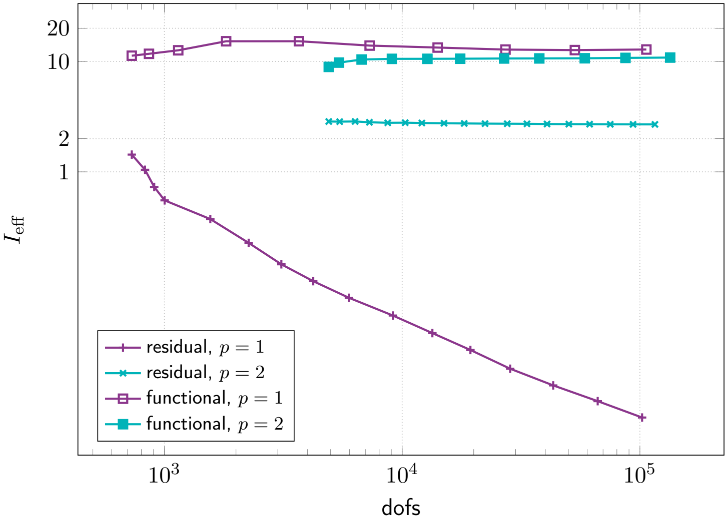
and $\sigma = -19\pi/4$, $\rho = \pi/4$ and $\gamma = 0.1$, i.e., $u \in H^{\mathbf{1.1}}(Q_i)$.



Example 3: Solution at $t = 1$



Example 3. Error rates for $u \in H^{1.1}(Q_i)$.



Example 3. Efficiency indices $I_{\text{eff}} = \frac{\left(\sum_{K\in\mathcal{T}_h} \eta_K^2\right)^{1/2}}{\|u-u_h\|_h}$.