Estimates of the distance to minimizers of nonlinear variational problems an applications to numerical analysis

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The plan

- Distance to the minimizer of an abstract variational problem.
    - Setting
    - Error measure and general error relations.
    - Special case: problems with linear source functionals
- Examples
- Nonlinear decomposition of a Banach space (Helmgholtz type theorem).

General variational problem

$$\inf_{w\in V} J(v), \quad J(v) = G(\Lambda w) + F(w) \ ^a$$

[a] This class includes, e.g., α-Laplacian, NonNewtonian fluids, nonlinear diffusion and reaction–diffusion, Linear and physically nonlinear elasticity, Elasto–plasticity, Models with unilateral and obstacle conditions...

$G : Y \to \mathbb{R}_+$: convex, continuous, coercive functional vanishing at zero element of Y (reflexive Banach space), $\Lambda : V \to Y$ bounded linear operator, $\Lambda^* : Y^* \to V^*$ Here $\Lambda : V \to Y$ is the differential operator (e.g., ∇ or $\nabla_{\rm sym}$),
Λ^* is the conjugate operator (e.g., ${\rm div}$ or ${\rm Div}$):

$$< \Lambda^* y^*, v >= \langle y^*, \Lambda v \rangle$$

Y and $Y^* \ \Rightarrow \ \langle y^*, y \rangle$, V and $V^* \ \Rightarrow < v^*, v >$.

Example

$$V = \overset{\circ}{W}{}^{1,\alpha}(\Omega),\; Y = L^{\alpha}(\Omega, \mathbb{R}^d),\; Y^* = L^{\alpha^*}(\Omega, \mathbb{R}^d),$$

$$\begin{aligned} & \frac{1}{\alpha} + \frac{1}{\alpha^*} = 1, \qquad \alpha \in (1, +\infty) \\ \Lambda &= \nabla, \quad \Lambda^* = -\mathrm{div}\,, \\ G(y) &= \frac{1}{\alpha}\int_\Omega |y|^\alpha\, dx, \quad F(v) = \int_\Omega fv\, dx \\ J(v) &= \frac{1}{\alpha}\int_\Omega |\nabla v|^\alpha dx - \int_\Omega fvdx. \end{aligned}$$

Euler equation for this problem is $\alpha -$ *Laplacian*:

$$\mathrm{div}\, |\nabla u|^{\alpha-2}\nabla u + f = 0,\; \mathrm{in}\,\Omega, \quad u = 0 \text{ on } \Gamma.$$

We generate a sequence of numerical solutions $u_k \in V$ and prove that $J(u_k) \to \inf J$ as $k \to +\infty$ (provided that all has been done correctly).

Question 1:

Which features of the exact minimizer u can be reconstructed and reliably controlled by this sequence?

Question 2:

How to control these features by computable quantities ?

We need some specific notions:

1. Fenchel conjugate functional to the functional $g : X \to X^*$:

$$g^*(\zeta^*) := \sup_{\zeta \in X} \{ < \zeta^*, \zeta > - g(\zeta) \}$$

Example: if $g(\zeta) = \frac{1}{\alpha}|\zeta|^\alpha$, then $g^*(\zeta^*) = \frac{1}{\alpha^*}|\zeta^*|^{\alpha^*}$. Properties and applications to convex variational problems are deeply studied (T. Rockafellar, J. Moreau,I. Ekeland and R. Themam...)

2. Compound functional is defined on $X \times X^*$:

$$D_g(\xi, \xi^*) := g(\xi) + g^*(\xi^*) - \langle \xi^*, \xi \rangle \geq 0 \,!$$

$D_g(\zeta^*, \zeta)$ possesses an important "vanishing property":

$$D_g(\zeta, \zeta^*) = 0 \;\Leftrightarrow\; \zeta^* \subset \partial g(\zeta) \text{ and } \zeta \subset \partial g^*(\zeta^*)$$

D_g is a nonnegative functional, which vanishes only in some special cases.

Special case: quadratic energy $\Rightarrow$ linear problems

If $g(\xi) = \frac{1}{2}|\xi|^2$ and $g^*(\xi^*) = \frac{1}{2}|\xi^*|^2$ then

$$D_g(\xi, \xi^*) = \frac{1}{2}|\xi|^2 + \frac{1}{2}|\xi^*|^2 - (\xi, \xi^*) = \frac{1}{2}|\xi - \xi^*|^2$$

For this reason basic error relations
for linear problems (and only for them!) are presented
in terms of norms!

1. Original (primal) problem

$$J(\ u\ ) = \inf J(v), \quad J(v) = G(\Lambda v) + F(v).$$

u is the exact solution (minimizer).

has a dual counterpart

$$\max_{y^* \in Y^*} I^*(y^*) \text{ where } I^*(y^*) := -G^*(y^*) - F^*(-\Lambda^* y^*),$$

p^* is the exact dual solution, maximizer.

For a wide class of problems

$$I^*(y^*) \leq I(p^*) = J(u) \leq J(v)$$

1. u and p^* satisfy two necessary and sufficient conditions:

$$(I)\ D_F(u, -\Lambda^* p^*) := \quad F(u) + F^*(-\Lambda^* p^*) + < \Lambda^* p^*, u >= 0,$$
$$(II)\ D_G(\Lambda u,\ p^*)\ := \quad G(\Lambda u) + G^*(p^*) - (p^*, \Lambda u) = 0$$

Hint: Linear Elasticity ($F(v) = \int_\Omega fvdx$)

$$(I) \quad \Rightarrow \quad \mathrm{Div}\, p^* + f = 0,$$
$$(II) \quad \Rightarrow \quad p^* = G'(\nabla_{\rm sym}(u)) = \mathbb{L}\nabla_{\rm sym}(u).$$

A variational numerical method approximates u or p^*, or both solutions simultaneously.

Let $y^* \in Y^*$ and $v \in V$ approximate p^* and u.

We introduce the following measure of the distance between $\{u, p^*\}$ and $\{v, y^*\}$:

$$\mu(\{u, p^*\}, \{v, y^*\}) := D_F(u, -\Lambda^* y^*) + D_G(\Lambda u, y^*) + D_F(v, -\Lambda^* p^*) + D_G(\Lambda v, p^*).$$

We have 4 nonnegative terms. The first pair compare u and y^* throughout Λ and Λ^*.
The second pair does the same for v and p^*.

It is clear that $\mu(\{u, p^*\}, \{v, y^*\}) \geq 0$. When it vanishes?

Since all the compounds are nonnegative, it must hold:

$$D_F(u, -\Lambda^* y^*) = 0, \qquad D_G(\Lambda u, y^*) = 0,$$
$$D_F(v, -\Lambda^* p^*) = 0, \qquad D_G(\Lambda v, p^*) = 0.$$

what amounts

$$\Lambda v \in \partial G^*(p^*) \quad \text{and} \quad y^* \in \partial G(\Lambda u),$$
$$-\Lambda^* y^* \in \partial F(u), \quad \text{and} \quad v \in \partial F^*(-\Lambda^* p^*).$$

These relations are equivalent to I and II!

$\boldsymbol{\mu}(\{u, p^*\}, \{v, y^*\}) = 0$ if and only if $\{v, y^*\}$ is equal to $\{u, p^*\}$!

$\boldsymbol{\mu}$ is a right measure!

Theorem (1)

For any $v \in V$ and $y^* \in Y^*$

$$\underbrace{\mu(v) + \mu^*(y^*)}_{\textit{error measure}} = \underbrace{D_G(\Lambda v, y^*) + D_F(v, -\Lambda^* y^*)}_{\textit{computable quantity}}$$

Here the measure is decomposed into two parts

$$\mu(v) = D_F(v , -\Lambda^* p^*) + D_G(\Lambda v , p^*),$$
$$\mu^*(y^*) = D_F(u , -\Lambda^* y^*) + D_G(\Lambda u , y^*).$$

Theorem (2)

$$\mu(v) + \mu^*(y^*) \quad = \quad \underbrace{J(v) \ - \ I^*(y^*)}_{\textit{duality gap}}.$$

This identity[a] shows that a variational problem automatically generates the measure μ!

If we minimize $J(v)$ (e.g., classical FEM approach)
or maximize $I^*(y^*)$ (e.g., dual FEM approach)
or do both (e.g., mixed FEM approach)
WE APPROXIMATE EXACT SOLUTIONS IN TERMS OF μ.

μ is the maximal measure of a variational problem.

[a] S.R. *Math. Comput.*, 2000; also exposed in the book form, Elsevier 2004

Conclusion:
a variational problem itself generates a natural measure of errors, which is provides maximum quantitative information on the quality of approximating sequences.

In general, components of $\boldsymbol{\mu}$ are nonconvex functionals, e.g.,

$$D_G(y, y^*) := \int_\Omega \left(\frac{1}{q}|y|^q + \frac{1}{q^*}|y^*|^{q^*} - yy^* \right) dx$$

is not a convex functional on $Y \times Y^*$. However, $\boldsymbol{\mu}\left(\{u, p^*\}, \{v, y^*\}\right)$ generates a system of convex sets (local topology) at the vicinity of the exact solutions pair $(\nabla u, p^*)$.

Illustrative example

$V = Y = \mathbb{R}$, $G(y) = \frac{1}{\alpha}|y|^\alpha$, $F(v) = \frac{1}{\beta}|v|^\beta$, $\alpha, \beta > 1$,
$\Lambda v = \kappa v$, $\Lambda^* y^* = \kappa y^*$, $G^*(y^*) = \frac{1}{\alpha^*}|y^*|^{\alpha^*}$, $F^*(v^*) = \frac{1}{\beta^*}|v^*|^{\beta^*}$,
$J(v) = \frac{1}{\alpha}|\kappa v|^\alpha + \frac{1}{\beta}|v|^\beta$, $u = 0$ is the minimizer.
$J^*(y^*) = -\frac{1}{\alpha^*}|y^*|^{\alpha^*} - \frac{|\kappa|^{\beta^*}}{\beta^*}|y^*|^{\beta^*}$, the maximizer p^* is also zero.
Then

$$\mathcal{D}_G(\Lambda v, y^*) = \frac{1}{\alpha}|\kappa v|^\alpha + \frac{1}{\alpha^*}|y^*|^{\alpha^*} - \kappa v y^*,$$

$$\mathcal{D}_G(\Lambda u, y^*) = \frac{1}{\alpha^*}|y^*|^{\alpha^*},\ \mathcal{D}_G(\Lambda v, p^*) = \frac{1}{\alpha}|\kappa v|^\alpha.$$

$$\mathcal{D}_F(v, -\Lambda^* p^*) = \frac{1}{\beta}|v|^\beta,\ \mathcal{D}_F(-\Lambda^* y^*, u) = \frac{1}{\beta^*}|-\kappa y^*|^{\beta^*}.$$

Hence the measure is given by the relation

$$\boldsymbol{\mu}(v, y^*; u, p^*) = \frac{|\kappa|^\alpha}{\alpha}|v|^\alpha + \frac{1}{\beta}|v|^\beta + \frac{1}{\alpha^*}|y^*|^{\alpha^*} + \frac{|\kappa|^{\beta^*}}{\beta^*}|y^*|^{\beta^*}.$$

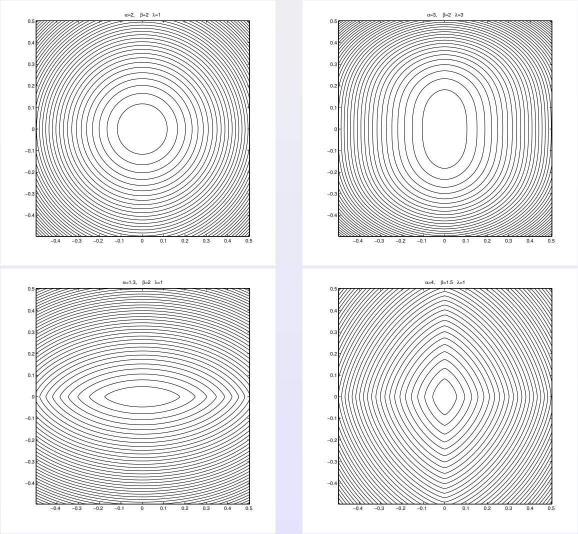

Figure: Level lines of $\boldsymbol{\mu}$ for $\alpha = 2$, $\beta = 2$, $\kappa = 1$ (top left), $\alpha = 3$, $\beta = 2$, $\kappa = 3$ (top right), $\alpha = 1.3$, $\beta = 2$, $\kappa = 1$ (bottom left) and $\alpha = 4$, $\beta = 1.5$, $\kappa = 1$ (bottom right)

Comment: other "nonlinear" error measures for the primal variable

Assumption: G is differentiable and uniformly convex , i.e.,

$$G\left(\frac{y_1+y_2}{2}\right)+\frac{1}{2}\Phi(y_1-y_2)\leq \frac{1}{2}G(y_1)+\frac{1}{2}G(y_2) \qquad \forall y_1,y_2\in Y$$

where $\Phi : Y \to \mathbb{R}^+$. Then we can introduce two other measures:

$$\mu^+(v) := \left\langle G'(\Lambda u) - G'(\Lambda v), \Lambda v - \Lambda u\right\rangle, \ (monotonicity\ measure)$$

$$\mu^-(v) := \Phi(\Lambda(v-u)) \quad (uniform\, convexity\, measure).$$

Theorem

$$\mu^-(\mathbf{v}) \leq \mu(\mathbf{v}) \leq \mu^+(\mathbf{v}) \qquad \forall \mathbf{v}\in \mathbf{V}.$$

$$J(v) = \int_\Omega (\frac{1}{2} A\nabla v \cdot \nabla v - fv)dx \ \to \ \mathrm{min}.$$

Nonlinear effects and free boundaries arise due to the set

$K := \{v \in V_0 := H^1_0 \mid \phi(x) \leq v(x) \leq \psi(x) \, \mathrm{a.e.\,in}\, \Omega\}, \ \phi, \psi \in H^2(\Omega)$.

Here $\Lambda v = \nabla v$, $\Lambda^* y^* = -\mathrm{div}\, y^*$,

$$G(\Lambda v) = \int_\Omega \tfrac{1}{2} A\nabla v \cdot \nabla v \, dx, \qquad F(v) = \int_\Omega -fv\, dx + \Psi(v) \,,$$

$$\Psi(v) = \begin{cases} 0 & \text{if } \phi \leq v \leq \psi \\ +\infty & \text{else} \end{cases} \qquad \text{Let } v^* \in L^2(\Omega)$$

$$F^*(v^*) = \sup_{v\in V}\{(v^*, v) - F(v)\} = \sup_{v\in K} \int_\Omega v(v^* + f)\, dx$$

$$= \int_\Omega (-\phi(v^* + f)_\ominus + \psi(v^* + f)_\oplus) dx,$$

Then for $v \in K$

$$D_F(v, -\Lambda^* p^*) = F(v) + F^*(\operatorname{div} p^*) - (\operatorname{div} p^*, v)$$
$$= \int_\Omega \big(-\phi(\operatorname{div} p^* + f)_\ominus + \psi(\operatorname{div} p^* + f)_\oplus \underbrace{-fv - \operatorname{div} p^* v}\big) dx.$$

On two obstacles p^* is known and it is defined by ψ and ϕ.

$$D_F(v, -\Lambda^* p^*) = \int\limits_{\Omega^u_\ominus} \mathrm{W}_\phi(x)(v-\phi)\, dx + \int\limits_{\Omega^u_\oplus} \mathrm{W}_\psi(x)(\psi - v)\, dx := \boldsymbol{\mu_{\phi\psi}}$$

where $\mathrm{W}_\phi(x) := -(\operatorname{div} A\nabla\phi + f)$ and $\mathrm{W}_\psi(x) := \operatorname{div} A\nabla\psi + f$ are two nonnegative weight functions

We have an extra measure that has been missed before: $\boldsymbol{\mu_{\phi\psi}(v)}$

.

It controls in a weak (integral) sense *weather or not the function v coincides with obstacles on true coincidence sets* $\Omega^u_\ominus$ *and* $\Omega^u_\oplus$

$$\boldsymbol{\mu}(v) := D_G(\nabla v, p^*) + \mu_{\phi\psi}(v)$$

If the functional G is generated by quadratic form, then

$$D_G(\nabla v, p^*) = \frac{1}{2}\|A\nabla v - p^*\|^2_{A^{-1}},$$

Error identity for the primal variable:

$$\frac{1}{2}\|\nabla(u - v)\|^2_A + \mu_{\phi\psi}(v) = J(v) - J(u)$$

Error identity yields the a posteriori error estimate for the full error measure:

$$D_G(\nabla v, p^*) + \mu_{\phi\psi}(v) \leq$$

$$\leq (1+\beta^{-1}) D_G(\nabla v, y^*) + \frac{1}{2} C_\Omega^2 (1+\beta) \|\mathrm{div}\, y^* + f + \lambda_1 - \lambda_2\|^2 + \int_\Omega (\lambda_1(v-\phi) + \lambda_2(\psi - v))\, dx$$

The estimate has no gap! Indeed, set $y^* = p^*$, and

$$\lambda_1 = -(\mathrm{div}\, p^* + f), \quad \lambda_2 = 0 \quad \text{on } \Omega_\phi^u,$$
$$\lambda_2 = \mathrm{div}\, p^* + f, \quad \lambda_1 = 0 \quad \text{on } \Omega_\psi^u,$$
$$\lambda_1 = 0, \qquad \lambda_2 = 0 \qquad \text{on } \Omega_0^u$$

Then, the second term vanishes. Tend β to $+\infty$, then the first term tends to $D_G(\nabla v, p^*)$, and the last term tends to $\mu_{\phi\psi}(v)$.

Minimization of the majorant with respect to $y^* \in H(\Omega, \mathrm{div}\,)$, λ_1, λ_2 in L^2_+, and $\beta > 0$ provides true value of $\boldsymbol{\mu}(v)$.

Practical reconstruction of λ_1 and λ_2:

$$\begin{aligned}
\lambda_1 &= (\operatorname{div} y^* + f)_\ominus, \quad \lambda_2 = 0 \quad \text{on } \Omega^v_\ominus, \\
\lambda_2 &= (\operatorname{div} y^* + f)_\oplus, \quad \lambda_1 = 0 \quad \text{on } \Omega^v_\oplus, \\
\lambda_1 &= 0, \qquad \lambda_2 = 0 \qquad \text{on } \Omega^v_0.
\end{aligned}$$

In other words, we define λ_1 and λ_2 using known sets

$$\Omega^v_\oplus := \{x \in \Omega \mid v(x) = \psi(x)\}, \; \Omega^v_\ominus := \{x \in \Omega \mid v(x) = \phi(x)\}.$$

See more about in S.R. and J. Valdman ZAMM, 2017.

$\mu_{\phi\psi}(v)$ is not enough informative to detect free boundaries

Example:
ϕ and ψ are harmonic functions, and $f = const < 0$.
In this case $\Omega^u_\oplus = \emptyset$ and

$$\mu_{\phi\psi}(v) = f \int\limits_{\Omega^u_\ominus} (v - \phi)\, dx = (v \geq \phi) = f \|v - \phi\|_{L^1(\Omega^u_\ominus)}.$$

L^1-norm of the distance to ϕ says nothing about *configuration of the free boundary.*
Concerning reliable approximation of free boundaries we arrive at a pessimistic conclusion:

$\mu_{\phi\psi}(v)$ is too weak to control configuration of the free boundary!
In general, energy based numerical methods are principally invalid for such type (profound) quantitative analysis!

Classical Helmgholtz Decomposition Theorem

A vector field y in $L^2(\Omega)$ is uniquely decomposed in the form

$$y = y_0 + y_\nabla, \text{ where } y_\nabla := \nabla w, \; w \in H^1(\Omega), \quad \operatorname{div} y_0 = 0$$

This result was firstly established for the vector fields in $L^2(\mathbb{R}^3)$, but also holds for a bounded Lipshitz domain Ω if we set suitable boundary conditions ($\Gamma = \Gamma_D \cup \Gamma_N$).
Define the sets:

$$\begin{aligned} Q_0 : \qquad & \operatorname{div} y_0 = 0, \quad y_0 \cdot n = 0 \text{ on } \Gamma_N, \\ V_0 : \qquad & w \in H^1(\Omega), \quad w = 0 \text{ on } \Gamma_D. \end{aligned}$$

We have **orthogonality** in the standard sense:

$$\int_\Omega y_0 \cdot y_\nabla \, dx = 0. \tag{1}$$

Assumptions:

- [A] $\Lambda : V \to Y$ and $\Lambda^* : Y^* \to V^*$ are bounded linear operators
- [B] $G : Y \to \mathbb{R}_+$ is convex, continuous, and Gateaux differentiable, $G(0_Y) = 0$.
- [C] $\|\Lambda v\|_Y$ generates an equivalent norm in V and $\|\Lambda v\|_Y \geq c\|v\|_V$.
- [D] growth conditions: $G(y) \geq C\|y\|^{1+\alpha}$, $\alpha > 0$.

Define two sets in Y^*:

$$Y^*_\Lambda(\Omega) := \left\{ y^* \in Y^*(\Omega) \mid \exists\, v \in V : \, D_G\left(\Lambda v, y^*\right) = 0 \right\}.$$

$y^* \in Y^*_\Lambda(\Omega)$ if it is representable via Λv and the *nonlinear* relation.

Another set is

$$Y^*_0(\Omega) \coloneqq \left\{ y^* \in Y^*(\Omega) \,|\, (y^*, \Lambda w) = 0 \; \forall\, w \in V(\Omega) \right\}.$$

Recall that $(y^*, \Lambda w) = \langle \Lambda^* y^*, w \rangle$, so that
$y^* \in Y^*_0(\Omega) \Leftrightarrow y^* \in \mathcal{N}(\Lambda^*) = \mathcal{R}^\perp(\Lambda)$.

Theorem (3*)

Let [A]–[D] holds. Then

- *The sets* $Y_0^*(\Omega)$ *and* $Y_\Lambda^*(\Omega)$ *are closed subsets of* $Y^*(\Omega)$
- $Y_0^*(\Omega) \cap Y_\Lambda^*(\Omega)$ *contains only zero element.*
- *For any function* $y^* \in Y^*(\Omega)$ *there exists a unique decomposition*

$$y^* = y_\Lambda^* + y_0^*$$

where $y_0^* \in Y_0^*(\Omega)$ *and* $y_\Lambda^* \in Y_\Lambda^*(\Omega)$.

Remark: Orthogonality condition has a different form: any element Λv and y_0^* are orthogonal in the sense of $Y^* \leftrightarrow Y$ pairing:

$$(y_0^*, \Lambda v) = 0.$$

* S.R., St. Petersburg. Math. J., 1999

Classical Helmgholtz decomposition is a very special case:

$$V = \overset{\circ}{H}{}^1(\Omega), \quad \Lambda = \nabla,$$
$$Y = Y^* = L^2(\Omega, \mathbb{R}^d) \text{ and these spaces are identified}$$

$D_G(\nabla v, y^*) = 0$ is equivalent to L^2 equality $y^* = \nabla v$.

Then the decomposition reads

$$y = y_0 + \nabla v$$

and orthogonality is simply in $(y, \nabla v)_{L^2} = 0$

.

- Intersection of Y^*_Λ and Y^*_0.
  If $y^* \in Y^*_\Lambda$, then there exists $w \in V$ such that
  $$D_G(\Lambda w, y^*) = G(\Lambda w) + G^*(y^*) - (y^*, \Lambda w) = 0.$$
  Since $y^* \in Y^*_0$, $(y^*, \Lambda w) = 0$.
  Hence $y^* = 0$.
- Decomposition of $y^* \in Y^*$ Consider the problem
  $$\inf_{v \in V} \{G(\Lambda v) - (y^*, \Lambda v)\}$$
  Minimizer v_{y^*} exists and is unique due to reflexivity+coercivity+strong convexity.
  It satisfies
  $$(G'(\Lambda v_{y^*}) - y^*, \Lambda v) = 0 \qquad \forall v \in V.$$
  Hence $y^*_0 := y^* - G'(\Lambda v_{y^*}) \in Y^*_0$

It is easy to see that the element $y^*_\Lambda := G'(\Lambda v_{y^*})$ belongs to Y^*_Λ. This immediately follows from the property of compounds:

$$G(\Lambda v_{y^*}) + G^*(y^*_\Lambda) - (y^*_\Lambda, \Lambda v_{y^*}) = 0.$$

Uniqueness of decomposition

The element $y^*_\Lambda = G'(v_{y^*})$ is unique. Hence nonuniqueness may arise only if there exist two different $y^*_{0,1}$ and $y^*_{0,2}$ such that

$$y^* = y^*_\Lambda + y^*_{0,1}, \quad y^* = y^*_\Lambda + y^*_{0,2}.$$

Then for any positive λ_1, λ_2 such that $\lambda_1 + \lambda_2 = 1$

$$\lambda_1 D_G(\Lambda v_{y^*}, y^* - y^*_{0,1}) + \lambda_2 D_G(\Lambda v_{y^*}, y^* - y^*_{0,2}) = 0$$

$$G(\Lambda v_{y^*}) + \lambda_1 G^*(y^* - y^*_{0,1}) + \lambda_2 G^*(y^* - y^*_{0,}) - (\Lambda v_{y^*}, y^* - \lambda_1 y^*_{0,1} - \lambda_2 y^*_{0,2}) = 0$$

Since

$$\lambda_1 G^*(y^* - y^*_{0,1}) + \lambda_2 G^*(y^* - y^*_{0,1}) \geq G^*(y^* - \lambda_1 y^*_{0,1} - \lambda^*_2 y_{0,2})$$

we conclude that

$$G(\Lambda v_{y^*}) + G^*(y^* - \lambda_1 y^*_{0,1} - \lambda^*_2 y_{0,2}) - (\Lambda v_{y^*}, y^* - \lambda_1 y^*_{0,1} - \lambda_2 y^*_{0,2}) \leq 0.$$

Above relation may hold as the equality only!
Due to properties of compounds, this means that

$$y^* - \lambda_1 y^*_{0,1} - \lambda^*_2 y_{0,2} = G'(\Lambda v_{y^*}) = y^*_\Lambda.$$

Such a relation cannot be true because the element y^*_Λ is uniquely defined.

Thank you for attention

Distance to the set of "equilibrated" fields Y^*_ℓ

Lemma

Assume that there exists a nonnegative continuous functional $H : V \to \mathbb{R}_+$ such that $G(\Lambda w) \geq H(w)$ for all $w \in V$.
Let $H^* : V^* \to \mathbb{R}_+$ is the Young– Fenchel conjugate to H.
Then for any $y^* \in Q^*$, the following estimate holds

$$\inf_{q^* \in Y^*_\ell} G^*(y^* - q^*)\, dx \leq H^*(\mathcal{R}(y^*))$$

where $\mathcal{R} : V \to \mathbb{R}$ is a linear functional
$\langle \mathcal{R}(y^*), w \rangle := \langle y^*, \Lambda w \rangle + < \ell, w >$, *that defines the set*

$$Y^*_\ell = \{ y^* \in Y^* \mid \langle \mathcal{R}(y^*), w \rangle = 0 \; \forall w \in V \}$$

Example: "Distamce Lemma in terms of L^α spaces", $V = \overset{\circ}{W}{}^{1,\alpha}(\Omega)$

"Energy functional": $G(\Lambda w) = \frac{1}{\alpha}\int_\Omega |\nabla w|^\alpha \, dx$,
"Dual Energy functional": $G^*(y^*) = \frac{1}{\alpha'}\int_\Omega |y^*|^{\alpha'} dx$.
Friedrichs type inequality: $\|w\|_\alpha \leq C_F \|\nabla w\|_\alpha$ yields the estimate

$$G(\nabla w) = \frac{1}{\alpha}\|\nabla w\|_\alpha^\alpha \, dx \geq \frac{1}{\alpha C_F^\alpha}\|w\|_\alpha^\alpha = H(w)$$

Majorant of the distance to Y_ℓ^* is given by H^*. Compute it!
For $w^* \in L^{\alpha'}(\Omega)$ it is simple:

$$H^*(w^*) = \sup_{w \in V}\left\{\int_\Omega w^* w \, dx - \frac{1}{\alpha C_F^\alpha}\|w\|^\alpha\right\} = \frac{C_F^{\alpha'}}{\alpha'}\|w^*\|_{\alpha'}^{\alpha'}.$$

Thus, if $\operatorname{div} y^* + \ell \in L^{\alpha'}$ then

$$\inf_{q^* \in Y_\ell^*} G^*(q^* - y^*) \leq \frac{C_F^{\alpha'}}{\alpha'}\|\operatorname{div} y^* + \ell\|_{\alpha'}^{\alpha'}.$$

General form of the error majorant of $\boldsymbol{\mu}(v) = D_G(\Lambda v, p^*)$

Introduce

$$\rho(\lambda, y^*) = \lambda G^* \left(\frac{y^*}{\lambda} \right) - G^*(y^*) + \langle y^*, \Lambda v \rangle + < \ell, v > .$$

If $y^* \to Y_\ell^*$ and $\lambda \to 1$, then $\rho(\lambda, y^*) \to 0$!

Theorem (S.R., RJNAMM, 2012)

For any $y^* \in Y^*$ *and* $\lambda \in (0, 1)$,

$$\boldsymbol{\mu}(v) \leq \boldsymbol{\mu}^+(v, y^*) := D_G(\Lambda v, y^*) + H^* \left(\frac{\mathcal{R}}{1 - \lambda} \right) + \rho(\lambda, y^*),$$

$$\boxed{\boldsymbol{\mu}(v) = \inf_{\lambda > 0, y^* \in Y^*} \boldsymbol{\mu}^+(v, y^*)}$$

If $y^* \in Q^*$, then $\mathcal{R}(y^*) = \Lambda^* y^* + \ell$.
We obtain

$$\boldsymbol{\mu}(v) \leq D_G(\Lambda v, y^*) + H^* \left(\frac{\Lambda^* y^* + \ell}{1 - \lambda} \right) + \lambda G^* \left(\frac{y^*}{\lambda} \right) - G^*(y^*) + < \Lambda^* y^* + \ell, v >$$

Here blue terms present a combined measure of the distance to Y^*_ℓ. If $\Lambda^* y^* + \ell$ is small, then these terms are small.
Since $\Lambda p^* + \ell = 0$, this measure also has no gap!

Example: μ and its majorant for α–Laplacian

$$
\begin{aligned}
\mu(v) \leq\ & \int_\Omega \Big(\frac{1}{\alpha}|\nabla v|^\alpha + \frac{1}{\alpha'}|y^*|^{\alpha'} - \nabla v \cdot y^*\Big) dx + \\
+ \ & \frac{C_F^{\alpha'}}{\alpha'(1-\lambda)^{\alpha'}} \|\mathrm{div}\, y^* - \ell\|_{\alpha'}^{\alpha'} + \\
+ \ & \left(\frac{1}{\lambda^{\alpha'}} - 1\right) \frac{1}{\alpha'} \|y^*\|^{\alpha'} + \int_\Omega (y^* \cdot \nabla v - \ell v) dx.
\end{aligned}
$$

Conclusion:
(a) The majorant is fully computable.
(b) if $\|\mathrm{div}\, y^* - \ell\|_{\alpha'}$ is small then λ can be set small, three last terms are small and the main part of the error majorant is $D(\nabla v, y^*)$,
(c) in this case, $D(\nabla v, y^*)$ is a good error indicator for mesh refinement.

Thank you for attention

First we prove the completeness of $Y^*_\Lambda(\Omega)$. Let $\{{y^*}_m\}$ be a sequence in $Y^*_\Lambda(\Omega)$ that converges to y^*. In this case, there exists a sequence $\{v_m\} \in V_0 + u_0$ such that

$$\int_\Omega \left(g^{**}(\nabla v_m) + g^*({y^*}_m) - \nabla v_m \cdot {y^*}_m\right) \, dx = 0\,. \qquad (2)$$

By using **??**, **??** and 2 we find that the sequence $\{v_m\}$ is bounded in $V(\Omega)$. Therefore, there exists a weakly converging subsequence which for the sake of simplicity is also denoted by v_m. Let $v \in V_0 + u_0$ be a weak limit of this sequence. Since the functional $\int_\Omega g^{**}(\nabla v_m) dx$ is weakly lower semicontinuous we get

$$\int_\Omega D\left(\nabla v, y^*\right) dx = \int_\Omega \left(g^{**}(\nabla v) + g^*(y^*) - \nabla v \cdot y^*\right) \, dx \leq 0\,. \quad (3)$$

By recalling that D is nonnegative we arrive at the conclusion that holds as equality what, in fact, means that $y^* \in Y^*_\Lambda(\Omega)$.
The completeness of $Y^*_f(\Omega)$ follows straightforwardly from its definition.

Let us show that $Y_\Lambda^*(\Omega) \cap Y_f^*(\Omega) = \{p^*\}$. For this purpose we use the identity

$$\int_\Omega D(\nabla w, y^*) = I^{**}(w) - \mathbf{l}^*(\mathbf{y}^*) + \int_{\boldsymbol{\Omega}} (\mathbf{p}^* - \mathbf{y}^*) \cdot \nabla(\mathbf{w} - \mathbf{u_0})\, \mathbf{dx} \quad \forall \mathbf{w} \in$$

Assume that y^* belongs the sets $Y_\Lambda^*(\Omega)$ and $Y_f^*(\Omega)$ simultaneously. The integral in the right hand side of 4 equals zero because $y^* \in Y_f^*(\Omega)$. Whence,

$$\inf_{w \in V_0 + u_0} \int_\Omega D(w, y^*)\, dx = \inf_{w \in V_0 + u_0} I^{**}(w) - \mathbf{l}^*(\mathbf{y}^*) = \inf \mathcal{P}^{**} - \mathbf{l}^*(\mathbf{y}^*)\,.$$

The left hand side of 5 equals zero because $y^* \in Y_\Lambda^*(\Omega)$. Thus,

$$\mathbf{l}^*(\mathbf{y}^*) \;=\; \inf \mathcal{P}^{**} \;=\; \sup \mathcal{P}^*\,,$$

so that y^* is a solution of the dual problem.
The reminder of the present is devoted to the proof of **??**. Prior to giving it, however, we note that the existence of $y^* f \in Y_f^*(\Omega)$ and

$y^*l \in Y^*_\Lambda(\Omega)$ such that $y^* = y^*f + y^*l$ readily follows from the existence of a minimizer $\bar{v}$ of the problem

$$\inf_{v \in V_0 + u_0} \int_\Omega \left(g^{**}(\nabla v) - y^* \cdot \nabla v + fv \right) dx \,.$$

Indeed, $\bar{v}$ meets the Euler's equation

$$\int_\Omega (y^* - \Lambda \nabla \bar{v}) \cdot \nabla w \, dx = \int_\Omega fw \, dx \quad \forall w \in V_0(\Omega)$$

what means that $y^* - \Lambda \nabla \bar{v} \in Y^*_f(\Omega)$. Since $\Lambda \nabla \bar{v} \in Y^*_\Lambda(\Omega)$ the existence of y^*f and y^*l follows. The uniqueness of such decomposition we prove by reductio ab absurdum. Assume that there are two different functions ${y^*}_{1f}$ and ${y^*}_{2f}$ in $Y^*_f(\Omega)$ such that

$$y^* - {y^*}_{1f} \in Y^*_\Lambda(\Omega),$$
$$y^* - {y^*}_{2f} \in Y^*_\Lambda(\Omega).$$

Then $V_0 + u_0$ contains two functions v_1 and v_2 which satisfy the equalities

$$\int_\Omega \left(g^{**}(\nabla v_1) + g^*(y^* - {y^*}_{1f})\right)\, dx = \int_\Omega \nabla v_1 \cdot (y^* - {y^*}_{1f})\, dx, \quad (6)$$

$$\int_\Omega \left(g^{**}(\nabla v_2) + g^*(y^* - {y^*}_{2f})\right)\, dx = \int_\Omega \nabla v_2 \cdot (y^* - {y^*}_{2f})\, dx. \quad (7)$$

We note that

$$\begin{aligned} \textstyle\int_\Omega \nabla v_i \cdot (y^* - {y^*}_{if})\, dx &= \int_\Omega \left(\nabla v_i \cdot y^* - \nabla u_0 \cdot {y^*}_{if} - {y^*}_{if} \cdot \nabla(v_i - u \\ &= \int_\Omega \left(\nabla v_i \cdot y^* - \nabla u_0 \cdot {y^*}_{if} - f(v_i - u_0)\right)\, dx \end{aligned}$$

Let us multiply 6 on λ_1 and 7 on λ_2, where

$$\lambda_1 + \lambda_2 = 1, \quad \lambda_i > 0 \quad i = 1, 2$$

and add these equalities. When taking into account 8 we obtain

$$\int_\Omega (\lambda_1 g^{**}(\nabla v_1) + \lambda_2 g^{**}(\nabla v_2) + \lambda_1 g^*(y^* - {y^*}_{1f}) + \lambda_2 g^*(y^* - {y^*}_{2f}$$

$$+ \int_\Omega (\nabla u_0 \cdot (\lambda_1 {y^*}_{1f} + \lambda_2 {y^*}_{2f}))\, dx + \int_\Omega f(\lambda_1 v_1 + \lambda_2 v_2 - u_0)\, dx =$$

$$= \int_\Omega (\lambda_1 \nabla v_1 + \lambda_2 \nabla v_2) \cdot y^*\, dx. \quad (9)$$

Since

$$\begin{aligned} \lambda_1 v_1 + \lambda_2 v_2 \quad &\in V_0 + u_0, \\ \lambda_1 {y^*}_{1f} + \lambda_2 {y^*}_{2f} \quad &\in Y^*_f(\Omega) \end{aligned}$$

we have

$$\int_\Omega f(\lambda_1 v_1 + \lambda_2 v_2 - u_0)\, dx = \int_\Omega (\lambda_1 {y^*}_{1f} + \lambda_2 {y^*}_{2f}) \cdot (\lambda_1 \nabla v_1 + \lambda_2 \nabla v_2 -$$

The function g^{**} is convex, the function g^* is strongly convex and

${y^*}_{1f} \neq {y^*}_{2f}$ by the assumption. Therefore,

$$\begin{aligned} \lambda_1 g^*(y^* - {y^*}_{1f}) + \lambda_2 g^*(y^* - {y^*}_{2f}) &> g^*(y^* - \lambda_1 {y^*}_{1f} - \lambda_2 {y^*}_{2f}) = \\ \lambda_1 g^{**}(\nabla v_1) + \lambda_2 g^{**}(\nabla v_2) &\geq g^{**}(\lambda_1 \nabla v_1 + \lambda_2 \nabla v_2) = g^{**}(\end{aligned}$$

Now 9, 10, 11 and 12 yields the strict inequality

$$\int_\Omega \left(g^{**}(\nabla \widehat{v}) + g^*(y^* - \widehat{y^*}) - (y^* - \widehat{y^*}) \cdot \nabla \widehat{v} \right) dx < 0 \,, \qquad (13)$$

where $\widehat{v} := \lambda_1 v_1 + \lambda_2 v_2$ and $\lambda_1 {y^*}_{1f} + \lambda_2 {y^*}_{2f}$. However, the integrand of 13 is nonnegative. Thus, we arrive at a contradiction which completes the proof.

Remark:

The above proof is not based on any specific properties of functions g^* and g^{**} other than convexity of g^* and strong convexity of g^{**}. For this reason, Theorem **??** has a general meaning and is applicable not only to the considered class of variational problems. In particular, if g and g^* are positively defined quadratic functions then $Y_f^*(\Omega)$ and $Y_\Lambda^*(\Omega)$ are linear manifolds. For example, if $g(\nabla v) = \frac{1}{2} |\nabla v|^2$, $u_0 = 0$ and $f = 0$, then $Y_f^*(\Omega)$ is the set of solenoidal functions and $Y_\Lambda^*(\Omega)$ is the set of gradients of scalar valued functions vanishing on the boundary $\partial\Omega$. It is well known that these two sets are orthogonal subspaces of the space $L^2\Omega, \mathbb{R}^n$ (see e.g. [**?**]).