Weierstrass Institute for
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Leibniz Association

Maximal parabolic regularity for the treatment of real world problems

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Strobl, July 2019

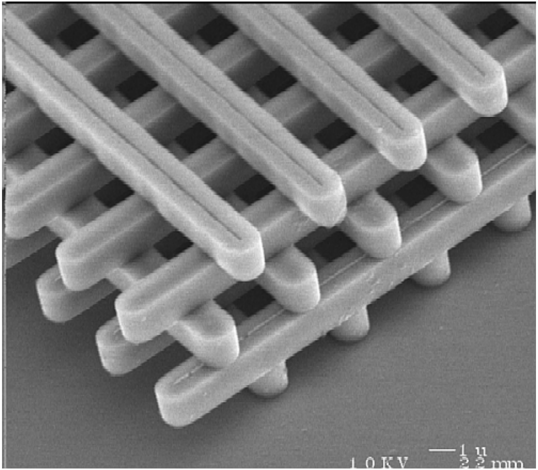

Figure: 3D photonic crystal. Courtesy Sandia National Laboratories.

Examples

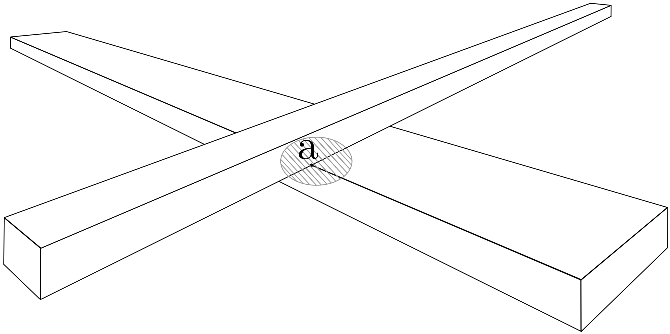

Figure: crucial part of the geometry: the point a is an obstruction against 'strong Lipschitz domain'

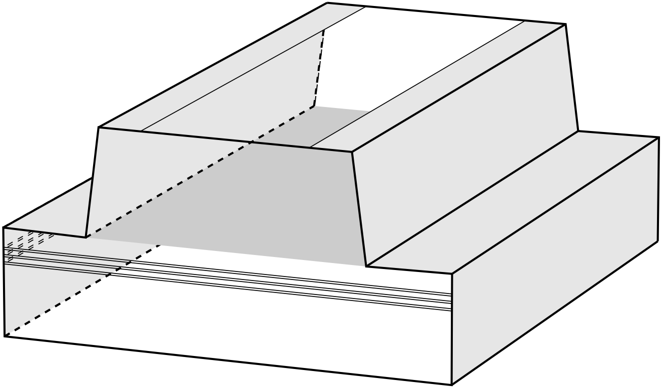

Figure: Schema of a ridge waveguide quantum well laser (detail 3.2$\mu m \times 1.5\mu m \times 4\mu m$)

Figure: the brighter parts form the Dirichlet boundary D

The first question of interest is the following:

Which Banach spaces are adequate to consider the corresponding elliptic/parabolic equations in?

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a (bounded) domain and $D \subset \partial\Omega$ be (relatively) closed (and a bit well-behaved). By $W^{1,q}_D(\Omega) =: W^{1,q}_D$ we denote the subspace of $W^{1,q}(\Omega)$ the elements of which have trace 0 on D and by $W^{-1,q}_D$ the space of antilinear forms on $W^{1,q'}_D$.

Denoting by $\langle \cdot, \cdot \rangle$ the usual $W^{1,2}_D \leftrightarrow W^{-1,2}_D$ duality, we define, for a coefficient function $\mu \in L^\infty(\Omega; \mathbb{R}^{d^2})$ the operator $-\nabla \cdot \mu\nabla : W^{1,2}_D \to W^{-1,2}_D$ as follows:

$$\langle -\nabla \cdot \mu\nabla\varphi, \psi \rangle := \int_\Omega \mu\nabla\varphi \cdot \nabla\overline{\psi}, \quad \varphi, \psi \in W^{1,2}_D. \tag{1}$$

and maintain the notation $-\nabla \cdot \mu\nabla$ for the restriction to any space $W^{-1,q}_D$, $q > 2$.

Overall, we suppose that the coefficient function μ takes *real* matrices as values and is *elliptic*, i.e. satisfies $\mu\xi \cdot \xi \geq \mu_\bullet |\xi|^2$ for all $\xi \in \mathbb{R}^d$.

BOUNDARY CONDITIONS:

Let $f \in W_D^{-1,2}$ be an element of the kind $f = f_\Omega + f_\partial$, where $f_\Omega \in L^2(\Omega)$ and $f_\partial \in L^2(\partial\Omega \setminus D)$. (Note that such f induce continuous antilinear forms on $W_D^{1,2}$, and, are, hence, admitted as right hand sides.)

Then, supposing so much smoothness on Ω that Gauss' theorem holds, the equation $-\nabla \cdot \mu\nabla\varphi = f$ implies

$$-\operatorname{div} \mu\nabla\varphi = f_\Omega$$

IN THE DISTRIBUTIONAL SENSE on Ω and

$$\nu \cdot \mu\nabla\varphi = f_\partial$$

on $\partial\Omega \setminus D$.

On the other hand, when restricting the operator $-\nabla \cdot \mu\nabla$ to an L^p space, say L^2, then 'boundary functionals'of type f_∂ do not belong to $L^p(\Omega)$. This says that f_∂ are enforced to be 0 then, what leads to the NATURAL BOUNDARY CONDITION $\nu \cdot \mu\nabla\varphi = 0.$ – if considering the equation in L^p.

This gives us the following general PHILOSOPHY : taking this or that Banach space for the consideration of the elliptic/parabolic equation is NOT a matter of taste (as many people think) but essentially depends of the kind of boundary conditions under consideration:

if one has to impose NATURAL boundary conditions (in other words: HOMOGENEOUS Neumann conditions $\nu \cdot \mu \nabla \varphi = 0$ on $\partial\Omega \setminus D$), then L^p spaces are adequate,

and if one is confronted with INHOMOGENEOUS Neumann conditions, then spaces of type $W_D^{-1,q}$ are adeaquate.

Besides: this is NOT the end of the story: one can have good reasons to consider the equation in interpolation spaces $[L^p, W_D^{-1,q}]_\theta$.

Let us recall that, for $p \in [2, \infty[$, the duality mapping $J_p : L^p \to L^{p'}$ is given by

$$\psi \mapsto \frac{1}{\|\psi\|_{L^{p'}}^{p'-2}} \psi |\psi|^{p-2}.$$

Theorem Denoting the restriction of $-\nabla \cdot \mu \nabla$ to L^p by A_p the NUMERICAL RANGE of A_p, namely the set

$$\{ \int_\Omega A_p \psi \overline{J_p \psi} : \psi \in dom(A_p) \}$$

lies in the sector

$$\mathcal{S}_p := \{\lambda \in \mathbb{C} : |\Im z| \le \kappa_p \frac{\|\mu\|_{L^\infty}}{\mu_\bullet} \Re z\}, \quad \kappa_p = \frac{1}{2} \frac{p}{\sqrt{p-1}} \tag{2}$$

if p is larger than half the space dimension d.

Warning This is NOT obvious at all and rests on highly non-trivial calculations of Cialdea/Maz'ya.

.Theorem Let A be a densely defined operator in the Banach space X. Let N be the numerical range of A and define $\Sigma := \mathbb{C} \setminus N$. If $\Sigma_0 \subset \Sigma$ is a connected component of Σ containing at least one resolvent point, then the spectrum of A is contained in the complement of Σ_0, and one has the resolvent estimate

$$\|(A-\lambda)^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{dist(N,\lambda)}. \tag{3}$$

Theorem Assume $p \in \{2\} \cup] \max(2, \frac{d}{2}), \infty[$. One has

$$\|(A_p+\lambda)^{-1}\|_{\mathcal{L}(L^p)} \leq \frac{1}{\lambda}, \quad \lambda > 0. \tag{4}$$

A_p does not have spectrum outside $\mathcal{S}_p$ and, satisfies the following estimates

$$\|(A_p+\lambda)^{-1}\|_{\mathcal{L}(L^p)} \leq \frac{\sqrt{1+\kappa_p^2}}{|\lambda|}; \quad \Re\lambda \geq 0 \tag{5}$$

and

$$\|(A_p+1+\lambda)^{-1}\|_{\mathcal{L}(L^p)} \leq \sqrt{2}\frac{\sqrt{1+\kappa_p^2}}{1+|\lambda|}; \quad \Re\lambda \geq 0. \tag{6}$$

Theorem For each $p \in [2, \infty[$, the operator A_p generates a semigroup of contractions on L^p. All these semigroups are even analytic.

Proof The contractivity assertion follows for $p \in \{2\} \cup] \max(2, \frac{d}{2}), \infty[$ by (4) and the Hille/Yosida thm. For the remaining p's the claim follows by interpolation.

The claim of analyticity follows for $p \in \{2\} \cup] \max(2, \frac{d}{2}), \infty[$ by the estimates (5) and (6) and for the other p's by interpolation.

Corollary The same assertions hold true for $p \in]1, 2[$.

Proof Duality.

Theorem Let B be an operator on L^2 which generates an analytic semigroup there. If B induces on all L^p, $(p \in]1, \infty[)$ *contractive* semigroups, with corresponding (negative) generators $-B_p$, then each B_p satisfies max. par. reg. on L^p.

Proof This is a pioneering result of

Lamberton: J. Funct. Anal. **72** (1987) 252 – 262

Theorem If the coefficient function μ is *real* and *elliptic* then each corresponding operator A_p does satisfy maximal parabolic regularity on L^p.

Proof This follows from the foregoing results.

Let B be a *positive* operator on a Banach space X, i.e. satisfying

$$\|(B+\lambda)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{1+\lambda}, \; \lambda \geq 0. \tag{7}$$

Then we define, for $\alpha \in]0,1[$ the operator $B^{-\alpha} \in \mathcal{L}(X)$ by

$$B^{-\alpha} := \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(A+t)^{-1} dt$$

The operators $B^{-\alpha}$ have the properties which one naively would expect.

Theorem Let A be a positive operator in a Banach space X. Let Y be another Banach space into which X is continuously and densely embedded, and assume that, for some $\alpha \in]0,1[$, $A^{-\alpha} \colon X \to X$ continuously extends to a topological isomorphism $B^{(\alpha)}$ from Y onto X. Define an operator $\tilde{A}$ on Y via

$$\tilde{A}^{-1}\psi = A^{\alpha-1}B^{(\alpha)}\psi, \quad \psi \in Y. \tag{8}$$

i)$\tilde{A}$ is an extension of A. $\tilde{A}$ is a positive operator *in* Y, i.e. $dom(\tilde{A}) \subset Y$.
iii) $\tilde{A}^{-\alpha} = B^{(\alpha)}$ with $\mathrm{dom}(\tilde{A}) = \mathrm{dom}(A^{1-\alpha})$.
iv) If $-A$ generates a semigroup on X, then $-\tilde{A}$ generates a semigroup on Y. If the first is analytic, then the second also is.

Theorem Adopt the suppositions of the foregoing thm. inclusively the definition of the operator $\tilde{A}$. If A satisfies max. par. reg. on X, then $\tilde{A}$ satisfies max. par. reg. on Y.

Proof Let us consider the operator which assigns to $f \in L^r(J;X)$ the function $\int_0^\cdot e^{-(\cdot-s)A} f(s)ds$ - formally denoted by $\left(\frac{\partial}{\partial t} + A\right)^{-1}$. Since A satisfies max. par. reg. on X, the operator $A\left(\frac{\partial}{\partial t} + A\right)^{-1}$ is a continuous one from $L^r(J;X)$ into itself. Exploiting the special form of $\left(\frac{\partial}{\partial t} + A\right)^{-1}$ it is clear that the following commutation property holds:

$$A^\alpha A\left(\frac{\partial}{\partial t} + A\right)^{-1} A^{-\alpha} = A\left(\frac{\partial}{\partial t} + A\right)^{-1}.$$

Let us now consider the normed space $\underline{X}$, being X as a set, but equipped with the *weaker* Y topology, and, correspondingly, $L^r(J;\underline{X})$. Then one has for any $f \in L^r(J;\underline{X})$

$$\tilde{A}\left(\frac{\partial}{\partial t} + \tilde{A}\right)^{-1} f = A\left(\frac{\partial}{\partial t} + A\right)^{-1} f =$$

$$A^\alpha A\left(\frac{\partial}{\partial t} + A\right)^{-1} A^{-\alpha} f = \tilde{A}^\alpha A\left(\frac{\partial}{\partial t} + A\right)^{-1} \tilde{A}^{-\alpha} f,$$

and, consequently,

$$\|\tilde{A}\big(\frac{\partial}{\partial t} + \tilde{A}\big)^{-1} f\|_{L^r(J;\underline{X})} \leq \|\tilde{A}^{\alpha}\|_{\mathcal{L}(L^r(J;X);L^r(J;\underline{X}))}$$

$$\|A\big(\frac{\partial}{\partial t} + A\big)^{-1}\|_{\mathcal{L}(L^r(J;X);L^r(J;X))} \|\tilde{A}^{-\alpha}\|_{\mathcal{L}(L^r(J;\underline{X});L^r(J;X))} \|f\|_{L^r(J;\underline{X})}$$

But, since X was assumed to be dense in Y, also $L^r(J;X)$ is dense in $L^r(J;Y)$. So the mapping

$$\tilde{A}\big(\frac{\partial}{\partial t} + \tilde{A}\big)^{-1} : L^r(J;\underline{X}) \to L^r(J;\underline{X})$$

extends by continuity in a unique manner to a *continuous* one from $L^r(J;Y)$ to $L^r(J;Y)$. This shows that $\big(\frac{\partial}{\partial t} + \tilde{A}\big)^{-1}$ satisfies max. par. reg. on Y.

What are the candidates for the spaces X, Y?

Naturally, $X = L^p$, $Y = W_D^{-1,p}$ – but: is it true that an operator power, say $(A_p + 1)^{-1/2}$, does continuously extend to a (bounded) operator from $W_D^{-1,p}$ *onto* L^p – thus being a topological isomorphism (by the open mapping thm.).

The answer is in the affirmative!!! (but highly non-trivial)

Assumption For every point $x \in \overline{\partial\Omega \setminus D}$ there is a bi-Lipschitzian boundary chart, i.e. there is an open neighbourhood $U = U_x$ of x and a bi-Lipschitzian mapping $\phi = \phi_x$ from U onto the (open) unit cube, such that $\phi(x) = 0$ and $\phi(U \cap \Omega$ equals a half cube.

.**Theorem** Under the above assumption and a very mild measure theoretic condition on D, there is a $q_0 > 2$, such that, for all $q \in]1, q_0[$ the operator

$$(A_q + 1)^{-1/2} : L^q \to W_D^{1,q} \tag{9}$$

is a topological isomorphism.

Corollary Under the same assumptions, the operator $(A_q + 1)^{-1/2} : L^q \to L^q$ extends by continuity to an isomorphism

$$(A_q \widetilde{+ 1)}^{-1/2} : W_D^{-1,q} \to L^q \tag{10}$$

for $q \in]q_0', \infty[$.

Theorem For $q \in]q_0', \infty[$ the operators $-\nabla \cdot \mu \nabla$ do satisfy max. par. reg. on $W_D^{-1,q}$.

Proof. By one of the foregoing thm.s the square root isomorphism 'transports' max.par.reg.from L^q to $W_D^{-1,q}$.

In the concept of max. par. reg. the relevance of the quality of the domain of the corresponding operator is not so clearly visible. In particular, this is the case when exploiting the equivalent condition

$$A\Big(\frac{\partial}{\partial t} + A\Big)^{-1} \in \mathcal{L}(L^r(J;X)).$$

Nevertheless, it is of HIGH RELEVANCE – at least when considering non-linear problems. In this spirit, we present now some elliptic regularity results:

Theorem Under extrem general geometric conditions on the domain Ω and the boundary part D, there is an $\epsilon > 0$, such that

$$-\nabla \cdot \mu \nabla + 1 : W^{1,q}_D \to W^{-1,q}_D \tag{11}$$

is a topological isomorphism for all $q \in]2-\epsilon, 2+\epsilon[$ – and this uniformly for all coefficient functions with a common L^∞ bound and a common coercivity constant.

Warning! IN GENERAL one can expect ϵ only minimally small.

The obstructions against larger ϵ's are

– non-smooth geometries

– discontinuous coefficient functions

– mixed boundary conditions (here $q \geq 4$ is false)

Theorem The domain of the operator $-\nabla \cdot \mu \nabla$, when considered on $W^{-1,q}_D$ $(q > d)$, and equipped with the graph norm, continuously embeds into L^∞ – also under extrem general geometric conditions on the domain Ω and the boundary part D.

Theorem The domain of the operator $-\nabla \cdot \mu \nabla$, when considered on $W^{-1,q}_D$ $(q > d)$, and equipped with the graph norm, continuously embeds into a Hoelder space $C^\alpha(\Omega)$ – also under very general geometric conditions on the domain Ω and the boundary part D.

Theorem Consider the dimension $d = 3$. Even then there is a broad zoo of geometries for the domains Ω and Dirichlet boundary parts D, such that (11) is a topological isomorphism even for q's larger than 3.

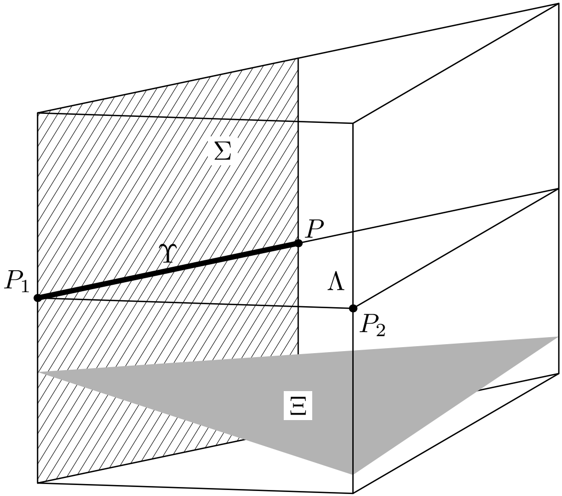

Figure: all sides of the prism carry the Dirichlet boundary condition, except the shaded half side

Finally, we have the following PERMANENCE PRINCIPLE:

Theorem Assume that

$$-\nabla \cdot \mu \nabla + 1 : W_D^{1,q} \to W_D^{-1,q} \tag{12}$$

is a topological isomorphism for some $q \in [2, 2^*]$, $2^* = \frac{2d}{d-2}$ being the Sobolev conjugated number of 2. Let w be a real, uniformly continuous function on Ω which admits a strictly positive lower bound. Then

$$-\nabla \cdot w\mu \nabla + 1 : W_D^{1,q} \to W_D^{-1,q} \tag{13}$$

also is a topological isomorphism.

Evidently, this is the door opener ? for the treatment of quasilinear parabolic equations.