A second order multipoint flux mixed finite element method on hybrid meshes

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Porous media modeling

Model equations for single-phase flow:

Conservation of mass	Darcy's law
$\operatorname{div} \mathbf{u} = f$	$\mathbf{u} = -K\nabla p$

Quantity of interest : p

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Second order form

$$-\operatorname{div}(K\nabla p) = f \qquad \text{in } \Omega$$
$$p = 0 \qquad \text{on } \partial\Omega.$$

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(ii) Not accurate for rough coefficients (local arithmetic averaging of K)

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Mixed form

$$K^{-1}\mathbf{u} + \nabla p = 0 \quad \text{in } \Omega$$
$$\operatorname{div} \mathbf{u} = f \quad \text{in } \Omega$$
$$p = 0 \quad \text{on } \partial\Omega.$$

(i) Handles rough coefficients better (local harmonic averaging of K)

(ii) Have to solve a full saddle point problem... or do you ? $\Rightarrow$ MFMFE

Variational formulation

$$
\begin{aligned}
K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\
\operatorname{div}\mathbf{u} &= f && \text{in } \Omega \\
p &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

Variational formulation

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\begin{aligned}
(K^{-1}\mathbf{u}, \mathbf{v}) - (p, \operatorname{div}\mathbf{v}) &= 0 && \forall \mathbf{v} \in H(\operatorname{div}, \Omega) \\
(\operatorname{div}\mathbf{u}, q) &= (f, q) && \forall q \in L^2(\Omega)
\end{aligned}
$$

Discrete variational formulation

$$\begin{aligned} K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= f && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Discrete variational formulation

$$\begin{aligned} (K^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div}\mathbf{v}_h) &= 0 && \forall \mathbf{v}_h \in \mathbf{V}_h \subseteq H(\operatorname{div}, \Omega) \\ (\operatorname{div}\mathbf{u}_h, q_h) &= (f, q_h) && \forall q_h \in Q_h \subseteq L^2(\Omega) \end{aligned}$$

Problem : we have to solve a full (indefinite) saddle point system ...

Mass lumping

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K^{-1}\mathbf{u} + \nabla p &= 0 && \text{in } \Omega \\
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Discrete variational formulation via *mass lumping* (MFMFE)

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(K^{-1}\mathbf{u}_h, \mathbf{v}_h)_h - (p_h, \operatorname{div} \mathbf{v}_h) &= 0 && \forall \mathbf{v}_h \in \mathbf{V}_h \subseteq H(\operatorname{div}, \Omega) \\
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For appropriate spaces $\mathbf{V}_h$, Q_h and $(\cdot,\cdot)_h$, the *lumped mass matrix* M_h is block-diagonal, and the variable $\mathbf{u}_h$ can be eliminated efficiently.

$$\begin{pmatrix} M_h & -C^\top \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u}_h \\ \mathsf{p}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathsf{f} \end{pmatrix} \quad \Longrightarrow \quad C M_h^{-1} C^\top \mathsf{p}_h = \mathsf{f}$$

The problem reduces to symmetric, positive definite cell-centered system for the pressure (CCFD)

Discretization

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M. Wheeler, I. Yotov *A multipoint flux mixed finite element method*. SIAM 2006

$V(T) = \mathsf{BDM}_1(T) := P_1(T)^2$

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$$(\mathbf{u}_h, \mathbf{v}_h)_h := \frac{|T|}{3}\sum_{i=1}^{3} \mathbf{u}_h(r_i)\mathbf{v}_h(r_i)$$

r_i vertex

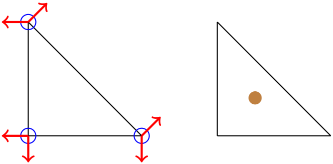

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points.

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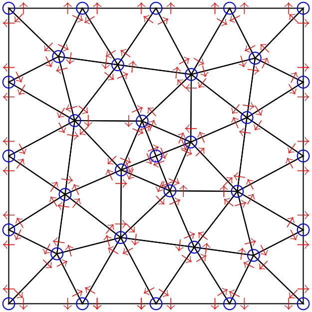
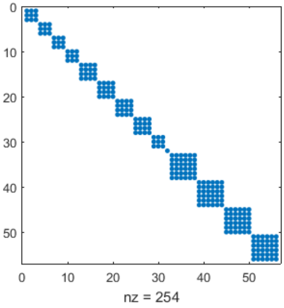
$Q(T) = P_0(T)$ $\qquad$ r_i vertex





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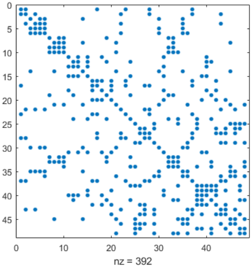

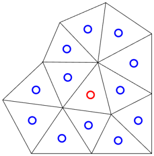

Figure: Matrix $CM_h^{-1}C^T$ (left), stencil of the method (right)

Convergence analysis

Summary of the convergence results

$$\|\mathbf{u} - \mathbf{u}_h\| = O(h) \qquad \text{and} \qquad \|\pi_h^0 p - p_h\| = O(h^2)$$

Relevant properties

(i) $P_0(T)^2 \subseteq \mathbf{V}(T)$ and $P_0(T) \subseteq Q(T)$ such that $\operatorname{div} \mathbf{V}(T) \subseteq Q(T)$

(ii) The quadrature rule is exact for $P_0(T)^2 \times \mathbf{V}(T)$

(iii) The quadrature rule induces a norm on $\mathbf{V}(T)$

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Wheeler-Yotov element : $\mathbf{V}(T) = \mathrm{BDM}_1(T) = P_1(T)^2$

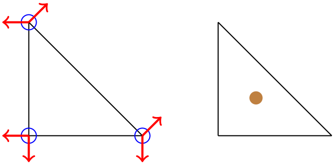

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points. The quadrature rule is exact for $P_1(T)$.

Higher order candidates

Natural extension of the first order estimates

$$\|\mathbf{u} - \mathbf{u}_h\| = O(h^2) \qquad \text{and} \qquad \|\pi_h^1 p - p_h\| = O(h^3)$$

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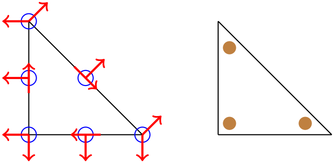
(iii) The quadrature rule induces a norm on $\mathbf{V}(T)$ ✗

First candidate : $\mathbf{V}(T) = \text{BDM}_2(T) = P_2(T)^2$



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Relevant properties

(i) $P_1(T)^2 \subseteq \mathbf{V}(T)$ and $P_1(T) \subseteq Q(T)$ such that $\operatorname{div} \mathbf{V}(T) \subseteq Q(T)$ ✓

(ii) The quadrature rule is exact for $P_1(T)^2 \times \mathbf{V}(T)$ ✓

(iii) The quadrature rule induces a norm on $\mathbf{V}(T)$ ✓

Second candidate : $\mathbf{V}(T) = \text{BDM}_2^+(T) = P_2(T)^2 \oplus b_3 \cdot [1,0]^T \oplus b_3 \cdot [0,1]^T$

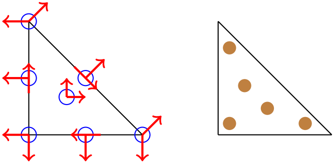

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points. The quadrature rule is exact for $P_3(T) \oplus b_3 \cdot P_1(T)$.

Higher order candidates

Natural extension of the first order estimates

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Third candidate : $\mathbf{V}(T) = \mathrm{RT}_1(T) := P_1(T)^2 + \mathbf{x} \cdot P_1^h(T)$

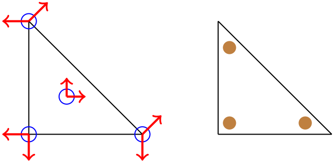

Figure: DOFs of $V(T)$ (left) and $Q(T)$ (right). Blue circles are quadrature points. The quadrature rule is exact for $P_2(T)$.

A new theory

Split the error in $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \leq \|\mathbf{u}-\Pi_h\mathbf{u}\|_{L^2(\Omega)} + \|\Pi_h\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$

$$(\Pi_h\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)_h - (\pi_h^1 p - p_h, \operatorname{div}\mathbf{v}_h) = (\Pi_h\mathbf{u}-\mathbf{u},\mathbf{v}_h) + \sigma_h(\Pi_h\mathbf{u}, v_h)$$
$$(\operatorname{div}(\Pi_h\mathbf{u}-\mathbf{u}_h), q_h) = 0$$

with $\sigma_h(\Pi_h\mathbf{u},\mathbf{v}_h) = (\Pi_h\mathbf{u},\mathbf{v}_h)_h - (\Pi_h\mathbf{u},\mathbf{v}_h)$.

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(I) $\operatorname{div}(\Pi_h\mathbf{u}-\mathbf{u}_h) = 0 \quad \Rightarrow \quad \Pi_h\mathbf{u}-\mathbf{u}_h \in P_1(T)^2$

(II) $\sigma_h(\mathbf{u}_h,\mathbf{v}_h) = 0$ if $\mathbf{u}_h, \mathbf{v}_h \in P_1(T)^2$

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Split the error in $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \leq \|\mathbf{u}-\Pi_h\mathbf{u}\|_{L^2(\Omega)} + \|\Pi_h\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}$

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Taking $\mathbf{v}_h = \Pi_h\mathbf{u}-\mathbf{u}_h$ and $q_h = \pi_h^1 p - p_h$, we obtain

$$\begin{aligned}
\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_h^2 &= (\Pi_h\mathbf{u}-\mathbf{u}, \Pi_h\mathbf{u}-\mathbf{u}_h) + \sigma_h(\Pi_h\mathbf{u}, \Pi_h\mathbf{u}-\mathbf{u}_h) \\
&= (\Pi_h\mathbf{u}-\mathbf{u}, \Pi_h\mathbf{u}-\mathbf{u}_h) + \sigma_h(\Pi_h\mathbf{u}-\pi_h^1\mathbf{u}, \Pi_h\mathbf{u}-\mathbf{u}_h) \\
&\leq \|\Pi_h\mathbf{u}-\mathbf{u}\|_0\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0 + c\|\Pi_h\mathbf{u}-\pi_h^1\mathbf{u}\|_0\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0 \\
&\leq Ch^2\|\mathbf{u}\|_{H^2(\mathcal{T}_h)}\|\Pi_h\mathbf{u}-\mathbf{u}_h\|_0
\end{aligned}$$

A new theory

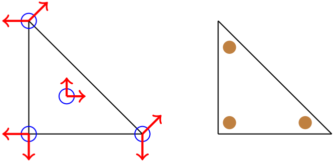
Theorem

$$\|\mathbf{u}-\mathbf{u}_h\| = O(h^2) \qquad \textit{and} \qquad \|\pi_h^0(p-p_h)\| = O(h^3)$$

Relevant properties

- (i) $P_1(T)^2 \subset \mathbf{V}(T)$ and $P_1(T) \subset Q(T)$ such that $\operatorname{div}\mathbf{V}(T) \subseteq Q(T)$ ✓
- (ii_a) $\exists\, \widetilde{\mathbf{V}}(T) \subset \mathbf{V}(T)$ s.t. $\mathbf{v} \in \mathbf{V}(T)$ with $\operatorname{div}\mathbf{v} \in \operatorname{div}\widetilde{\mathbf{V}}(T)$ imply $\mathbf{v} \in \widetilde{\mathbf{V}}(T)$ ✓
- (ii_b) The quadrature rule is exact for $P_1(T)^2 \times \widetilde{\mathbf{V}}(T)$ ✓
- (iii) The quadrature rule induces a norm on $\mathbf{V}(T)$ ✓

Third candidate : $\mathbf{V}(T) = \mathrm{RT}_1(T) := P_1(T)^2 + \mathbf{x}\cdot P_1^h(T)$



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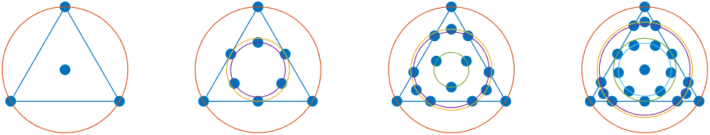
(iii) The theory can be used to design even higher order approximations, but finding appropriate spaces and quadrature formulas gets increasingly difficult.



Similar concept in the paper by Geevers, et al, 2018

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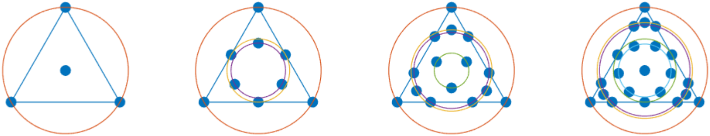
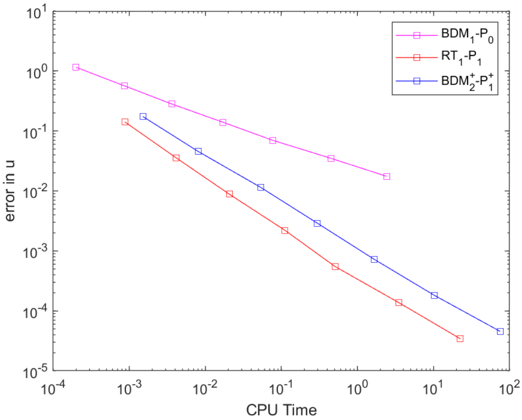


Similar concept in the paper by Geevers, et al, 2018

(iv) Application to wave propagation

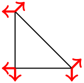
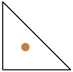
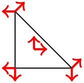
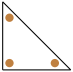
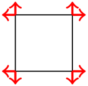
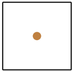
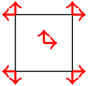
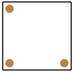
$$\begin{aligned} \partial_t \mathbf{u} + \nabla p &= f && \text{in } \Omega \\ \partial_t p + \operatorname{div} \mathbf{u} &= g && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Comparison



The $\mathrm{RT}_1 - \mathrm{P}_1$ pair is about 3x faster than the $\mathrm{BDM}_2^+ - \mathrm{P}_1^+$ pair.

Hybrid meshes

	dim $\mathbf{V}(T)$	dim $Q(T)$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\|\pi_h^0 p - p_h\|_0$	DOFs for $\mathbf{u}_h$	DOFs for p_h
$\mathrm{BDM}_1 - \mathrm{P}_0$	6+0	1	$O(h)$	$O(h^2)$		
$\mathrm{RT}_1 - \mathrm{P}_1$	6+2	3	$O(h^2)$	$O(h^3)$		
$\mathrm{BDM}_1 - \mathrm{P}_0$	8+0	1	$O(h)$	$O(h^2)$		
$\mathrm{BDFM}_2 - \mathrm{P}_1$	8+2	3	$O(h^2)$	$O(h^3)$		

Numerical tests

$$p = \sin(\pi x)\sin(\pi y) \qquad K = \begin{pmatrix} 4 + (x+2)^2 + y^2 & 1 + \sin(xy) \\ 1 + \sin(xy) & 2 \end{pmatrix}$$

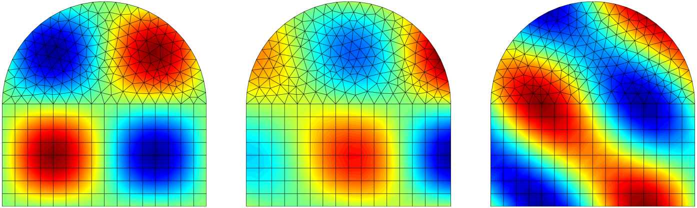


Figure: Snapshots of the pressure p_h (left) and the two velocity components $u_{x,h}$, $u_{y,h}$ (middle, right) for the second order approximation.

h	DOF u	DOF p	$\|u - u_h\|$	eoc	$\|\pi_h^0(p - p_h)\|$	eoc
2^{-1}	164	84	0.078309	—	0.033106	—
2^{-2}	724	396	0.013097	2.57	0.002864	3.53
2^{-3}	2498	1386	0.002275	2.52	0.000391	2.87
2^{-4}	9738	5466	0.000484	2.23	0.000049	2.99
2^{-5}	40230	22770	0.000099	2.28	0.000005	3.13

Table: Degrees of freedom, relative discretization errors, and convergence rates for the second order multipoint flux finite element method.

Summary

- Introduced the multipoint flux mixed finite element method (MFMFE)
- Presented the first order approximation introduced by Wheeler and Yotov
- Proposed an extension to second order approximations

H. Egger, B. Radu *A second order multipoint flux mixed finite element method on hybrid meshes*, TU Darmstadt, 12/2018 arXive: 1812.03938

A few additional remarks

- Extension to the $3D$ case has also been done.
- The framework can be used to design even higher order approximations
- We can devise local post-processing strategies for the pressure
- The techniques can also be applied for the wave and Maxwell's equations

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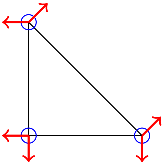
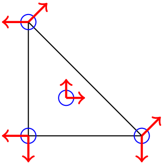
Thank you for your attention

Acknowledgement : The work of Bogdan Radu is supported by the 'Excellence Initiative' of the German Federal and State Governments and the Graduate School of Computational Engineering at Technische Universität Darmstadt

Triangles

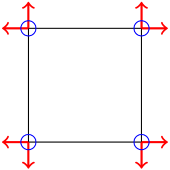


(a) BDM_1 first order element

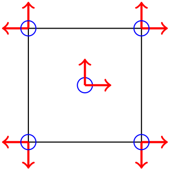


(b) RT_1 second order element

Parallelograms

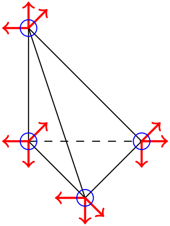


(a) BDM_1 first order element

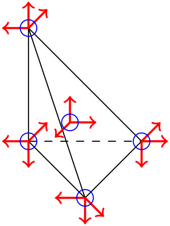


(b) $BDFM_2$ second order element

Tetrahedra

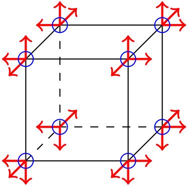


(a) BDM_1 first order element

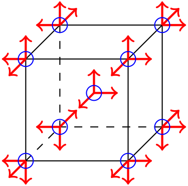


(b) RT_1 second order element

Hexahedra



(a) $eBDDF_1$ first order element



(b) ? - second order element