On Abstract Friedrichs Systems and Some of their Applications.

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We consider problems of the following *normal form*

$$\left(\partial_{t}\mathcal{M}+A\right)U=F,$$

where A is maximal accretive in a real Hilbert space H, $\mathcal{M}$ a space-time operator referred to as "material law operator", such that $\partial_t\mathcal{M}$ is positive definit. We focus here on a special case: A skew-selfadjoint and

$$\mathcal{M}=M_0+\partial_t^{-1}M_1,$$

where M_k, $k=0,1$, is a continuous linear operator in H.

Key idea to make sense of ∂_t^{-1}: exponential weight function $t\mapsto\exp\left(-\rho\,t\right)$, $\rho\in\mathbb{R}$, generates a weighted L^2-type solution space $H_{\rho,0}\left(\mathbb{R},H\right)$ (inner product $\langle\,\cdot\,|\,\cdot\,\rangle_{\rho,0,0}$, norm: $|\,\cdot\,|_{\rho,0,0}$), H a **real** Hilbert space,

$$(\varphi,\psi)\mapsto\int_{\mathbb{R}}\langle\varphi\left(t\right)|\,\psi\left(t\right)\rangle_{H}\exp\left(-2\rho\,t\right)dt.$$

Time-differentiation ∂_{t} as the maximal operator in $H_{\rho,0}\left(\mathbb{R},H\right)$ induced by the usual derivative.

Introduction

Time-differentiation ∂_t is a *normal* operator in $H_{\rho,0}(\mathbb{R}, H)$

$$\partial_t = \mathfrak{sym}(\partial_t) + \mathfrak{skew}(\partial_t) = \frac{1}{2}(\partial_t + \partial_t^*) + \frac{1}{2}(\partial_t - \partial_t^*)$$

with $\mathfrak{sym}(\partial_t)$ self-adjoint and , $\mathfrak{skew}(\partial_t)$ skew-selfadjoint and commuting resolvents:

$$\mathfrak{sym}(\partial_t) = \rho.$$

For $\rho \in \mathbb{R} \setminus \{0\}$: continuous invertibility of ∂_t. In particular, for $\rho \in]0, \infty[$:

$$\mathfrak{sym}(\partial_t) = \rho > 0.$$

Well-Posedness Condition (WPC): M_0 selfadjoint and

$$\boxed{\rho M_0 + \mathfrak{sym}(M_1) \geq c_0 > 0}$$

for all sufficiently large $\rho \in]0, \infty[$.

Introduction

Dynamic abstract Friedrichs system (Friedrichs 1954,1958):

$$(\partial_t \mathscr{M} + A)\, U = (\partial_t M_0 + M_1 + A)\, U = F$$

$$\begin{aligned}
&\partial_t M_0 + M_1 + A = \\
&= (\rho M_0 + \mathfrak{sym}\,(M_1)) + ((\partial_t - \rho)\, M_0 + \mathfrak{skew}\,(M_1) + A) \\
&= E_0 + \mathscr{A}\,.
\end{aligned}$$

E_0 symmetric strictly positive definite, $\mathscr{A}$ skew-selfadjoint in $H_{\rho,0}(\mathbb{R}, H)$. W.l.o.g. $E_0 = 1$, since we have the congruence

$$\sqrt{E_0}\left(1 + \sqrt{E_0^{-1}}\mathscr{A}\sqrt{E_0^{-1}}\right)\sqrt{E_0} = E_0 + \mathscr{A},$$

and note that

$$\sqrt{E_0^{-1}}\mathscr{A}\sqrt{E_0^{-1}}$$

remains skew-selfadjoint.

The Basic Solution Theorem

Theorem

Let A be skew-selfadjoint and M_0, M_1 satisfy ***Assumption*** $(\mathbf{WPC})$*. Then we have for all sufficiently large $\rho \in]0, \infty[$ that for every $f \in H_{\rho,0}(\mathbb{R}, H)$ there is a unique solution $U \in H_{\rho,0}(\mathbb{R}, H)$ of the problem*

$$\overline{(\partial_t M_0 + M_1 + A)} U = f.$$

The solution operator $\left(\overline{\partial_t M_0 + M_1 + A}\right)^{-1}$ is continuous and causal on $H_{\rho,0}(\mathbb{R}, H)$.

Causal? For every $a \in \mathbb{R}$ we have: If $F \in H_{\rho,0}(\mathbb{R}, H)$ vanishes on the time interval $]-\infty, a[$, then so does $\left(\overline{\partial_t M_0 + M_1 + A}\right)^{-1} F$.

Causality follows from $(\mathbf{WPC})$ and the observation that for all $a \in \mathbb{R}$

$$\left[\chi_{]-\infty,a]}, \partial_t\right] = \delta_{\{a\}} \geq 0,$$

where $\delta_{\{a\}} u = u(a)\, \delta_{\{a\}}$.

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An Illustrative Example

Frequently,

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix},$$

where G is a closed densely defined linear operator.

We recall that we will here consider only simple material laws

$$\mathcal{M} = M_0 + \partial_t^{-1} M_1,$$

i.e. on the case associated with abstract Friedrichs systems:

$$\boxed{(\partial_t M_0 + M_1 + A)\, U = F.}$$

An Illustrative Example

Consider a material law with

$$\mathcal{M} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} + \partial_t^{-1} \begin{pmatrix} (1-\varepsilon_1) & 0 \\ 0 & (1-\varepsilon_2) \end{pmatrix}, \ \varepsilon_1, \varepsilon_2 \in \{0,1\}.$$

- $\varepsilon_1 = 1, \varepsilon_2 = 1$: $\begin{pmatrix} \partial_t & -G^* \\ G & \partial_t \end{pmatrix} \sim \begin{pmatrix} \partial_t^2 + G^*G & 0 \\ G & \partial_t \end{pmatrix}$ by a formal row operation ("hyperbolic").
- $\varepsilon_1 = 1, \varepsilon_2 = 0$: $\begin{pmatrix} \partial_t & -G^* \\ G & 1 \end{pmatrix} \sim \begin{pmatrix} \partial_t + G^*G & 0 \\ G & 1 \end{pmatrix}$ by a formal row operation ("parabolic"). Note that $\varepsilon_1 = 0, \varepsilon_2 = 1$ is analogous.
- $\varepsilon_1 = 0, \varepsilon_2 = 0$: $\begin{pmatrix} 1 & -G^* \\ G & 1 \end{pmatrix} \sim \begin{pmatrix} 1 + G^*G & 0 \\ G & 1 \end{pmatrix}$ by a formal row operation ("elliptic").

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The Basic Case

As mentioned earlier, the skew-selfadjointness frequently stems from the form

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix},$$

where G is a closed densely defined operator.

For example:

- $G = \mathring{\text{grad}}, \mathring{\text{curl}}, \mathring{\text{div}}, \text{grad}, \text{curl}, \text{div}, \iota^{-1}_{\left(\mathring{H}_1, L^2\right)} (= \text{"1"})$,
- $G = \begin{pmatrix} \text{grad} \\ \gamma_D \end{pmatrix}$, $\gamma_D : \text{dom}\,(\text{grad}) \subseteq L^2(\partial\Omega) \to L^2(\partial\Omega)$ the Dirichlet boundary trace map,
- $G = \begin{pmatrix} \mathring{\text{grad}} & 0 \\ 0 & \mathring{\text{Grad}} \end{pmatrix}$ or $G = \begin{pmatrix} \overline{\iota^*_{\Omega_1} \mathring{\text{grad}}} \\ \overline{\iota^*_{\Omega \setminus \overline{\Omega_1}} \mathring{\text{Grad}}} \end{pmatrix}$ used for coupling acoustics and elasticity

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Skew-Selfadjointness: Weak = Strong

More generally

$$A = \begin{pmatrix} X & -G^* \\ G & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} + \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix},$$

with X, Y skew-selfadjoint yields a skew-Hermitean A. When is A skew-selfadjoint?

Tools:

- Transmutator: L, R continuous linear operators, C closed linear operator

  $$[L, C, R] := LC - CR$$

  assumed to be defined on dom (C).

- The commutator

  $$[L, C] := [L, C, L]$$

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Let A_k, $k = 1, 2$, be closed densely defined operators from H to K, $\mathrm{dom}\,(A_1 + A_2) = \mathrm{dom}\,(A_1) \cap \mathrm{dom}\,(A_2)$ dense in H.

Theorem

Let $(L_\varepsilon)_{\varepsilon \in]0,1[}$, $(R_\varepsilon)_{\varepsilon \in]0,1[}$ be bounded families of continuous linear mappings in K and H, respectively, and $[L_\varepsilon, A_1 + A_2, R_\varepsilon]$ defined on $\mathrm{dom}\,(A_1) \cap \mathrm{dom}\,(A_2)$ such that $\overline{[L_\varepsilon, A_1 + A_2, R_\varepsilon]} \in \mathscr{L}(H, K)$. Moreover,

- $L_\varepsilon^* \left[\mathrm{dom}\,((A_1 + A_2)^*)\right] \subseteq \mathrm{dom}\,(A_1^* + A_2^*)$,
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Then

$$(A_1 + A_2)^* = \overline{A_1^* + A_2^*}.$$

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Let A_1, A_2 be skew-selfadjoint, then under the assumptions of the previous theorem we have

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Let $(L_\varepsilon)_{\varepsilon \in]0,1[}$, $(R_\varepsilon)_{\varepsilon \in]0,1[}$ be bounded families of continuous linear mappings in K and H, respectively, and $[L_\varepsilon, A_1 + A_2, R_\varepsilon]$ defined on $\mathrm{dom}\,(A_1) \cap \mathrm{dom}\,(A_2)$ such that $\overline{[L_\varepsilon, A_1 + A_2, R_\varepsilon]} \in \mathscr{L}(H, K)$. Moreover,

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Then
$$(A_1 + A_2)^* = \overline{A_1^* + A_2^*}.$$

Corollary

Let A_1, A_2 be skew-selfadjoint, then under the assumptions of the previous theorem we have
$$\overline{A_1 + A_2} \text{ skew-selfadjoint.}$$

Skew-Selfadjointness: Weak = Strong

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An application: (acoustics in moving media; inspired by discussions with Martin Berggren and Linus Hägg; joint work with Sascha Trostorff and Marcus Waurick) assuming that

$\mathfrak{sym}\left(\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)$ is continuous

$$\partial_t\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\begin{pmatrix}0 & \operatorname{grad}\\ \mathring{\operatorname{div}} & 0\end{pmatrix}=$$
$$=\partial_t\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}+\frac{1}{2}\mathfrak{sym}\left(\alpha\partial_3\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)+A_1+A_2$$

with

$$A_1=\mathfrak{skew}\overline{\left(\partial_3\alpha\begin{pmatrix}\rho_* & 0\\ 0 & \kappa^{-1}\end{pmatrix}\right)},\ \text{(skew-selfadjoint for suitable } \alpha,\Omega\text{)}$$

$$A_2=\begin{pmatrix}0 & \operatorname{grad}\\ \mathring{\operatorname{div}} & 0\end{pmatrix},$$

$$R_\varepsilon=L_\varepsilon=(1+\varepsilon\partial_3)^{-1}.$$

The End

Thank You for Your Attention!