MULTIPLE-NETWORK POROELASTIC SYSTEMS: STABLE PARAMETER-ROBUST PRECONDITIONERS

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Outline

1. Multiple-network poroelastic theory (MPET)
2. Parameter-robust stability of the continuous model
3. Stable parameter-robust preconditioners for the discrete problem

The equations of multiple network poroelasticity

For homogeneous isotropic linear elastic porous media, the MPET model in an open domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $i = 1, \ldots, n$, reads:

$$
\begin{aligned}
-\mathsf{div}\boldsymbol{\sigma} + \sum_{i=1}^{n} \alpha_i \nabla p_i &= \boldsymbol{f} \quad \text{in} \quad \Omega \times (0,T), \\
\boldsymbol{v}_i &= -K_i \nabla p_i \quad \text{in} \quad \Omega \times (0,T), \\
-\alpha_i \mathsf{div}\dot{\boldsymbol{u}} - \mathsf{div}\boldsymbol{v}_i - c_{p_i}\dot{p}_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{ij}(p_i - p_j) &= g_i \quad \text{in} \quad \Omega \times (0,T),
\end{aligned}
$$

$$
\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon}(\boldsymbol{u}) + \lambda \mathsf{div}(\boldsymbol{u})\boldsymbol{I}, \quad \boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T).
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$\boldsymbol{u}$ displacement, $\boldsymbol{v}_i$ fluxes, p_i pressures

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Equilibrium equation

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Darcy's law

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Mass Conservation (continuity equation)

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$\boldsymbol{\sigma}$ Effective stress tensor (Hooke's law), $\boldsymbol{\epsilon}$ Strain tensor

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Biot-Willis parameter(couple n pore pressures p with the displacement)

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The hydraulic conductivities,which give an indication of the general permeability of a porous medium.

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Storage coefficients are connected to compressibility of each fluid

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The network transfer coefficients couple the network pressures

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Body force density

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Forced fluid extractions or injections into the medium

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The Lamé parameters related to the modulus of elasticity (Young's modulus)
$\lambda := \frac{\nu E}{(1+\nu)(1-2\nu)}$, $\mu := \frac{E}{2(1+\nu)}$, $\nu \in \left[0, \frac{1}{2}\right)$

Boundary and initial conditions

Boundary conditions:

$$\begin{aligned}
p_i(\boldsymbol{x},t) &= p_{i,D}(\boldsymbol{x},t) && \text{for } \boldsymbol{x} \in \Gamma_{p_i,D}, \quad t>0,\\
\boldsymbol{v}_i(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) &= q_{i,N}(\boldsymbol{x},t) && \text{for } \boldsymbol{x} \in \Gamma_{p_i,N}, \quad t>0,\\
\boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{u}_D(\boldsymbol{x},t) && \text{for } \boldsymbol{x} \in \Gamma_{\boldsymbol{u},D}, \quad t>0,\\
(\boldsymbol{\sigma}(\boldsymbol{x},t) - \sum_{i=1}^{n} \alpha_i p_i \boldsymbol{I})\,\boldsymbol{n}(\boldsymbol{x}) &= \boldsymbol{g}_N(\boldsymbol{x},t) && \text{for } \boldsymbol{x} \in \Gamma_{\boldsymbol{u},N}, \quad t>0,
\end{aligned}$$

$i=1,\dots,n$, where $\Gamma_{p_i,D} \cap \Gamma_{p_i,N} = \emptyset$, $\overline{\Gamma}_{p_i,D} \cup \overline{\Gamma}_{p_i,N} = \Gamma = \partial\Omega$ and $\Gamma_{\boldsymbol{u},D} \cap \Gamma_{\boldsymbol{u},N} = \emptyset$, $\overline{\Gamma}_{\boldsymbol{u},D} \cup \overline{\Gamma}_{\boldsymbol{u},N} = \Gamma$.

Initial conditions:

$$\begin{aligned}
p_i(\boldsymbol{x},0) &= p_{i,0}(\boldsymbol{x}) && \text{for } \boldsymbol{x} \in \Omega, \quad i=1,\dots,n,\\
\boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0(\boldsymbol{x}) && \text{for } \boldsymbol{x} \in \Omega.
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\end{aligned}
$$

p_0 is equal to the hydrostatic pressure and $\boldsymbol{u}_0$ is the solution of the uncoupled elasticity equation with the corresponding boundary conditions.

The backward Euler method

After implicit time discretization one has to solve a static problem in each time step:

$$
\begin{aligned}
-2\mu \operatorname{div} \boldsymbol{\epsilon}(\boldsymbol{u}^k) - \lambda \nabla \operatorname{div} \boldsymbol{u}^k + \sum_{i=1}^{n} \alpha_i \nabla p_i^k &= \boldsymbol{f}^k, \\
K_i^{-1} \boldsymbol{v}_i^k + \nabla p_i^k &= \boldsymbol{0}, \qquad i = 1, \dots, n, \\
-\alpha_i \operatorname{div} \boldsymbol{u}^k - \tau \operatorname{div} \boldsymbol{v}_i^k - c_{p_i} p_i^k - \tau \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{ij} (p_i^k - p_j^k) &= g_i^k, \qquad i = 1, \dots, n,
\end{aligned}
$$

where $\boldsymbol{u}^k$, $\boldsymbol{v}_i^k$, p_i^k for $i = 1, \dots, n$ yield approximations of $\boldsymbol{u}$, $\boldsymbol{v}_i$, p_i at a given time $t_k = t_{k-1} + \tau$:

$$
\begin{array}{lcll}
\boldsymbol{u}(x, t_k) & \approx & \boldsymbol{u}^k \in \boldsymbol{U} & := \{\boldsymbol{u} \in H^1(\Omega)^d : \boldsymbol{u} = \boldsymbol{u}_D \text{ on } \Gamma_{\boldsymbol{u},D}\}, \\
\boldsymbol{v}_i(x, t_k) & \approx & \boldsymbol{v}_i^k \in \boldsymbol{V}_i & := \{\boldsymbol{v}_i \in H(\operatorname{div}, \Omega) : \boldsymbol{v}_i \cdot \boldsymbol{n} = q_{i,N} \text{ on } \Gamma_{p_i,N}\}, \\
p_i(x, t_k) & \approx & p_i^k \in P_i & := L^2(\Omega).
\end{array}
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\end{aligned}
$$

Recasting the model

We make the rather general and reasonable assumptions that $\lambda > 0$, $R_i^{-1} > 0$, $\alpha_{p_i} \geq 0$, $\alpha_{ij} \geq 0$ for $1 \leq i, j \leq n$:

$$-\mathrm{div}\boldsymbol{\epsilon}(\boldsymbol{u}) - \lambda\nabla\mathrm{div}\boldsymbol{u} + \sum_{i=1}^{n}\nabla p_i = \boldsymbol{f}, \tag{1a}$$

$$R_i^{-1}\boldsymbol{v}_i + \nabla p_i = \boldsymbol{0}, \tag{1b}$$

$$-\mathrm{div}\boldsymbol{u} - \mathrm{div}\boldsymbol{v}_i - (\alpha_{p_i} + \alpha_{ii})p_i + \sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_{ij}p_j = g_i. \tag{1c}$$

Substitutions: $R_i^{-1} = \tau^{-1}K_i^{-1}\alpha_i^2$, $\alpha_{p_i} = c_{p_i}/\alpha_i^2$, $\beta_{ii} = \sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}$, $\alpha_{ij} = \tau\beta_{ij}/(\alpha_i\alpha_j)$

Matrix form

$$
\left[\begin{array}{ccccc|cccc}
-\mathrm{div}\epsilon-\lambda\nabla\mathrm{div} & 0 & \dots & \dots & 0 & \nabla & \dots & \dots & \nabla \\
0 & R_1^{-1}I & 0 & \dots & 0 & \nabla & 0 & \dots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\
0 & 0 & \dots & 0 & R_n^{-1}I & 0 & \dots & 0 & \nabla \\
\hline
-\mathrm{div} & -\mathrm{div} & 0 & \dots & 0 & \tilde{\alpha}_{11}I & \alpha_{12}I & \dots & \alpha_{1n}I \\
\vdots & 0 & \ddots & & \vdots & \alpha_{21}I & \ddots & & \alpha_{2n}I \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & \vdots \\
-\mathrm{div} & 0 & \dots & 0 & -\mathrm{div} & \alpha_{n1}I & \alpha_{n2}I & \dots & \tilde{\alpha}_{nn}I
\end{array}\right]
\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v}_1 \\ \vdots \\ \vdots \\ \boldsymbol{v}_n \\ p_1 \\ \vdots \\ \vdots \\ p_n \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \\ \vdots \\ \vdots \\ \boldsymbol{0} \\ g_1 \\ \vdots \\ \vdots \\ g_n \end{bmatrix}
$$

where $\tilde{\alpha}_{ii} = -\alpha_{p_i} - \alpha_{ii}, i = 1, \cdots, n.$

Weak formulation

Find $(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$, such that for any $(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$ there holds

$$(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{w})) + \lambda(\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{w}) - \sum_{i=1}^{n}(p_i, \mathrm{div}\boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}) \tag{2a}$$

$$(R_i^{-1}\boldsymbol{v}_i, \boldsymbol{z}_i) - (p_i, \mathrm{div}\boldsymbol{z}_i) = 0, \tag{2b}$$

$$-(\mathrm{div}\boldsymbol{u}, q_i) - (\mathrm{div}\boldsymbol{v}_i, q_i) + \tilde{\alpha}_{ii}(p_i, q_i) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij}(p_j, q_i) = (g_i, q_i), \tag{2c}$$

where (2b) and (2c) are for $i = 1, \dots, n$.
We define

$$R^{-1} = \max\{R_1^{-1}, \dots, R_n^{-1}\}, \quad \lambda_0 = \max\{1, \lambda\}. \tag{3}$$

Parameter-matrix-dependent norms

We further introduce the $n \times n$ parameter matrices

$$\Lambda_1 = \begin{bmatrix} \alpha_{11} & -\alpha_{12} & \dots & -\alpha_{1n} \\ -\alpha_{21} & \alpha_{22} & \dots & -\alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1} & -\alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} \alpha_{p_1} & 0 & \dots & 0 \\ 0 & \alpha_{p_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{p_n} \end{bmatrix},$$

$$\Lambda_3 = \begin{bmatrix} R & 0 & \dots & 0 \\ 0 & R & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R \end{bmatrix}, \qquad \Lambda_4 = \begin{bmatrix} \frac{1}{\lambda_0} & \dots & \dots & \frac{1}{\lambda_0} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{\lambda_0} & \dots & \dots & \frac{1}{\lambda_0} \end{bmatrix}.$$

Λ_1 is symmetric positive semidefinite (SPSD). Λ_2 is SPSD.
Λ_3 is symmetric positive definite (SPD). Λ_4 is SPSD with eigenvalues $\lambda_i = 0,\ i = 1, \dots, n-1$ and $\lambda_n = \frac{n}{\lambda_0}$.

Parameter-matrix dependent norms

Now we introduce the SPD matrix

$$\Lambda = \sum_{i=1}^{4} \Lambda_i.$$

and the parameter-matrix-dependent norms $\|\cdot\|_{\boldsymbol{U}}$, $\|\cdot\|_{\boldsymbol{V}}$, $\|\cdot\|_{\boldsymbol{P}}$ induced by the following *inner products*:

$$(\boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{U}} = (\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{w})) + \lambda(\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{w}), \tag{4a}$$

$$(\boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{V}} = \sum_{i=1}^{n} (R_i^{-1}\boldsymbol{v}_i, \boldsymbol{z}_i) + (\Lambda^{-1}\mathrm{Div}\boldsymbol{v}, \mathrm{Div}\boldsymbol{z}), \tag{4b}$$

$$(\boldsymbol{p}, \boldsymbol{q})_{\boldsymbol{P}} = (\Lambda\boldsymbol{p}, \boldsymbol{q}), \tag{4c}$$

where $\boldsymbol{p}^T=(p_1, \ldots, p_n)$, $\boldsymbol{v}^T=(\boldsymbol{v}_1^T, \ldots, \boldsymbol{v}_n^T)$, $(\mathrm{Div}\boldsymbol{v})^T=(\mathrm{div}\boldsymbol{v}_1, \ldots, \mathrm{div}\boldsymbol{v}_n)$.

The bilinear form

Related to the problem (2) we introduce

$$\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) = (\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{w})) + \lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{w}) - \sum_{i=1}^{n}(p_i,\operatorname{div}\boldsymbol{w}) + \sum_{i=1}^{n}(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{z}_i)$$
$$- \sum_{i=1}^{n}(p_i,\operatorname{div}\boldsymbol{v}_i) - \sum_{i=1}^{n}(\operatorname{div}\boldsymbol{u},q_i) - \sum_{i=1}^{n}(\operatorname{div}\boldsymbol{v}_i,q_i) + \sum_{i=1}^{n}\tilde{\alpha}_{ii}(p_i,q_i) + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\alpha_{ij}(p_j,q_i).$$

$$\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) = (\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{w})) + \lambda(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{w}) - \sum_{i=1}^{n}(p_i,\operatorname{div}\boldsymbol{w}) + \sum_{i=1}^{n}(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{z}_i)$$
$$- (\boldsymbol{p},\mathsf{Div}\boldsymbol{v}) - \sum_{i=1}^{n}(\operatorname{div}\boldsymbol{u},q_i) - (\mathsf{Div}\boldsymbol{v},\boldsymbol{q}) + ((\Lambda_1+\Lambda_2)\boldsymbol{p},\boldsymbol{q}).$$

The uniform inf-sup condition for the MPET equations

Theorem 1 [Hong & Kraus & Lymbery & P. 2019]

There exists a constant C_b independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}$, $i, j = 1, \ldots, n$ and the network scale n, such that for any $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$, $(\boldsymbol{w}; \boldsymbol{z}; \boldsymbol{q}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$

$$|\mathcal{A}((\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}), (\boldsymbol{w}; \boldsymbol{z}; \boldsymbol{q}))| \leq C_b(\|\boldsymbol{u}\|_{\boldsymbol{U}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{p}\|_{\boldsymbol{P}})(\|\boldsymbol{w}\|_{\boldsymbol{U}} + \|\boldsymbol{z}\|_{\boldsymbol{V}} + \|\boldsymbol{q}\|_{\boldsymbol{P}}).$$

Theorem 2 [Hong & Kraus & Lymbery & P. 2019]

There exists a constant $\omega > 0$ independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}$ for all $i, j \in \{1, \ldots, n\}$, and independent of the number of networks n, such that

$$\inf_{(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}} \sup_{(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}} \frac{\mathcal{A}((\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}), (\boldsymbol{w}; \boldsymbol{z}; \boldsymbol{q}))}{(\|\boldsymbol{u}\|_{\boldsymbol{U}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{p}\|_{\boldsymbol{P}})(\|\boldsymbol{w}\|_{\boldsymbol{U}} + \|\boldsymbol{z}\|_{\boldsymbol{V}} + \|\boldsymbol{q}\|_{\boldsymbol{P}})} \geq \omega.$$

Sketch of the proof

Lemma 1 : There exists a constant $\beta_d > 0$ such that

$$\inf_{q \in P_i} \sup_{\boldsymbol{v} \in \boldsymbol{V}_i} \frac{(\mathrm{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\mathrm{div}} \|q\|} \geq \beta_d, \quad i = 1, \dots, n.$$

Lemma 2 : There exists a constant $\beta_s > 0$ such that

$$\inf_{(q_1, \cdots, q_n) \in P_1 \times \cdots \times P_n} \sup_{\boldsymbol{u} \in \boldsymbol{U}} \frac{(\mathrm{div} \boldsymbol{u}, \sum_{i=1}^{n} q_i)}{\|\boldsymbol{u}\|_1 \| \sum_{i=1}^{n} q_i \|} \geq \beta_s.$$

Sketch of the proof :

For any $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}) = (\boldsymbol{u}; \boldsymbol{v}_1, \dots, \boldsymbol{v}_n; p_1, \dots, p_n) \in \boldsymbol{U} \times \boldsymbol{V}_1 \times \cdots \times \boldsymbol{V}_n \times P_1 \times \cdots \times P_n$,

$\boldsymbol{\psi} \in \boldsymbol{V}$ such that $\mathrm{Div} \boldsymbol{\psi} = \sqrt{R} \boldsymbol{p}$ and $\|\boldsymbol{\psi}\|_{\mathrm{div}} \leq \beta_d^{-1} \sqrt{R} \|\boldsymbol{p}\|$,

$\boldsymbol{u}_0 \in \boldsymbol{U}$ such that $\mathrm{div} \boldsymbol{u}_0 = \frac{1}{\sqrt{\lambda_0}} (\sum_{i=1}^{n} p_i)$, $\|\boldsymbol{u}_0\|_1 \leq \beta_s^{-1} \frac{1}{\sqrt{\lambda_0}} \| \sum_{i=1}^{n} p_i \|$.

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Sketch of the proof :

For any $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}) = (\boldsymbol{u}; \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n; p_1, \ldots, p_n) \in \boldsymbol{U} \times \boldsymbol{V}_1 \times \cdots \times \boldsymbol{V}_n \times P_1 \times \cdots \times P_n$,

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$\boldsymbol{u}_0 \in \boldsymbol{U}$ such that $\mathrm{div}\boldsymbol{u}_0 = \frac{1}{\sqrt{\lambda_0}} (\sum_{i=1}^{n} p_i)$, $\|\boldsymbol{u}_0\|_1 \leq \beta_s^{-1} \frac{1}{\sqrt{\lambda_0}} \| \sum_{i=1}^{n} p_i \|$.

Sketch of the proof

1- Choose: $\boldsymbol{w} = \delta \boldsymbol{u} - \frac{1}{\sqrt{\lambda_0}} \boldsymbol{u}_0, \quad \boldsymbol{z} = \delta \boldsymbol{v} - \sqrt{R}\psi, \quad \boldsymbol{q} = -\delta \boldsymbol{p} - \Lambda^{-1}\text{Div}\boldsymbol{v}$.
where δ is a positive constant to be determined later.

2- Verify the boundedness of $(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})$ by $(\boldsymbol{u};\boldsymbol{v};\boldsymbol{p})$ in the combined norm. Then we obtain the boundedness estimate
$\|\boldsymbol{w}\|_{\boldsymbol{U}} + \|\boldsymbol{z}\|_{\boldsymbol{V}} + \|\boldsymbol{q}\|_{\boldsymbol{P}} \leq (\delta + 1 + \beta_d^{-1} + \beta_s^{-1})\big(\|\boldsymbol{u}\|_{\boldsymbol{U}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{p}\|_{\boldsymbol{P}}\big)$.

3- Verify the coercivity of $\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q}))$.

$$\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) \geq (\delta - \beta_s^{-2} - 1)(\boldsymbol{\epsilon}(\boldsymbol{u}),\boldsymbol{\epsilon}(\boldsymbol{u})) + (\delta - 2)\lambda(\text{div}\boldsymbol{u},\text{div}\boldsymbol{u}) + \frac{1}{2}(\Lambda_4\boldsymbol{p},\boldsymbol{p})$$
$$+ (\delta - \beta_d^{-2})\sum_{i=1}^{n}(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i) + \frac{3}{4}(\Lambda_3\boldsymbol{p},\boldsymbol{p}) + \frac{1}{2}(\Lambda^{-1}\text{Div}\boldsymbol{v},\text{Div}\boldsymbol{v}) + (\delta - 1)((\Lambda_1 + \Lambda_2)\boldsymbol{p},\boldsymbol{p}).$$

4- Choose $\delta := \max\left\{\beta_s^{-2} + \frac{1}{2} + 1, \beta_d^{-2} + \frac{1}{2}, 2 + \frac{1}{2}\right\}$.
$\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) \geq \frac{1}{2}\big(\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 + \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{p}\|_{\boldsymbol{P}}^2\big).\square$

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$$+ (\delta - \beta_d^{-2})\sum_{i=1}^{n}(R_i^{-1}\boldsymbol{v}_i,\boldsymbol{v}_i) + \frac{3}{4}(\Lambda_3 \boldsymbol{p},\boldsymbol{p}) + \frac{1}{2}(\Lambda^{-1}\text{Div}\boldsymbol{v},\text{Div}\boldsymbol{v}) + (\delta - 1)((\Lambda_1 + \Lambda_2)\boldsymbol{p},\boldsymbol{p}).$$

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$\mathcal{A}((\boldsymbol{u};\boldsymbol{v};\boldsymbol{p}),(\boldsymbol{w};\boldsymbol{z};\boldsymbol{q})) \geq \frac{1}{2}\big(\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 + \|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{p}\|_{\boldsymbol{P}}^2\big).\square$

Stability result

Corollary 1

Let $(\boldsymbol{u}; \boldsymbol{v}; \boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times P$ be the solution of (2). Then there holds the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{U}} + \|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{p}\|_{\boldsymbol{P}} \leq C_1(\|\boldsymbol{f}\|_{\boldsymbol{U}^*} + \|\boldsymbol{g}\|_{\boldsymbol{P}^*}),$$

for some positive constant C_1 that is independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}, \alpha_{ij}, i, j = 1, \ldots, n$ and the network scale n, where $\|\boldsymbol{f}\|_{\boldsymbol{U}^*} = \sup\limits_{\boldsymbol{w} \in \boldsymbol{U}} \frac{(\boldsymbol{f}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{U}}}, \quad \|\boldsymbol{g}\|_{\boldsymbol{P}^*} = \sup\limits_{\boldsymbol{q} \in \boldsymbol{P}} \frac{(\boldsymbol{g}, \boldsymbol{q})}{\|\boldsymbol{q}\|_{\boldsymbol{P}}} = \|\Lambda^{-\frac{1}{2}} \boldsymbol{g}\|.$

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Hong, Q, Kraus, J, Lymbery, M, Philo, F. Conservative discretizations and parameter-robust preconditioners for Biot and multiple-network flux-based poroelasticity models. Numer Linear Algebra Appl. 2019; e2242. https://doi.org/10.1002/nla.2242

Uniform norm-equivalent block-diagonal preconditioner

$\Lambda = \sum_{i=1}^{4} \Lambda_i = (\gamma_{ij})_{n\times n}, \Lambda^{-1} = (\tilde{\gamma}_{ij})_{n\times n}$ and define

$$\begin{aligned}(\mathscr{B}_{\boldsymbol{u}}\cdot,\cdot) &= (\cdot,\cdot)_{\boldsymbol{U}}\\ (\mathscr{B}_{\boldsymbol{v}}\cdot,\cdot) &= (\cdot,\cdot)_{\boldsymbol{V}}\\ (\mathscr{B}_{\boldsymbol{p}}\cdot,\cdot) &= (\cdot,\cdot)_{\boldsymbol{P}}\end{aligned} \qquad \text{then} \qquad \mathscr{B} := \begin{bmatrix} \mathscr{B}_{\boldsymbol{u}}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathscr{B}_{\boldsymbol{v}}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathscr{B}_{\boldsymbol{p}}^{-1} \end{bmatrix},$$

where

$$\mathscr{B}_{\boldsymbol{u}} = -\operatorname{div}\boldsymbol{\epsilon} - \lambda\nabla\operatorname{div}, \qquad \mathscr{B}_{\boldsymbol{p}} = \begin{bmatrix} \gamma_{11}I & \gamma_{12}I & \dots & \gamma_{1n}I \\ \gamma_{21}I & \gamma_{22}I & \dots & \gamma_{2n}I \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1}I & \gamma_{n2}I & \dots & \gamma_{nn}I \end{bmatrix},$$

and

$$\mathscr{B}_{\boldsymbol{v}} = \begin{bmatrix} R_1^{-1}I & 0 & \dots & 0 \\ 0 & R_2^{-1}I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_n^{-1}I \end{bmatrix} - \begin{bmatrix} \tilde{\gamma}_{11}\nabla\operatorname{div} & \tilde{\gamma}_{12}\nabla\operatorname{div} & \dots & \tilde{\gamma}_{1n}\nabla\operatorname{div} \\ \tilde{\gamma}_{21}\nabla\operatorname{div} & \tilde{\gamma}_{22}\nabla\operatorname{div} & \dots & \tilde{\gamma}_{2n}\nabla\operatorname{div} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1}\nabla\operatorname{div} & \tilde{\gamma}_{n2}\nabla\operatorname{div} & \dots & \tilde{\gamma}_{nn}\nabla\operatorname{div} \end{bmatrix}.$$

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K.-A. Mardal and R. Winther - Preconditioning discretizations of systems of partial differential equations. Numer. Linear Algebra Appl., 18(1):1–40, 2011.

Discretization

Let $\mathcal{T}_h$ be a shape-regular triangulation of mesh-size h. Define the set of all interior edges (or faces) by $\mathcal{E}_h^I$ and the set of all boundary edges (or faces) by $\mathcal{E}_h^B$. Let $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B$.

Denote by $e = \partial T_1 \cap \partial T_2$ the common boundary (interface) , and by $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ the unit normal vectors to e that point to the exterior of T_1 and T_2.

Trace operators:

Let $q \in H^1(\mathcal{T}_h)$, $\boldsymbol{v} \in H^1(\mathcal{T}_h)^d$ and $\boldsymbol{\tau} \in H^1(\mathcal{T}_h)^{d\times d}$, for any $e \in \mathcal{E}_h^I$: the averages are defined as

$$\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}|_{\partial T_1 \cap e} \cdot \boldsymbol{n}_1 - \boldsymbol{v}|_{\partial T_2 \cap e} \cdot \boldsymbol{n}_2), \quad \{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}|_{\partial T_1 \cap e}\boldsymbol{n}_1 - \boldsymbol{\tau}|_{\partial T_2 \cap e}\boldsymbol{n}_2)$$

and the jumps are given by

$$[q] = q|_{\partial T_1 \cap e} - q|_{\partial T_2 \cap e}, \quad [\boldsymbol{v}] = \boldsymbol{v}|_{\partial T_1 \cap e} - \boldsymbol{v}|_{\partial T_2 \cap e}.$$

When $e \in \mathcal{E}_h^B$ then the above quantities are defined as

$$\{\boldsymbol{v}\} = \boldsymbol{v}|_e \cdot \boldsymbol{n}, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}|_e \boldsymbol{n}, \quad [q] = q|_e, \quad [\boldsymbol{v}] = \boldsymbol{v}|_e.$$

Discretization

Let $\mathcal{T}_h$ be a shape-regular triangulation of mesh-size h. Define the set of all interior edges (or faces) by $\mathcal{E}_h^I$ and the set of all boundary edges (or faces) by $\mathcal{E}_h^B$. Let $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B$.
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$$\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}|_{\partial T_1\cap e} \cdot \boldsymbol{n}_1 - \boldsymbol{v}|_{\partial T_2\cap e} \cdot \boldsymbol{n}_2), \quad \{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}|_{\partial T_1\cap e}\boldsymbol{n}_1 - \boldsymbol{\tau}|_{\partial T_2\cap e}\boldsymbol{n}_2)$$

and the jumps are given by

$$[q] = q|_{\partial T_1\cap e} - q|_{\partial T_2\cap e}, \quad [\boldsymbol{v}] = \boldsymbol{v}|_{\partial T_1\cap e} - \boldsymbol{v}|_{\partial T_2\cap e}.$$

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$$\{\boldsymbol{v}\} = \boldsymbol{v}|_e \cdot \boldsymbol{n}, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}|_e\boldsymbol{n}, \quad [q] = q|_e, \quad [\boldsymbol{v}] = \boldsymbol{v}|_e.$$

Discretization

Let $\mathcal{T}_h$ be a shape-regular triangulation of mesh-size h. Define the set of all interior edges (or faces) by $\mathcal{E}_h^I$ and the set of all boundary edges (or faces) by $\mathcal{E}_h^B$. Let $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^B$.
Denote by $e = \partial T_1 \cap \partial T_2$ the common boundary (interface) , and by $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ the unit normal vectors to e that point to the exterior of T_1 and T_2.

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FEM spaces

The finite element spaces that we consider here are given by

$$\begin{aligned} \boldsymbol{U}_h &= \{\boldsymbol{u} \in H(\mathrm{div};\Omega) : \boldsymbol{u}|_K \in \boldsymbol{U}(K),\ K \in \mathcal{T}_h;\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ \boldsymbol{V}_{i,h} &= \{\boldsymbol{v} \in H(\mathrm{div};\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}_i(K),\ K \in \mathcal{T}_h;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ P_{i,h} &= \{q \in L^2(\Omega) : q|_K \in Q_i(K),\ K \in \mathcal{T}_h;\ \int_\Omega q dx = 0\}, \end{aligned}$$

for $i = 1,\dots,n$, $\boldsymbol{V}_h = \boldsymbol{V}_{1,h} \times \cdots \times \boldsymbol{V}_{n,h}$, $\boldsymbol{P}_h = P_{1,h} \times \cdots \times P_{n,h}$.

The local spaces $\boldsymbol{U}(K)/\boldsymbol{V}_i(K)/Q_i(K)$ are defined by

- $BDM_l(K)/RT_{l-1}(K)/P_{l-1}(K)$,
- $BDFM_l(K)/RT_{l-1}(K)/P_{l-1}(K)$.

To ensure the existence and uniqueness of the approximation and to preserve the divergence condition pointwise which gives a conservation of mass. Note that for all these choices the important condition:
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DG discretization

The DG discretization of the variational Problem (2) is given by: Find $(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h,) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$, such that for any $(\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$

$$a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + \lambda(\operatorname{div} \boldsymbol{u}_h, \operatorname{div} \boldsymbol{w}_h) - \sum_{i=1}^{n} (p_{i,h}, \operatorname{div} \boldsymbol{w}_h) = (\boldsymbol{f}, \boldsymbol{w}_h), \tag{5a}$$

$$(R_i^{-1} \boldsymbol{v}_{i,h}, \boldsymbol{z}_{i,h}) - (p_{i,h}, \operatorname{div} \boldsymbol{z}_{i,h}) = 0, \tag{5b}$$

$$\begin{aligned} &- (\operatorname{div} \boldsymbol{u}_h, q_{i,h}) - (\operatorname{div} \boldsymbol{v}_{i,h}, q_{i.h}) \\ &\quad + \tilde{\alpha}_{ii}(p_{i,h}, q_{i,h}) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij}(p_{j,h}, q_{i,h}) = (g_i, q_{i,h}), \end{aligned} \tag{5c}$$

where (5b) and (5c) are for $i = 1, \ldots, n$ and

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{w}) = & \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{w}) dx - \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\epsilon}(\boldsymbol{u})\} \cdot [\boldsymbol{w}_t] ds \\ & - \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\epsilon}(\boldsymbol{w})\} \cdot [\boldsymbol{u}_t] ds + \sum_{e \in \mathcal{E}_h} \int_e \eta h_e^{-1} [\boldsymbol{u}_t] \cdot [\boldsymbol{w}_t] ds, \end{aligned} \tag{6}$$

$\tilde{\alpha}_{ii} = -\alpha_{p_i} - \alpha_{ii}$, and η is a stabilization parameter independent of the parameters λ, R_i^{-1}, α_{p_i}, α_{ij}, $i, j = 1, \ldots, n$, the network scale n and h.

Mesh-dependent norms

For any $\boldsymbol{u} \in H^1(\mathcal{T}_h)^d$, we introduce the mesh dependent norms:

$$\begin{aligned} \|\boldsymbol{u}\|_h^2 &= \sum_{K\in\mathcal{T}_h} \|\varepsilon(\boldsymbol{u})\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{u}_t]\|_{0,e}^2, \\ \|\boldsymbol{u}\|_{1,h}^2 &= \sum_{K\in\mathcal{T}_h} \|\nabla\boldsymbol{u}\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{u}_t]\|_{0,e}^2, \end{aligned}$$

Next, for $\boldsymbol{u} \in H^2(\mathcal{T}_h)^d$, we define the "DG"-norm

$$\|\boldsymbol{u}\|_{DG}^2 = \sum_{K\in\mathcal{T}_h} \|\nabla\boldsymbol{u}\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{u}_t]\|_{0,e}^2 + \sum_{K\in\mathcal{T}_h} h_K^2|\boldsymbol{u}|_{2,K}^2,$$

and, finally, the mesh dependent norm

$$\|\boldsymbol{u}\|_{\boldsymbol{U}_h}^2 = \|\boldsymbol{u}\|_{DG}^2 + \lambda\|\operatorname{div}\boldsymbol{u}\|^2.$$

Discrete bilinear form

Related to the discrete problem (5) we introduce

$$\begin{aligned}\mathcal{A}_h((\boldsymbol{u}_h;\boldsymbol{v}_h;\boldsymbol{p}_h),(\boldsymbol{w}_h;\boldsymbol{z}_h;\boldsymbol{q}_h)) &= a_h(\boldsymbol{u}_h,\boldsymbol{w}_h)+\lambda(\operatorname{div}\boldsymbol{u}_h,\operatorname{div}\boldsymbol{w}_h)-\sum_{i=1}^{n}(p_{i,h},\operatorname{div}\boldsymbol{w}_h)\\ &+\sum_{i=1}^{n}(R_i^{-1}\boldsymbol{v}_{i,h},\boldsymbol{z}_{i,h})-(\boldsymbol{p}_h,\mathsf{Div}\boldsymbol{z}_h)\\ &-(\mathsf{div}\boldsymbol{u}_h,\sum_{i=1}^{n}q_{i,h})-(\mathsf{Div}\boldsymbol{v}_h,\boldsymbol{q}_h)-((\Lambda_1+\Lambda_2)\boldsymbol{p}_h,\boldsymbol{q}_h).\end{aligned}$$

Discrete parameter-robust continuity and stability

Theorem 3 [Hong & Kraus & Lymbery & P. 2019]

There exists a constant C_{bd} independent of the parameters λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i, j \in \{1, \ldots, n\}$, the network scale n and the mesh size h such that the inequality

$$|\mathcal{A}_h((\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h), (\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h))| \leq C_{bd}(\|\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h\|_{\boldsymbol{S}_h})(\|\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h\|_{\boldsymbol{S}_h})$$

is fulfilled for any $(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h, (\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$.

Theorem 4 [Hong & Kraus & Lymbery & P. 2019]

There exits a positive constant β_0 independent of the parameters λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i, j \in \{1, \ldots, n\}$, the network scale n and the mesh size h, such that

$$\inf_{(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h) \in \boldsymbol{S}_h} \sup_{(\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h) \in \boldsymbol{S}_h} \frac{\mathcal{A}_h((\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h), (\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h))}{(\|\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h\|_{\boldsymbol{S}_h})(\|\boldsymbol{w}_h; \boldsymbol{z}_h; \boldsymbol{q}_h\|_{\boldsymbol{S}_h})} \geq \beta_0.$$

where $\boldsymbol{S}_h := \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$ and $\|\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h\|_{\boldsymbol{S}_h} =: \|\boldsymbol{u}_h\|_{\boldsymbol{U}_h} + \|\boldsymbol{v}_h\|_{\boldsymbol{V}} + \|\boldsymbol{p}_h\|_{\boldsymbol{P}}$.

This result shows the full parameter-robust stability of the discrete MPET problem in its flux-based formulation.

Sketch of the proof

The proof uses similar arguments of theorem 2 by taking the following results and properties:

1- the discrete version of Korn's inequality yields:

$$\|\boldsymbol{u}\|_{DG} \eqsim \|\boldsymbol{u}\|_h \eqsim \|\boldsymbol{u}\|_{1,h}, \text{ for all } \boldsymbol{u} \in \boldsymbol{U}_h.$$

2- the bilinear form $a_h(\cdot,\cdot)$ from (6) is continuous and coercive:

$$|a_h(\boldsymbol{u},\boldsymbol{w})| \lesssim \|\boldsymbol{u}\|_{DG}\|\boldsymbol{w}\|_{DG}, \quad \text{for all} \quad \boldsymbol{u},\ \boldsymbol{w} \in H^2(\mathcal{T}_h)^d.$$

$$a_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \geq \alpha_a \|\boldsymbol{u}_h\|_h^2, \quad \text{for all} \quad \boldsymbol{u}_h \in \boldsymbol{U}_h,$$

3- the LBB conditions:

$$\inf_{q_{i,h}\in P_{i,h}} \sup_{\boldsymbol{u}_h\in\boldsymbol{U}_h} \frac{(\operatorname{div}\boldsymbol{u}_h, q_{i,h})}{\|\boldsymbol{u}_h\|_{1,h}\|q_{i,h}\|} \geq \beta_{sd}, \inf_{q_{i,h}\in P_{i,h}} \sup_{\boldsymbol{v}_{i,h}\in\boldsymbol{V}_{i,h}} \frac{(\operatorname{div}\boldsymbol{v}_{i,h}, q_{i,h})}{\|\boldsymbol{v}_{i,h}\|_{\operatorname{div}}\|q_{i,h}\|} \geq \beta_{dd}, i = 1,\dots,n,$$

where α_a, β_{sd} and β_{dd} are positive constants independent of the parameters $\lambda, R_i^{-1}, \alpha_{p_i}$, $\alpha_{ij}, i,j = 1,\dots,n$, the network scale n and the mesh size h.

Stability result

Corollary 1

Let $(\boldsymbol{u}_h; \boldsymbol{v}_h; \boldsymbol{p}_h) \in \boldsymbol{U}_h \times \boldsymbol{V}_h \times \boldsymbol{P}_h$ be the solution of (5a)-(5c), then the estimate

$$\|\boldsymbol{u}_h\|_{\boldsymbol{U}_h} + \|\boldsymbol{v}_h\|_{\boldsymbol{V}} + \|\boldsymbol{p}_h\|_{\boldsymbol{P}} \leq C_2(\|\boldsymbol{f}\|_{\boldsymbol{U}_h^*} + \|\boldsymbol{g}\|_{\boldsymbol{P}^*}) \tag{7}$$

holds where

$$\|\boldsymbol{f}\|_{\boldsymbol{U}_h^*} = \sup_{\boldsymbol{w}_h \in \boldsymbol{U}_h} \frac{(\boldsymbol{f}, \boldsymbol{w}_h)}{\|\boldsymbol{w}_h\|_{\boldsymbol{U}_h}}, \quad \|\boldsymbol{g}\|_{\boldsymbol{P}^*} = \sup_{\boldsymbol{q}_h \in \boldsymbol{P}_h} \frac{(\boldsymbol{g}, \boldsymbol{q}_h)}{\|\boldsymbol{q}_h\|_{\boldsymbol{P}}}$$

and C_2 is a constant independent of λ, R_i^{-1}, α_{p_i}, α_{ij} for all $i, j \in \{1, \ldots, n\}$, the network scale n and the mesh size h.

Uniform norm-equivalent block-diagonal preconditioner

$$\mathscr{B}_h := \begin{bmatrix} \mathscr{B}_{h,\boldsymbol{u}}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathscr{B}_{h,\boldsymbol{v}}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathscr{B}_{h,\boldsymbol{p}}^{-1} \end{bmatrix}, \tag{8}$$

where

$$\mathscr{B}_{h,\boldsymbol{u}} = -\operatorname{div}_h \boldsymbol{\epsilon}_h - \lambda \nabla_h \operatorname{div}_h,$$

$$\mathscr{B}_{h,\boldsymbol{v}} = \begin{bmatrix} R_1^{-1} I_h & 0 & \dots & 0 \\ 0 & R_2^{-1} I_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_n^{-1} I_h \end{bmatrix} - \begin{bmatrix} \tilde{\gamma}_{11} \nabla_h \mathrm{div}_h & \tilde{\gamma}_{12} \nabla_h \mathrm{div}_h & \dots & \tilde{\gamma}_{1n} \nabla_h \mathrm{div}_h \\ \tilde{\gamma}_{21} \nabla_h \mathrm{div}_h & \tilde{\gamma}_{22} \nabla_h \mathrm{div}_h & \dots & \tilde{\gamma}_{2n} \nabla_h \mathrm{div}_h \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1} \nabla_h \mathrm{div}_h & \tilde{\gamma}_{n2} \nabla_h \mathrm{div}_h & \dots & \tilde{\gamma}_{nn} \nabla_h \mathrm{div}_h \end{bmatrix}$$

and

$$\mathscr{B}_{h,\boldsymbol{p}} = \begin{bmatrix} \gamma_{11} I_h & \gamma_{12} I_h & \dots & \gamma_{1n} I_h \\ \gamma_{21} I_h & \gamma_{22} I_h & \dots & \gamma_{2n} I_h \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} I_h & \gamma_{n2} I_h & \dots & \gamma_{nn} I_h \end{bmatrix}.$$

The diagonal preconditioner

Denote $D_R^{-1} = \mathrm{diag}(R_1^{-1}, R_2^{-1}, \dots, R_n^{-1})$ and $\mathscr{D}_R^{-1} = D_R^{-1} \otimes I_h$, we have

$$(\mathscr{B}_{h,\boldsymbol{v}} \boldsymbol{v}_h, \boldsymbol{z}_h) = (\boldsymbol{v}_h, \boldsymbol{z}_h)_{\boldsymbol{V}} = (\mathscr{D}_R^{-\frac{1}{2}} \boldsymbol{v}_h, \mathscr{D}_R^{-\frac{1}{2}} \boldsymbol{z}_h) + (\Lambda^{-1} \mathsf{Div} \boldsymbol{v}_h, \mathsf{Div} \boldsymbol{z}_h).$$

Let $D_{\boldsymbol{v}}^{-1} = Q_{\boldsymbol{v}} (D_R^{\frac{1}{2}} \Lambda^{-1} D_R^{\frac{1}{2}}) Q_{\boldsymbol{v}}^T = \mathrm{diag}(\bar{\mu}_1, \dots, \bar{\mu}_n)$ be diagonalize $(D_R^{\frac{1}{2}} \Lambda^{-1} D_R^{\frac{1}{2}})$.

Now, by the change of variables $\bar{\boldsymbol{v}}_h = \mathcal{Q}_{\boldsymbol{v}} \mathscr{D}_R^{-\frac{1}{2}} \boldsymbol{v}_h, \bar{\boldsymbol{z}}_h = \mathcal{Q}_{\boldsymbol{v}} \mathscr{D}_R^{-\frac{1}{2}} \boldsymbol{z}_h$, where $\mathcal{Q}_{\boldsymbol{v}} = Q_{\boldsymbol{v}} \otimes I_h$ we get :

$$(\mathcal{Q}_{\boldsymbol{v}} \mathscr{D}_R^{\frac{1}{2}} \mathscr{B}_{h,\boldsymbol{v}} \mathscr{D}_R^{\frac{1}{2}} \mathcal{Q}_{\boldsymbol{v}}^T \bar{\boldsymbol{v}}_h, \bar{\boldsymbol{z}}_h) = (\bar{\boldsymbol{v}}_h, \bar{\boldsymbol{z}}_h) + (D_{\boldsymbol{v}}^{-1} \mathsf{Div} \bar{\boldsymbol{v}}_h, \mathsf{Div} \bar{\boldsymbol{z}}_h).$$

We denote

$$\mathscr{B}_{h,\bar{\boldsymbol{v}}} := \begin{bmatrix} I_h & 0 & \dots & 0 \\ 0 & I_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_h \end{bmatrix} - \begin{bmatrix} \bar{\mu}_1 \nabla_h \mathrm{div}_h & 0 & \dots & 0 \\ 0 & \bar{\mu}_2 \nabla_h \mathrm{div}_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{\mu}_n \nabla_h \mathrm{div}_h \end{bmatrix},$$

which means that we only need to solve n **decoupled** elliptic $H(\mathrm{div})$ problems discretized by RT elements to get $\bar{\boldsymbol{v}}_h$, and we obtain the original $\boldsymbol{v}_h$ from back substitution. $\boldsymbol{v}_h = \mathscr{D}_R^{\frac{1}{2}} \mathcal{Q}_{\boldsymbol{v}}^T \bar{\boldsymbol{v}}_h$.

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$$(\mathscr{B}_{h,\boldsymbol{v}} \boldsymbol{v}_h, \boldsymbol{z}_h) = (\boldsymbol{v}_h, \boldsymbol{z}_h)_{\boldsymbol{V}} = (\mathscr{D}_R^{-\frac{1}{2}} \boldsymbol{v}_h, \mathscr{D}_R^{-\frac{1}{2}} \boldsymbol{z}_h) + (\Lambda^{-1}\mathsf{Div}\boldsymbol{v}_h, \mathsf{Div}\boldsymbol{z}_h).$$

Let $D_{\boldsymbol{v}}^{-1} = Q_{\boldsymbol{v}}(D_R^{\frac{1}{2}}\Lambda^{-1}D_R^{\frac{1}{2}})Q_{\boldsymbol{v}}^T = \mathsf{diag}(\bar{\mu}_1, \dots, \bar{\mu}_n)$ be diagonalize $(D_R^{\frac{1}{2}}\Lambda^{-1}D_R^{\frac{1}{2}})$.
Now, by the change of variables $\bar{\boldsymbol{v}}_h = \mathcal{Q}_{\boldsymbol{v}}\mathscr{D}_R^{-\frac{1}{2}}\boldsymbol{v}_h, \bar{\boldsymbol{z}}_h = \mathcal{Q}_{\boldsymbol{v}}\mathscr{D}_R^{-\frac{1}{2}}\boldsymbol{z}_h$, where $\mathcal{Q}_{\boldsymbol{v}} = Q_{\boldsymbol{v}} \otimes I_h$ we get :

$$(\mathcal{Q}_{\boldsymbol{v}}\mathscr{D}_R^{\frac{1}{2}}\mathscr{B}_{h,\boldsymbol{v}}\mathscr{D}_R^{\frac{1}{2}}\mathcal{Q}_{\boldsymbol{v}}^T\bar{\boldsymbol{v}}_h, \bar{\boldsymbol{z}}_h) = (\bar{\boldsymbol{v}}_h, \bar{\boldsymbol{z}}_h) + (D_{\boldsymbol{v}}^{-1}\mathsf{Div}\bar{\boldsymbol{v}}_h, \mathsf{Div}\bar{\boldsymbol{z}}_h).$$

We denote

$$\mathscr{B}_{h,\bar{\boldsymbol{v}}} := \begin{bmatrix} I_h & 0 & \dots & 0 \\ 0 & I_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_h \end{bmatrix} - \begin{bmatrix} \bar{\mu}_1\nabla_h \mathrm{div}_h & 0 & \dots & 0 \\ 0 & \bar{\mu}_2\nabla_h \mathrm{div}_h & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{\mu}_n\nabla_h \mathrm{div}_h \end{bmatrix},$$

which means that we only need to solve n **decoupled** elliptic $H(\mathrm{div})$ problems discretized by RT elements to get $\bar{\boldsymbol{v}}_h$, and we obtain the original $\boldsymbol{v}_h$ from back substitution. $\boldsymbol{v}_h = \mathscr{D}_R^{\frac{1}{2}}\mathcal{Q}_{\boldsymbol{v}}^T\bar{\boldsymbol{v}}_h$.

The diagonal preconditioner

we have

$$(\mathscr{B}_{h,\boldsymbol{p}}\boldsymbol{p}_h, \boldsymbol{q}_h) = (\boldsymbol{p}_h, \boldsymbol{q}_h)_{\boldsymbol{P}} = (\Lambda \boldsymbol{p}_h, \boldsymbol{q}_h).$$

Let $D_{\boldsymbol{p}} = Q_{\boldsymbol{p}} \Lambda Q_{\boldsymbol{p}}^T = \text{diag}(\mu_1, \ldots, \mu_n)$ be diagonalize Λ and by the change of

variables $\bar{\boldsymbol{p}}_h = \mathcal{Q}_{\boldsymbol{p}}\boldsymbol{p}_h, \bar{\boldsymbol{q}}_h = \mathcal{Q}_{\boldsymbol{p}}\boldsymbol{q}_h$ we get :

$$(\mathcal{Q}_{\boldsymbol{p}}\mathscr{B}_{h,\boldsymbol{p}}\mathcal{Q}_{\boldsymbol{p}}^T\bar{\boldsymbol{p}}_h, \bar{\boldsymbol{q}}_h) = (\mathcal{Q}_{\boldsymbol{p}}\Lambda\mathcal{Q}_{\boldsymbol{p}}^T\bar{\boldsymbol{p}}_h, \bar{\boldsymbol{q}}_h).$$

We denote

$$\mathscr{B}_{h,\bar{\boldsymbol{p}}} := \begin{bmatrix} \mu_1 I_h & 0 & \ldots & 0 \\ 0 & \mu_2 I_h & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_n I_h \end{bmatrix}$$

we obtain the original $\boldsymbol{p}_h$ from back substitution. $\boldsymbol{p}_h = \mathcal{Q}_{\boldsymbol{p}}^T\bar{\boldsymbol{p}}_h.$

Reference

Numerical experiments can be found in the following preprint/paper:

1- Fixed stress split algorithm:

Q. Hong, J. Kraus, M. Lymbery, M.F. Wheeler. Parameter-robust convergence analysis of fixed-stress split iterative method for multiple-permeability poroelasticity systems. arXiv:1812.11809

2- MINRES algorithm:

Q. Hong, J. Kraus, M. Lymbery, F. Philo. Conservative discretizations and parameter-robust preconditioners for Biot and multiple-network flux-based poroelasticity models. Numer Linear Algebra Appl. 2019; e2242. https://doi.org/10.1002/nla.2242.

Thank you for your attention!