Relating Dirichlet/Neumann fields in bounded and unbounded Lipschitz domains

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UNIVERSITÄT DUISBURG ESSEN

Open-Minded

Motivation

time-harmonic Maxwell equations: $\Omega \subset \mathbb{R}^3$ exterior domain, $\Gamma := \partial\Omega = \overline{\Gamma_1 \cup \Gamma_2}$

$$-\operatorname{rot} H + i\omega\varepsilon E = F \quad \text{on} \quad \Omega, \qquad n \times E = 0 \quad \text{on} \quad \Gamma_1,$$
$$\operatorname{rot} E + i\omega\mu H = G \quad \text{on} \quad \Omega, \qquad n \times H = 0 \quad \text{on} \quad \Gamma_2.$$

↪ rewrite: $(\mathrm{M} - \omega)\, u_\omega = f \quad , \quad \mathrm{M} := \begin{pmatrix} 0 & -i\varepsilon^{-1} \operatorname{rot} \\ i\mu^{-1} \operatorname{rot} & 0 \end{pmatrix}$

↪ polynomially weighted Sobolev spaces $\rightsquigarrow$ sol. op. $u_\omega = \mathcal{L}_\omega f$

long time goal: low frequency asymptotics, i.e., $\lim_{\omega \to 0} \mathcal{L}_\omega = ?$

↪ form suggests "Neumann-series": ($\mathcal{L}_0$ static solution operator)

$$(\mathrm{M} - \omega)\, u_\omega = f \rightsquigarrow (1 - \omega\mathcal{L}_0) u_\omega = \mathcal{L}_0 (\mathrm{M} - \omega)\, u_\omega = \mathcal{L}_0 f$$
$$\rightsquigarrow u_\omega = (1 - \omega\mathcal{L}_0)^{-1} \mathcal{L}_0 f = \sum_{j=0}^{\infty} \omega^j \mathcal{L}_0^j \mathcal{L}_0 f$$

$\Longrightarrow$ iteration of static solution operator!

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(divergence free and **no** Dirichlet-Neumann-field)

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↪ (static) solution theory only in weighted Sobolev spaces!

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Lebesgue/Sobolev spaces

usual: $\mathsf{L}^2(\Omega),\ \mathsf{H}^1(\Omega)$

$$\mathsf{R}(\Omega) := \Big\{\, u \in \mathsf{L}^2(\Omega) \;\Big|\; \operatorname{rot} u \in \mathsf{L}^2(\Omega) \,\Big\} \qquad \Big(\, \mathsf{H}(\mathrm{curl}\,;\Omega) \,\Big)$$

$$\mathsf{D}(\Omega) := \Big\{\, u \in \mathsf{L}^2(\Omega) \;\Big|\; \operatorname{div} u \in \mathsf{L}^2(\Omega) \,\Big\} \qquad \Big(\, \mathsf{H}(\mathrm{div}\,;\Omega) \,\Big)$$

weighted: $t \in \mathbb{R}$ and $\rho = \rho(x) = (1 + |x|^2)^{\frac{1}{2}},\ x \in \mathbb{R}^3$

$$\mathsf{L}^2_{\mathrm{t}}(\Omega) := \Big\{\, u \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \;\Big|\; \rho^t u \in \mathsf{L}^2(\Omega) \,\Big\} \qquad \mathsf{H}^1_{\mathrm{t}}(\Omega) := \Big\{\, u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\Big|\; \nabla u \in \mathsf{L}^2_{\mathrm{t+1}}(\Omega) \,\Big\}$$

$$\mathsf{R}_{\mathrm{t}}(\Omega) := \Big\{\, u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\Big|\; \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{t+1}}(\Omega) \,\Big\} \qquad \mathsf{D}_{\mathrm{t}}(\Omega) := \Big\{\, u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\Big|\; \operatorname{div} u \in \mathsf{L}^2_{\mathrm{t+1}}(\Omega) \,\Big\}$$

boundary conditions: $\mathrm{V}_{\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_{\mathrm{i}}}(\Omega)}^{\|\cdot\|_{\mathrm{V}(\Omega)}},\quad \mathrm{V} \in \{\, \mathsf{H}^1,\ \mathsf{R},\ \mathsf{D},\ \mathsf{H}^1_{\mathrm{t}},\ \mathsf{R}_{\mathrm{t}},\ \mathsf{D}_{\mathrm{t}} \},$

$$\mathsf{C}^\infty_{\Gamma_{\mathrm{i}}}(\Omega) := \Big\{\, \varphi|_\Omega \;\Big|\; \varphi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3) \ \text{ and } \ \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_{\mathrm{i}}) > 0 \,\Big\}$$

kernels: *"zero in the lower left"*, e.g.,

$$_0\mathsf{R}(\Omega),\ _0\mathsf{D}_{\Gamma_{\mathrm{i}}}(\Omega),\ _0\mathsf{R}_{\mathrm{t}}(\Omega),\ _0\mathsf{D}_{\mathrm{t},\Gamma_{\mathrm{i}}}(\Omega),\ \ldots$$

Known Results

$${}_0\mathsf{D}_\Gamma(\Omega) = \left\{ u \in \mathsf{D}_\Gamma(\Omega) \;\middle|\; \operatorname{div} u = 0 \right\} \qquad {}_0\mathsf{R}_\Gamma(\Omega) = \left\{ u \in \mathsf{R}_\Gamma(\Omega) \;\middle|\; \operatorname{rot} u = 0 \right\}$$

$$\mathcal{N}(\Omega) := \mathcal{H}_{\emptyset,\Gamma}(\Omega) = {}_0\mathsf{R}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_\Gamma(\Omega) \qquad \mathcal{D}(\Omega) := \mathcal{H}_{\Gamma,\emptyset}(\Omega) = {}_0\mathsf{R}_\Gamma(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}(\Omega)$$

$\longrightarrow$ full boundary conditions: Dirichlet-fields $(\Gamma_{\mathrm{t}} = \Gamma, \Gamma_{\mathrm{n}} = \emptyset)$

Theorem (Picard 82', 85', 86')

There exists a finite set $\mathring{\mathrm{B}}_{\mathrm{e}} \subset {}_0\mathsf{R}_\Gamma(\Omega)$ *of compactly supported functions, such that their projections along* $\overline{\nabla \mathsf{H}^1_\Gamma(\Omega)}$ *form a basis of Dirichlet-fields* $\mathcal{D}(\Omega) := \mathcal{H}_{\Gamma,\emptyset}(\Omega)$ *and*

$$\mathring{\mathrm{B}}_{\mathrm{e}}(\Omega)^\perp \cap \mathcal{D}(\Omega) = \{0\}.$$

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Main Result

$$\mathsf{H}^1_{\Gamma_t}(\Omega) = \overline{\mathsf{C}^\infty_{\Gamma_t}(\Omega)}^{\|\cdot\|_{\mathsf{H}^1(\Omega)}} \qquad {}_0\mathsf{R}_{\Gamma_t}(\Omega) = \left\{ u \in \mathsf{R}_{\Gamma_t}(\Omega) \;\middle|\; \mathrm{rot}\, u = 0 \right\}$$

$${}_0\mathsf{D}_{\Gamma_n}(\Omega) = \left\{ u \in \mathsf{D}_{\Gamma_n}(\Omega) \;\middle|\; \mathrm{div}\, u = 0 \right\} \qquad \mathcal{D}(\Omega) := \mathcal{H}_{\Gamma,\emptyset}(\Omega) = {}_0\mathsf{R}_\Gamma(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}(\Omega)$$

TODAYS QUESTION:

Is there a "suitable" substitution $\mathrm{B}_\mathsf{R}(\Omega)$ for $\mathcal{H}_{\Gamma_t,\Gamma_n}(\Omega)$?

suitable: finite, compactly supported, $\mathrm{B}_\mathrm{R}(\Omega) \subset {}_0\mathsf{R}_t(\Omega)$ and

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$\longrightarrow$ mixed boundary conditions: ?????

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$\longrightarrow$ mixed boundary conditions: Yes, there is!

Theorem

There exists a finite set $\mathrm{B}_{\mathsf{R}}(\Omega) \subset {}_0\mathsf{R}_{\Gamma_t}(\Omega)$ of compactly supported functions, such that their projections along $\overline{\nabla \mathsf{H}^1_{\Gamma_t}(\Omega)}$ form a basis of $\mathcal{H}_{\Gamma_t,\Gamma_n}(\Omega)$ and

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Ingredients

$$\mathsf{H}^1_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \nabla u \in \mathsf{L}^2_{\mathrm{t+1}}(\Omega) \right\} \qquad \mathsf{R}_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{t+1}}(\Omega) \right\}$$

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$$\Omega_r = \Omega \cap \mathrm{U}(r), \qquad \check{\mathrm{U}}(r) = \mathbb{R}^3 \setminus \overline{\mathrm{U}(r)}, \qquad \Sigma = \partial \mathrm{B}_r(0), \qquad \hat{\Gamma}_{\mathrm{i}} = \Gamma_{\mathrm{i}} \cup \Sigma \;\; (i = 1, 2)$$

Weck's (local) selection theorem:

$$\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega) \cap \mathsf{D}_{\Gamma_{\mathrm{n}}}(\Omega) \hookrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$$

$$\Longrightarrow \; {}_0\mathsf{R}_{\hat{\Gamma}_{\mathrm{t}}}(\Omega_r) \cap \mathsf{D}_{\Gamma_{\mathrm{n}}}(\Omega_r) \hookrightarrow \mathsf{L}^2(\Omega_r)$$

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$$* \quad \boxed{\dim \mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega) < \infty}$$

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$$* \quad \overline{\nabla \mathsf{H}^1_{\Gamma_i}(\Omega_r)} = \nabla \mathsf{H}^1_{\Gamma_i}(\Omega_r) \quad , \quad \overline{\operatorname{rot} \mathsf{R}_{\Gamma_i}(\Omega_r)} = \mathsf{R}_{\Gamma_i}(\Omega_r)$$

Notations and Assumptions

$$\mathsf{R}(\Omega) = \left\{ u \in \mathsf{L}^2(\Omega) \;\middle|\; \operatorname{rot} u \in \mathsf{L}^2(\Omega) \right\} \qquad \mathsf{D}(\Omega) = \left\{ u \in \mathsf{L}^2(\Omega) \;\middle|\; \operatorname{div} u \in \mathsf{L}^2(\Omega) \right\}$$

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Assumptions: $\Omega \subset \mathbb{R}^3$ exterior Lipschitz domain, $\Gamma := \partial\Omega = \overline{\Gamma_t \cup \Gamma_n}$

$r > 0$ such that $\Omega \cap \mathrm{U}_r$ contains "essential topology"

Notations:

- special domains: $\mathrm{U} := \mathrm{B}_r(0)$, $\Omega_r := \Omega \cap \mathrm{U}$, $\check{\mathrm{U}} = \mathbb{R}^3 \setminus \overline{\mathrm{U}}$
- "artificial boundary": $\Sigma := \partial \mathrm{U}$, $\hat{\Gamma}_t := \Gamma_t \cup \Sigma$
- extensions and restrictions

$$\widetilde{\mathcal{H}_{\hat{\Gamma}_t,\Gamma_n}(\Omega_r)} := \text{"extend } h \in \mathcal{H}_{\hat{\Gamma}_t,\Gamma_n}(\Omega_r) \text{ by zero to } \Omega\text{"}$$

$$\underset{\sim}{\mathcal{H}_{\Gamma_t,\Gamma_n}}(\Omega) := \text{"restrict } h \in \mathcal{H}_{\Gamma_t,\Gamma_n}(\Omega) \text{ to } \Omega_r\text{"}$$

- Helmholtz-decompositions: $\left(A\colon D(A) \subset H_1 \longrightarrow H_2 \text{ lddc} \rightsquigarrow H_2 = R(A) \oplus N(A^*) \right)$

$$\mathsf{L}^2(\Omega) = \overline{\nabla \mathsf{H}^1_{\Gamma_t}(\Omega)} \oplus {}_0\mathsf{D}_{\Gamma_n}(\Omega) = \nabla \mathsf{H}^1_{-1,\Gamma_t}(\Omega) \oplus {}_0\mathsf{D}_{\Gamma_n}(\Omega)$$

$$\mathsf{L}^2(\Omega_r) = \overline{\operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega_r)} \oplus {}_0\mathsf{R}_{\hat{\Gamma}_t}(\Omega_r) = \operatorname{rot} \mathsf{R}_{\Gamma_n}(\Omega_r) \oplus {}_0\mathsf{R}_{\Gamma_t}(\Omega_r)$$

Proof

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Lemma

The projection of $\widetilde{\mathcal{H}_{\hat{\Gamma}_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega_r)} \subset {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega)$ *along* $\overline{\nabla \mathsf{H}^1_{\Gamma_{\mathrm{t}}}(\Omega)}$ *on* $\mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega)$ *is injective.*

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Lemma

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$$\Omega_r = \Omega \cap \mathrm{U}_r, \qquad \check{\mathrm{U}}_r = \mathbb{R}^3 \setminus \overline{\mathrm{U}}_r, \qquad \Sigma = \partial \mathrm{U}_r, \qquad \hat{\Gamma}_{\mathrm{t}} = \Gamma_{\mathrm{t}} \cup \Sigma$$

Lemma

The projection of $\mathcal{H}_{\widetilde{\hat{\Gamma}_{\mathrm{t}},\Gamma_{\mathrm{n}}}}(\Omega_r) \subset {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega)$ *along* $\overline{\nabla \mathsf{H}^1_{\Gamma_{\mathrm{t}}}(\Omega)}$ *on* $\mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega)$ *is injective.*

Lemma

The projection of $\mathcal{H}_{\underset{\sim}{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}}(\Omega) \subset {}_0\mathsf{D}_{\Gamma_{\mathrm{n}}}(\Omega_r)$ *along* $\overline{\operatorname{rot} \mathsf{R}_{\hat{\Gamma}_{\mathrm{t}}}(\Omega_r)}$ *on* $\mathcal{H}_{\hat{\Gamma}_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega_r)$ *is injective.*

$$\Longrightarrow \boxed{\dim \mathcal{H}_{\hat{\Gamma}_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega_r) = \dim \mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega)}$$

Proof

$$\mathsf{H}^1_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \nabla u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\} \qquad \mathsf{R}_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \mathrm{rot}\, u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\}$$

$$\mathsf{D}_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \mathrm{div}\, u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\} \qquad \mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega) = {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_{\mathrm{n}}}(\Omega)$$

$$\Omega_r = \Omega \cap \mathrm{U}(r), \qquad \check{\mathrm{U}}(r) = \mathbb{R}^3 \setminus \overline{\mathrm{U}(r)}, \qquad \Sigma = \partial \mathrm{B}_r(0), \qquad \hat{\Gamma}_{\mathrm{i}} = \Gamma_{\mathrm{i}} \cup \Sigma \;\; (i = 1, 2)$$

Proof

$$\mathsf{H}^1_{\mathrm{t}}(\Omega) = \Big\{\, u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\Big|\; \nabla u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \,\Big\} \qquad \mathsf{R}_{\mathrm{t}}(\Omega) = \Big\{\, u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\Big|\; \mathrm{rot}\, u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \,\Big\}$$

$$\mathsf{D}_{\mathrm{t}}(\Omega) = \Big\{\, u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\Big|\; \mathrm{div}\, u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \,\Big\} \qquad \mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega) = {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega) \cap \varepsilon^{-1}\,{}_0\mathsf{D}_{\Gamma_{\mathrm{n}}}(\Omega)$$

$$\Omega_r = \Omega \cap \mathrm{U}(r), \qquad \check{\mathrm{U}}(r) = \mathbb{R}^3 \setminus \overline{\mathrm{U}(r)}, \qquad \Sigma = \partial \mathrm{B}_r(0), \qquad \hat{\Gamma}_{\mathrm{i}} = \Gamma_{\mathrm{i}} \cup \Sigma \quad (\, i = 1, 2\,)$$

easy choice: take a basis $\mathcal{B}$ of $\mathcal{H}_{\hat{\Gamma}_1,\Gamma_{\mathrm{n}}}(\Omega_r)$ $\longrightarrow$ $\boxed{\mathrm{B}_{\mathsf{R}}(\Omega) := \widetilde{\mathcal{B}}}$ (zero extension!)

Proof

$$H^1_t(\Omega) = \left\{ u \in L^2_t(\Omega) \mid \nabla u \in L^2_{t+1}(\Omega) \right\} \qquad R_t(\Omega) = \left\{ u \in L^2_t(\Omega) \mid \operatorname{rot} u \in L^2_{t+1}(\Omega) \right\}$$

$$D_t(\Omega) = \left\{ u \in L^2_t(\Omega) \mid \operatorname{div} u \in L^2_{t+1}(\Omega) \right\} \qquad \mathcal{H}_{\Gamma_t,\Gamma_n}(\Omega) = {}_0R_{\Gamma_t}(\Omega) \cap \varepsilon^{-1}{}_0D_{\Gamma_n}(\Omega)$$

$$\Omega_r = \Omega \cap U(r), \qquad \check{U}(r) = \mathbb{R}^3 \setminus \overline{U(r)}, \qquad \Sigma = \partial B_r(0), \qquad \hat{\Gamma}_i = \Gamma_i \cup \Sigma \quad (i = 1, 2)$$

easy choice: take a basis $\mathcal{B}$ of $\mathcal{H}_{\hat{\Gamma}_1,\Gamma_n}(\Omega_r)$ $\longrightarrow$ $\boxed{B_R(\Omega) := \widetilde{\mathcal{B}}}$ (zero extension!)

$$\Longrightarrow B_R(\Omega) \subset {}_0R_{\Gamma_t}(\Omega)$$

Proof

$$\mathsf{H}^1_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \nabla u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\} \qquad \mathsf{R}_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\}$$

$$\mathsf{D}_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \operatorname{div} u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\} \qquad \mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega) = {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_{\mathrm{n}}}(\Omega)$$

$$\Omega_r = \Omega \cap \mathrm{U}(r), \qquad \check{\mathrm{U}}(r) = \mathbb{R}^3 \setminus \overline{\mathrm{U}(r)}, \qquad \Sigma = \partial \mathrm{B}_r(0), \qquad \hat{\Gamma}_{\mathrm{i}} = \Gamma_{\mathrm{i}} \cup \Sigma \;\; (i = 1, 2)$$

<u>easy choice:</u> take a basis $\mathcal{B}$ of $\mathcal{H}_{\hat{\Gamma}_1,\Gamma_{\mathrm{n}}}(\Omega_r)$ $\longrightarrow$ $\boxed{\mathrm{B}_{\mathsf{R}}(\Omega) := \widetilde{\mathcal{B}}}$ (zero extension!)

$\Longrightarrow \mathrm{B}_{\mathsf{R}}(\Omega) \subset {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega)$

$\Longrightarrow \mathrm{B}_{\mathsf{R}}(\Omega)$ is finite.

Proof

$$\mathsf{H}^1_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \nabla u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\} \qquad \mathsf{R}_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\}$$

$$\mathsf{D}_{\mathrm{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \;\middle|\; \operatorname{div} u \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega) \right\} \qquad \mathcal{H}_{\Gamma_{\mathrm{t}},\Gamma_{\mathrm{n}}}(\Omega) = {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega) \cap \varepsilon^{-1}\,{}_0\mathsf{D}_{\Gamma_{\mathrm{n}}}(\Omega)$$

$$\Omega_r = \Omega \cap \mathrm{U}(r), \qquad \check{\mathrm{U}}(r) = \mathbb{R}^3 \setminus \overline{\mathrm{U}(r)}, \qquad \Sigma = \partial \mathrm{B}_r(0), \qquad \hat{\Gamma}_{\mathrm{i}} = \Gamma_{\mathrm{i}} \cup \Sigma \;\; (i = 1, 2)$$

easy choice: take a basis $\mathcal{B}$ of $\mathcal{H}_{\hat{\Gamma}_1,\Gamma_{\mathrm{n}}}(\Omega_r)$ $\longrightarrow$ $\boxed{\mathrm{B}_{\mathsf{R}}(\Omega) := \widetilde{\mathcal{B}}}$ (zero extension!)

$\Longrightarrow$ $\mathrm{B}_{\mathsf{R}}(\Omega) \subset {}_0\mathsf{R}_{\Gamma_{\mathrm{t}}}(\Omega)$

$\Longrightarrow$ $\mathrm{B}_{\mathsf{R}}(\Omega)$ is finite.

$\Longrightarrow$ elements in $\mathrm{B}_{\mathsf{R}}(\Omega)$ are compactly supported.

Proof

$$\mathsf{H}^1_{\mathsf{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathsf{t}}(\Omega) \mid \nabla u \in \mathsf{L}^2_{\mathsf{t}+1}(\Omega) \right\} \qquad \mathsf{R}_{\mathsf{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathsf{t}}(\Omega) \mid \operatorname{rot} u \in \mathsf{L}^2_{\mathsf{t}+1}(\Omega) \right\}$$

$$\mathsf{D}_{\mathsf{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathsf{t}}(\Omega) \mid \operatorname{div} u \in \mathsf{L}^2_{\mathsf{t}+1}(\Omega) \right\} \qquad \mathcal{H}_{\Gamma_{\mathsf{t}},\Gamma_{\mathsf{n}}}(\Omega) = {}_0\mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \varepsilon^{-1}\,{}_0\mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)$$

$$\Omega_r = \Omega \cap \mathrm{U}(r), \qquad \check{\mathrm{U}}(r) = \mathbb{R}^3 \setminus \overline{\mathrm{U}(r)}, \qquad \Sigma = \partial \mathrm{B}_r(0), \qquad \hat{\Gamma}_{\mathrm{i}} = \Gamma_{\mathrm{i}} \cup \Sigma \quad (i = 1, 2)$$

<u>easy choice:</u> take a basis $\mathcal{B}$ of $\mathcal{H}_{\hat{\Gamma}_1,\Gamma_{\mathsf{n}}}(\Omega_r)$ $\longrightarrow$ $\boxed{\mathrm{B}_{\mathsf{R}}(\Omega) := \widetilde{\mathcal{B}}}$ (zero extension!)

$\Longrightarrow$ $\mathrm{B}_{\mathsf{R}}(\Omega) \subset {}_0\mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega)$

$\Longrightarrow$ $\mathrm{B}_{\mathsf{R}}(\Omega)$ is finite.

$\Longrightarrow$ elements in $\mathrm{B}_{\mathsf{R}}(\Omega)$ are compactly supported.

$\Longrightarrow$ projections along $\nabla \mathsf{H}^1_{-1,\Gamma_{\mathsf{t}}}(\Omega)$ form a basis of $\mathcal{H}_{\Gamma_{\mathsf{t}},\Gamma_{\mathsf{n}}}(\Omega)$.

Proof

$$\mathsf{H}^1_{\mathsf{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathsf{t}}(\Omega) \;\middle|\; \nabla u \in \mathsf{L}^2_{\mathsf{t}+1}(\Omega) \right\} \qquad \mathsf{R}_{\mathsf{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathsf{t}}(\Omega) \;\middle|\; \operatorname{rot} u \in \mathsf{L}^2_{\mathsf{t}+1}(\Omega) \right\}$$

$$\mathsf{D}_{\mathsf{t}}(\Omega) = \left\{ u \in \mathsf{L}^2_{\mathsf{t}}(\Omega) \;\middle|\; \operatorname{div} u \in \mathsf{L}^2_{\mathsf{t}+1}(\Omega) \right\} \qquad \mathcal{H}_{\Gamma_{\mathsf{t}},\Gamma_{\mathsf{n}}}(\Omega) = {}_0\mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_{\mathsf{n}}}(\Omega)$$

$$\Omega_r = \Omega \cap \mathrm{U}(r), \qquad \check{\mathrm{U}}(r) = \mathbb{R}^3 \setminus \overline{\mathrm{U}(r)}, \qquad \Sigma = \partial \mathrm{B}_r(0), \qquad \hat{\Gamma}_{\mathsf{i}} = \Gamma_{\mathsf{i}} \cup \Sigma \quad (i = 1, 2)$$

easy choice: take a basis $\mathcal{B}$ of $\mathcal{H}_{\hat{\Gamma}_1,\Gamma_n}(\Omega_r)$ $\longrightarrow$ $\boxed{\mathrm{B}_{\mathsf{R}}(\Omega) := \widetilde{\mathcal{B}}}$ (zero extension!)

$\Longrightarrow$ $\mathrm{B}_{\mathsf{R}}(\Omega) \subset {}_0\mathsf{R}_{\Gamma_{\mathsf{t}}}(\Omega)$

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$\Longrightarrow$ elements in $\mathrm{B}_{\mathsf{R}}(\Omega)$ are compactly supported.

$\Longrightarrow$ projections along $\nabla \mathsf{H}^1_{-1,\Gamma_{\mathsf{t}}}(\Omega)$ form a basis of $\mathcal{H}_{\Gamma_{\mathsf{t}},\Gamma_{\mathsf{n}}}(\Omega)$.

Lemma

It holds:

$$\mathrm{B}_{\mathsf{R}}(\Omega)^{\perp} \cap \mathcal{H}_{\Gamma_{\mathsf{t}},\Gamma_{\mathsf{n}}}(\Omega) = \{0\}.$$