LOCAL THB-MULTIGRID

Fast solvers for large-scale locally refined IgA-systems.

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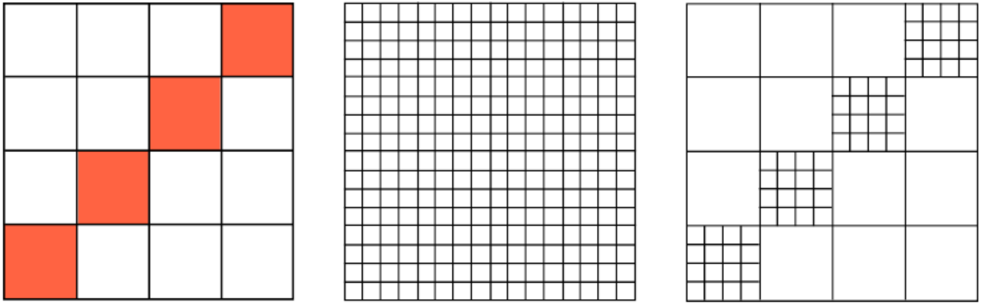
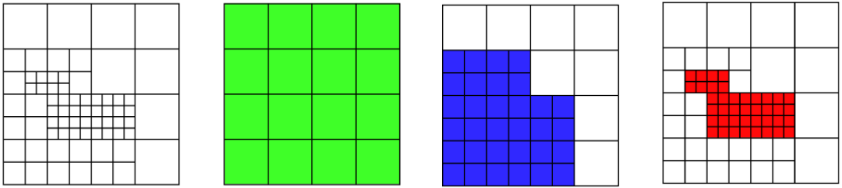
THB-SPLINES

Discretization based on B-splines



Hierarchical meshes



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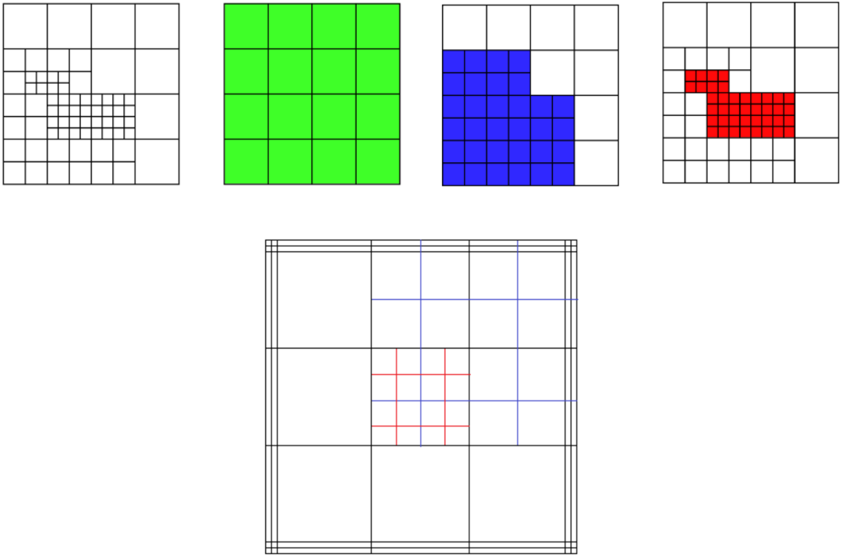
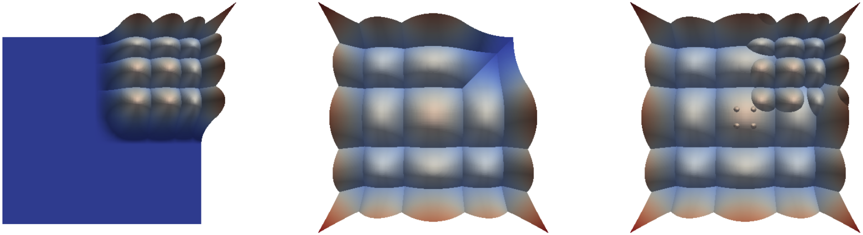


THB-splines: Dropping the tensor product structure



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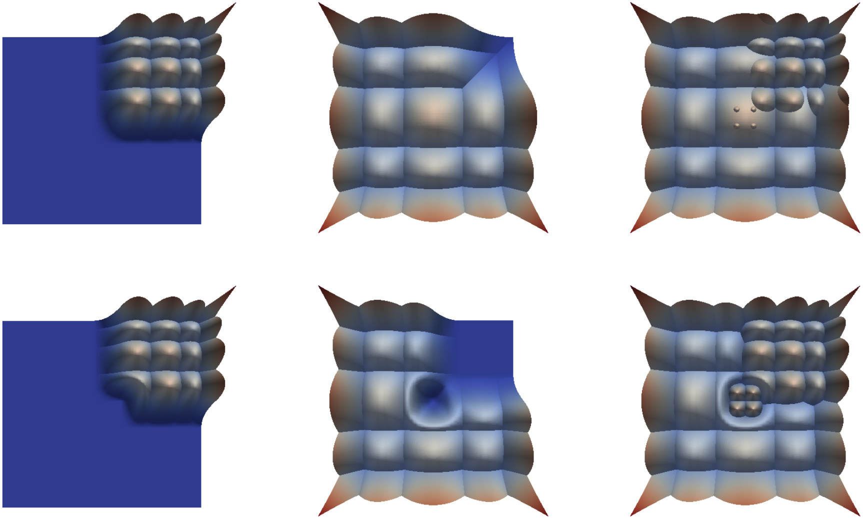

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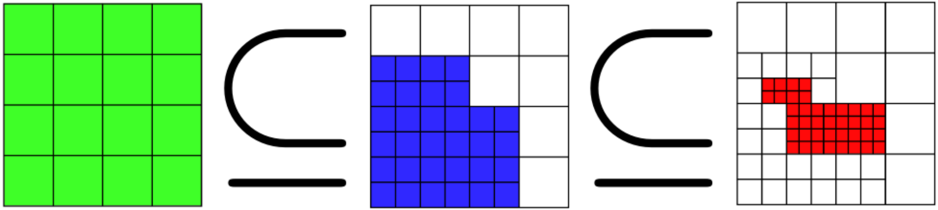

THB-splines: Important properties

- **non-negative**
- **linearly independent**
- form a **partition of unity**
- **nested enlargement**
- **strongly stable** with respect to the L^∞-norm

The model problem

- Let $V = H_0^1(D), D = (0,1)^2$.

Variational problem

Find $u \in V$ such that

$$a(u, v) = \langle F, v \rangle, \qquad \forall v \in V.$$

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- Assume $F \in V^*$ and $a(.,.) : V \times V \to \mathbb{R}$ is **bilinear, symmetric, continuous and coercive** $\implies$ well-posedness

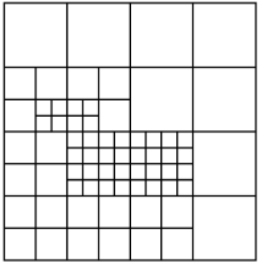
The model scheme

- Let $\mathbb{T} \subset V$ be a conforming THB-discretization.

THB-scheme

Find $u \in \mathbb{T}$ such that

$$a(u, v) = \langle F, v \rangle, \qquad \forall v \in \mathbb{T}.$$



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SUBSPACE CORRECTION METHODS FOR THB-SPLINES

Iterative solving

- **Iterative procedure** to solve **THB-scheme** for some initial guess $u^0 \in \mathbb{T}$.

Successive corrections

If $u^{l-1} \in \mathbb{T}$ is given then we can define $u^l = u^{l-1} + \hat{e}$ where $\hat{e} \in \mathbb{T}$ is an **approximate solution** of the residual equation

$$a(e, v) = \langle F, v \rangle - a(u^{l-1}, v), \qquad \forall v \in \mathbb{T}.$$

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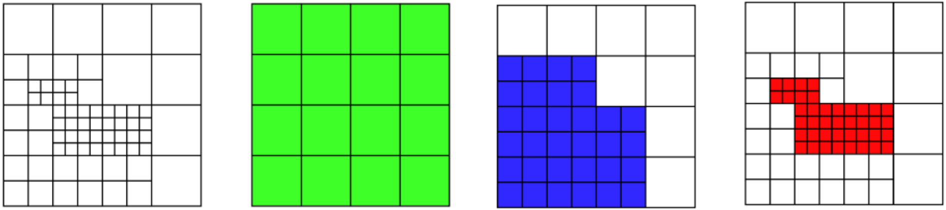
- Residual equation in general as difficult to solve as the original problem.

Divide and conquer.

- Let $\mathbb{T}_i \subseteq \mathbb{T}, i = 1, 2, \ldots, J$ be **subspaces** with

$$\mathbb{T} = \sum_{i=1}^{J} \mathbb{T}_i.$$



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- Let $\mathbb{T}_i \subseteq \mathbb{T}, i = 1, 2, \ldots, J$ be **subspaces** with

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- Let $a_i(.,.)$ be **continuous and coercive** approximations of $a(.,.)$ restricted on $\mathbb{T}_i \implies$ well-posedness of the subspace problems

SSC: Successive subspace corrections

Algorithm SSC INPUT: $u^0 \in \mathbb{T}$

for $l = 1, 2, \ldots$

$u_0^{l-1} = u^{l-1}$

for $i = 1, \ldots, J$

Let $e_i \in \mathbb{T}_i$ solve

$$a_i(e_i, v_i) = \langle F, v_i \rangle - a(u_{i-1}^{l-1}, v_i) \qquad \forall v_i \in \mathbb{T}_i$$

$u_i^{l-1} = u_{i-1}^{l-1} + e_i$

endfor

$u^l = u_J^{l-1}$

endfor

Algorithm Symmetric SSC INPUT: $u^0 \in \mathbb{T}$

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Equivalence of Symmetric SSC and Multigrid V-cycle

Theorem ((Xu, 1992))

*The **Symmetric SSC method** and the **multigrid V-cycle** (eg. ref. (Hackbusch, 1985)) are equivalent.*

*Ie., the error propagation operator (**EPO**) of the **Multigrid V-cycle** coincides with the EPO of the **symmetric SSC method**.*

Theorem ((Xu, 1992))

The EPO of the Symmetric SSC is self-adjoint (wrt. $a(.,.)$) and is the product of the EPO of the SSC method and its adjoint (wrt. $a(.,.)$)

Error propagation operators

- Let $T_i : \mathbb{T} \to \mathbb{T}_i$ be defined by

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- By the well-posedness of the subspace problems one sees, that $T_i|_{\mathbb{T}_i} : \mathbb{T}_i \to \mathbb{T}_i$ is **isomorphic.**
- If $a_i(.,.) = a(.,.)$ one has $T_i = P_i$, where P_i is the **energy projection**,

$$a(P_i v, v_i) = a(v, v_i), \qquad v \in \mathbb{T}, v_i \in \mathbb{T}_i.$$

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- From the definition of SSC it follows, that $u - u_i^{l-1} = (I - T_i)(u - u_{i-1}^{l-1})$, hence,

$$u - u^l = E(u - u^{l-1}) = \ldots = E^l(u - u^0), \qquad E = (I - T_J)(I - T_{J-1}) \cdots (I - T_1),$$

ie. SSC is the **multiplicative Schwarz method.**

Assumptions for proving uniform convergence

By the symmetry of $a(.,.)$ we have the **energy norm** $\|.\|_a := a(.,.)^{1/2}$.

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Contraction of subspace error operators

$$\exists \rho < 1 \forall i \in \{1, \dots, J\} : \|I - T_i\|_{a_i} \leq \rho \quad \text{where} \quad \|.\|_{a_i}^2 := a_i(.,.).$$

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This assumption admits the application of the **Xu-Zikatanov identity** (Xu and Zikatanov, 2002).

Stable decomposition

For any $v \in \mathbb{T}$ there exists a decomposition $v = \sum_{i=1}^{J} v_i, v_i \in \mathbb{T}_i$ such that

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Strengthened Cauchy-Schwarz (SCS) inequality

For any $u_i, v_i \in \mathbb{T}_i$

$$|\sum_{i=1}^{J} \sum_{j=i+1}^{J} a(u_i, v_j)| \leq K_2 \Big(\sum_{i=1}^{J} \|u_i\|_a^2 \Big)^{1/2} \Big(\sum_{j=1}^{J} \|v_i\|_a^2 \Big)^{1/2}.$$

Convergence of SSC

Theorem (Chen et al., 2012)

*Let $\mathbb{T} = \sum \widetilde{\mathbb{T}}_i$ be a space decomposition that is **stable** and satisfies the **SCS inequality** and let the subspace error operators $T_i : \mathbb{T} \to \widetilde{\mathbb{T}}_i$ be **contractions**. Then one has*

$$\|E\|_a^2 \leq 1 - \frac{1-\rho^2}{2K_1(1+(1+\rho)^2 K_2^2)}.$$

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This means, that SSC **converges uniformly** with a constant that is independent of J. In other words, a **symmetrized SSC-preconditioned conjugate gradient method** is h-robust.

FINDING A SUITABLE DECOMPOSITION

Admissible grids

Definition

For each point $x \in D$ in the **computational domain** $D = (0,1)^d$, let $\delta_{\boldsymbol{x}}$ be the largest difference between the levels of THB-basis functions that are non-zero in x. The **mesh level disparity** δ is defined as

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Fixing the mesh level disparity yields so-called **admissible THB-spline spaces**:

- The number of non-zero basis functions in any $x \in D$ is **uniformly bounded.**
- The (non-empty) measures of the supports of two THB-basis functions is **uniformly equivalent** to the measure of their intersection.

- **Stability and the SCS inequality** for an arbitrary THB-spline space $\mathbb{T}$ does not hold for any decomposition

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- **"Uniformly relate"** measures of supports between basis functions in $\mathbb{T}_i$ supported in any $x \in D$ (ie. by uniformly bounding δ).
- **"Uniformly relate"** measures of supports of basis functions in "$\mathbb{T}_{i+1} \setminus \mathbb{T}_i$" (which cannot be guaranteed by the adaptive method).

Virtual hierarchy

Definition

The **virtual decomposition** of $\mathbb{T}$ is defined as

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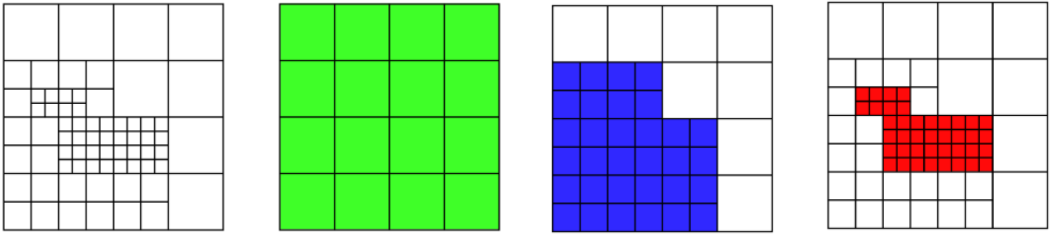
This construction **preserves the mesh level disparity** of $\mathbb{T}$ for all $\mathbb{T}_i$.



$$\mathbb{T}_{\text{new}} = \mathbb{T}_1 + \mathbb{T}_2 + \mathbb{T}_3$$

Enforcing locality

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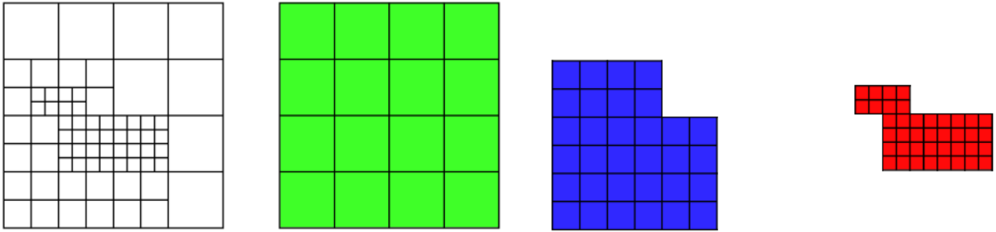
- In order to obtain **stability and SCS** for this decomposition the **ordering of the subspaces** $\widetilde{\mathbb{T}}_i$ is important.



$$\mathbb{T}_{\text{new}} = \widetilde{\mathbb{T}}_1 + \widetilde{\mathbb{T}}_2 + \widetilde{\mathbb{T}}_3$$

Stability of the decomposition

Theorem

For any $v \in \mathbb{T}$ *there exist* $v_i \in \widetilde{\mathbb{T}}_i$ *such that* $v = \sum_{i=1}^{J} v_i$ *and*

$$\sum_{i=1}^{J} \|v_i\|_a^2 \leq C\|v\|_a^2,$$

with C *independent of* J.

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Proof.

Based on the result for B-splines (Buffa et al., 2013), the mentioned properties of the **hierarchical QI** and the **discrete Hardy-inequality**. □

SCS inequality

Theorem

Let $u_i, v_i \in \widetilde{\mathbb{T}}_i$*. Then we have*

$$|\sum_{i=1}^{J} \sum_{j=i+1}^{J} a(u_i, v_j)| \leq C\Big(\sum_{i=1}^{J} \|u_i\|_a^2\Big)^{1/2} \Big(\sum_{j=1}^{J} \|v_j\|_a^2\Big)^{1/2},$$

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with C *independent of* J*.*

Proof.

Based on the result for B-splines (Cho and Vazquez, 2018) and lengthy, but elementary manipulations. □

Computational complexity

- governed by the **computational complexity of the subspace solvers**.
- **Uniform sparsity** of subspace systems on $\widetilde{\mathbb{T}}_i \implies$ using Gauss-Seidel relaxations as "solver" takes $\mathcal{O}(|\widetilde{\mathbb{T}}_i|)$ iterations.

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- **Optimal complexity**

Violation of the subspace error operator contraction assumption

Contraction of subspace error operators

$$\exists \rho < 1 \forall i \in \{1, \ldots, J\} : \|I - T_i\|_{a_i} \leq \rho \quad \text{where} \quad \|.\|_{a_i}^2 := a_i(.,.).$$

- **No uniform bound** for **Gauss-Seidel relaxations**

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- **No uniform bound** for **Gauss-Seidel relaxations**
- **Xu-Zikatanov formula** admits weaker assumption:

Weak contraction assumption

$$\exists \omega \in (0, 2) \forall v \in \mathbb{T} : \|T_i v\|_a^2 \leq \omega a(T_i v, v).$$

- **Remark:** $\|I - T_i\|_a \leq 1 \leftrightarrow a(T_i v, T_i v) \leq 2a(T_i v, v), v \in \mathbb{T}$

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- **Elementary computations from Xu-Zikatanov formula** yield convergence, if one can **show l^2-L^2-stability of THB-splines**

$$\exists c \neq c(|\mathbb{T}|) > 0 \forall w = \sum_{\tau \in \mathcal{T}} c_\tau(w)\tau \in \mathbb{T} : \sum_{\tau \in \mathcal{T}} |c_\tau(w)|^2 \leq c\|w\|_0^2.$$

Conclusion and future work

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- **Convergence of subspace correction methods** (for spectrally equivalent subspace solvers)

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- l^2-L^2**-stability of THB-splines** $\implies$ local multigrid convergence
- Convergence analysis for **jumping coefficients**
- **Nested iteration schemes**
- Application to **mixed problems**