Convergence analysis of the Adini element on a Shishkin mesh for a singularly perturbed fourth-order problem in two dimensions

Xiangyun Meng

Joint work with: Martin Stynes

Beijing Computational Science Research Center

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Talk overview

Plate bending problem
- A fourth-order problem
- A fourth-order singularly perturbed problem

Finite element method
- Conforming finite element method
- Nonconforming finite element method

Adini element for a fourth-order singularly perturbed problem
- Boundary-layer structure of typical solutions
- Convergence results
- Numerical results

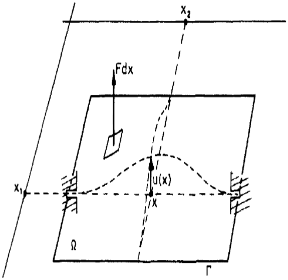
Outline

A fourth-order problem: plate bending problem

Boundary value problem:

$$\Delta^2 u(x,y) = f(x,y) \quad \text{in } \Omega := (0,1)^2,$$
$$u(x,y) = \frac{\partial u(x,y)}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

A linear model for a clamped thin elastic plate.
$u(x,y)$: displacement of the plate.
$f(x,y)$: transverse loading.



A fourth-order singularly perturbed problem

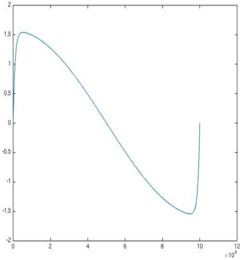
Boundary value problem:

$$\varepsilon^2 \Delta^2 u(x,y) - \Delta u(x,y) = f(x,y) \quad \text{in } \Omega := (0,1)^2,$$
$$u(x,y) = \frac{\partial u(x,y)}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where $0 < \varepsilon \ll 1$ and $f(x,y) \in L^2(\Omega)$ is a smooth function.
ε^2 is the ratio of bending rigidity to tensile stiffness in the plate.

Aim and difficulty

- Aim: solve the problem by robust numerical method that work for all values of the singular perturbation parameter ε, even it is very small.
- Difficulty: layers in derivative of typical solution (1D graph, $\varepsilon = 10^{-2}$)



Weak form of the boundary value problem

Weak form: Find $u \in H_0^2(\Omega)$, such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^2(\Omega).$$

Bilinear form $a(w, v)$ defined as:

$$a(w, v) := \varepsilon^2 \sum_{i,j=1}^{2} \left(\frac{\partial^2 w}{\partial x_i \partial x_j}, \frac{\partial^2 v}{\partial x_i \partial x_j} \right) + (\nabla w, \nabla v) \quad \forall w, v \in H^2(\Omega).$$

The following semi-norm is naturally associated with the problem:

$$\|v\|_\varepsilon := \left(\varepsilon^2 |v|_2^2 + |v|_1^2 \right)^{1/2} \quad \forall v \in H_0^2(\Omega);$$

by the Poincaré inequality it is a norm on $H_0^2(\Omega)$.

Outline

Conforming finite element method

- For fourth order problems, conforming finite element space $V_h \subseteq H^2$, which means $V_h \subseteq C^1$.
- For triangular mesh, Argyris element, 21 parameters, polynomials of degree 5.
- For tetrahedral mesh, Zenicek element, 220 parameters, polynomials of degree 9.
- We are interested in decreasing the number of nodal parameters and the degree of polynomials.

Nonconforming finite element method

- For fourth order problems, nonconforming finite element space $V_h \not\subseteq H^2$, which means $V_h \not\subseteq C^1$.
- Do not introduce new variables.
- Decrease the number of nodal parameters and the degree of polynomials.
- We need to analysis more items (consistency error).

A nonconforming element: Adini element

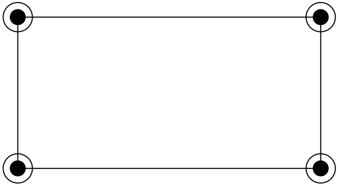
- element K is a rectangle.
- shape function space is

$$
\begin{aligned}
P_K &= Q_1(K) \oplus x^2 Q_1(K) \oplus y^2 Q_1(K) \\
&= P_3(K) \oplus \{xy^3, x^3 y\},
\end{aligned}
$$

where $Q_1 = \{x^i y^j | \ i, j \leq 1\}$ and $P_3 = \{x^i y^j | \ i + j \leq 3\}$.

- degrees of freedom are

$$
\mathbf{D}_K(v) = \left\{ v(a_i), \frac{\partial v(a_i)}{\partial x}, \frac{\partial v(a_i)}{\partial y}, i = 1, 2, 3, 4 \right\}.
$$



Anisotropic interpolation error estimates

Lemma

Let Π_N be the interpolation operator using the degrees of freedom of Adini element, $K = 2h_1 \times 2h_2$, $\phi \in H^3(K)$. Not only

$$|\phi - \Pi_N \phi|_{1,K} \leq Ch^2 |\phi|_{3,K} ,$$

but also we have

$$\left\| \frac{\partial(\phi - \Pi_N \phi)}{\partial x} \right\|_{0,K} \leq C \left(h_1^2 \left\| \frac{\partial^3 \phi}{\partial x^3} \right\|_{0,K} + h_1 h_2 \left\| \frac{\partial^3 \phi}{\partial x^2 \partial y} \right\|_{0,K} + h_2^2 \left\| \frac{\partial^3 \phi}{\partial x \partial y^2} \right\|_{0,K} \right).$$

Stability of the Adini interpolation operator

Define

$$\|f\|_{1,\infty,K} := \max_{|\alpha|\leq 1} \|D_w^\alpha f\|_{L^\infty(K)}.$$

Lemma

There exists a constant C, which is independent of K and ϕ, such that

$$\|\Pi_N \phi\|_{1,\infty,K} \leq C\|\phi\|_{1,\infty,K}$$

for each mesh rectangle K and each $\phi \in C^1(K)$.

Thanks to the Adini element's symmetrical nature.

Outline

Behaviour of the solution

Theorem

There is a constant C, independent of ε and f, such that

$$|u|_2 \leq C\varepsilon^{-1/2}\|f\|_0 \quad \textit{and} \quad |u|_3 \leq C\varepsilon^{-3/2}\|f\|_0$$

for all $f \in L^2$.

To give a sketch of proof, we need the following reduced problems:
The reduced equation:

$$\begin{aligned} -\Delta u^0(x,y) &= f(x,y) \quad \text{in } \Omega := (0,1)^2, \\ u^0(x,y) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The reduced weak form: Find $u^0 \in H_0^1(\Omega)$, such that

$$(\nabla u^0, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

Behaviour of the solution: sketch of proof

Subtract reduced equation from fourth order singularly perturbed equation:

$$\Delta^2 u(x, y) = \varepsilon^{-2}\Delta(u - u^0),$$

then

$$\|u\|_3 \leq C\varepsilon^{-2}|u - u^0|_1.$$

Subtract reduced weak form from fourth order singularly perturbed weak form:

$$\varepsilon^2(\Delta u, \Delta u) + (\nabla(u - u^0), \nabla(u - u^0)) = -\varepsilon^2 \int_{\partial\Omega} \Delta u \frac{\partial u^0}{\partial n} \mathrm{d}s - \varepsilon^2(\Delta u, f),$$

then

$$\varepsilon^2|u|_2^2 + |u - u^0|_1^2 \leq C\varepsilon\|f\|_0^2.$$

Above all

$$|u|_2 + \varepsilon|u|_3 \leq C\varepsilon^{-1/2}\|f\|_0.$$

Solution decomposition

The boundary value problem has a solution u which can be decomposed as

$$u = S + \sum_{i=1}^{4} E_i + E_{12} + E_{23} + E_{34} + E_{41},$$

S: smooth part;
E_i: boundary layer part along side i of $\bar{\Omega}$;
E_{ij}: corner layer part at the corner (i, j).

Remark

We will assume the pointwise bounds from our experience with second-order singularly perturbed problems.
The pointwise bounds of Assumption are reasonable, because they are consistent with the Sobolev-norm global estimates.

Solution decomposition

There exists a constant C such that

$$\left\| \frac{\partial^{i+j} S}{\partial x^i \partial y^j} \right\|_{0,\Omega} \leq C,$$

and for all $(x, y) \in \overline{\Omega}$ one has

$$\left| \frac{\partial^{i+j} E_1(x,y)}{\partial x^i \partial y^j} \right| \leq C\varepsilon^{1-j} e^{-y/\varepsilon},$$

$$\left| \frac{\partial^{i+j} E_2(x,y)}{\partial x^i \partial y^j} \right| \leq C\varepsilon^{1-i} e^{-x/\varepsilon},$$

$$\left| \frac{\partial^{i+j} E_{12}(x,y)}{\partial x^i \partial y^j} \right| \leq C\varepsilon^{1-i-j} e^{-x/\varepsilon} e^{-y/\varepsilon},$$

for $0 \leq i + j \leq 4$ and similarly for the remaining components of the decomposition.

Shishkin mesh

Define the transition point parameter (positive constant α will be chosen later):

$$\lambda = \alpha\varepsilon \ln N.$$

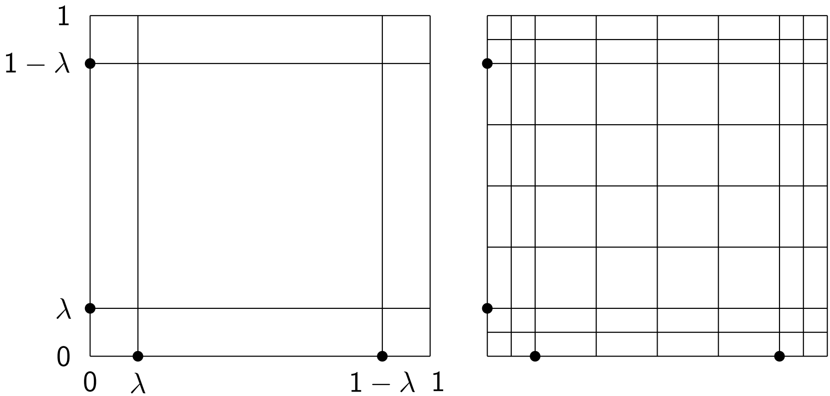


Figure: A rectangular Shishkin mesh with $N = 8$

Finite element problem

Adini finite element discretisation is: find $u_N \in V_{N0}$ such that

$$a_N(u_N, v_N) = (f, v_N) \quad \forall v_N \in V_{N0},$$

where we define

$$a_N(w_N, v_N) := \sum_{K \in \mathcal{T}_N} \left[\varepsilon^2 \sum_{i,j=1}^{2} \left(\frac{\partial^2 w_N}{\partial x_i \partial x_j}, \frac{\partial^2 v_N}{\partial x_i \partial x_j} \right)_K + (\nabla w_N, \nabla v_N)_K \right].$$

For any function v defined on Ω that lies in $H^2(K)$ for all $K \in \mathcal{T}_N$, define "broken" semi-norms by

$$|v|_{1,N}^2 := \sum_{K \in \mathcal{T}_N} |v|_{1,K}^2 \quad \text{and} \quad |v|_{2,N}^2 := \sum_{K \in \mathcal{T}_N} |v|_{2,K}^2,$$

and

$$\|v\|_{\varepsilon,N} := \left(\sum_{K \in \mathcal{T}_N} \|v\|_{\varepsilon,K}^2 \right)^{1/2} \quad \text{where} \quad \|v\|_{\varepsilon,K}^2 := \varepsilon^2 |v|_{2,K}^2 + |v|_{1,K}^2.$$

The second Strang Lemma

Lemma

There exists a constant C such that

$$\|u - u_N\|_{\varepsilon,N} \leq C \left(\inf_{v_N \in V_{N0}} \|u - v_N\|_{\varepsilon,N} + \sup_{w_N \in V_{N0}} \frac{|F_N(u, w_N)|}{\|w_N\|_{\varepsilon,N}} \right),$$

where

$$F_N(u, w_N) := \varepsilon^2 \sum_{K \in \mathcal{T}_N} \int_{\partial K} \frac{\partial^2 u}{\partial n_K^2} \frac{\partial w_N}{\partial n_K} \, \mathrm{d}s,$$

with t_K, n_K denote unit tangential and normal vectors on ∂K.

Approximation error estimate

Theorem

There exists a constant C, which is independent of ε and N, such that

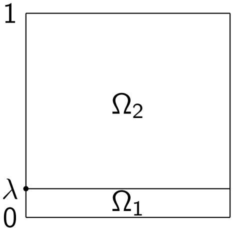
$$\inf_{v_N \in V_{N0}} \|u - v_N\|_{\varepsilon,N} \le \|u - \Pi_N u\|_{\varepsilon,N}$$
$$\le C\left[\varepsilon^{1/2}(N^{-1}\ln N)^2 + \varepsilon N^{1-\alpha} + N^{-\alpha} + N^{-3}\right],$$

choose $\alpha \ge 3$ then

$$\inf_{v_N \in V_{N0}} \|u - v_N\|_{\varepsilon,N} \le C\left[\varepsilon^{1/2}(N^{-1}\ln N)^2 + N^{-3}\right].$$

Approximation error estimate: sketch of proof

$$\|E_1 - \Pi_N E_1\|^2_{\varepsilon,N} = \|E_1 - \Pi_N E_1\|^2_{\varepsilon,N,\Omega_1} + \|E_1 - \Pi_N E_1\|^2_{\varepsilon,N,\Omega_2}$$



- $\|E_1 - \Pi_N E_1\|^2_{\varepsilon,N,\Omega_1}$: Anisotropic interpolation error estimates of the Adini interpolation operator.
- $\|E_1 - \Pi_N E_1\|^2_{\varepsilon,N,\Omega_2} \leq \|E_1\|^2_{\varepsilon,\Omega_1} + \|\Pi_N E_1\|^2_{\varepsilon,N,\Omega_1}$: Stability of the Adini interpolation operator.

Consistency error estimate

Theorem

There exists a constant C, which is independent of ε and N, such that

$$\sup_{w_N \in V_{N0}} \frac{|F_N(u, w_N)|}{\|w_N\|_{\varepsilon,N}} \leq C \min\left\{\varepsilon^{1/2}, \varepsilon^{-3/2} N^{-2}\right\}.$$

Consistency error estimate: sketch of proof

Use the properties of Adini element (weak continuous and symmetry), we have second order convergence rate:

$$
\begin{aligned}
F_N(u, w_N) &= \varepsilon^2 \sum_{K \in \mathcal{T}_N} \int_{\partial K} \frac{\partial^2 u}{\partial n_K^2} \frac{\partial w_N}{\partial n_K} \, \mathrm{d}s \\
&\leq C\varepsilon^2 \sum_{K \in \mathcal{T}_N} \left(h_1^2 \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{0,K} \left\| \frac{\partial^2 w_N}{\partial x^2} \right\|_{0,K} + h_1^2 \left\| \frac{\partial^3 u}{\partial x \partial y^3} \right\|_{0,K} \left\| \frac{\partial w_N}{\partial x \partial y} \right\|_{0,K} \right. \\
&\qquad \left. + h_2^2 \left\| \frac{\partial^4 u}{\partial x^4} \right\|_{0,K} \left\| \frac{\partial^2 w_N}{\partial y^2} \right\|_{0,K} + h_2^2 \left\| \frac{\partial^3 u}{\partial x^3 \partial y} \right\|_{0,K} \left\| \frac{\partial^2 w_N}{\partial x \partial y} \right\|_{0,K} \right).
\end{aligned}
$$

Then for singularly perturbed problem on Shishkin mesh, we have:

$$
\begin{aligned}
\frac{|F_N(u, w_N)|}{\|w_N\|_{\varepsilon,N}} &\leq C\varepsilon \left[\sum_{K \in \mathcal{T}_N} \left(h_1^4 \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{0,K}^2 + h_1^4 \left\| \frac{\partial^4 u}{\partial x \partial y^3} \right\|_{0,K}^2 \right. \right. \\
&\qquad \left. \left. + h_2^4 \left\| \frac{\partial^4 u}{\partial x^4} \right\|_{0,K}^2 + h_2^4 \left\| \frac{\partial^4 u}{\partial x^3 \partial y} \right\|_{0,K}^2 \right) \right]^{1/2} \\
&\leq C\varepsilon^{-3/2} N^{-2}.
\end{aligned}
$$

In standard estimate, $\varepsilon^{-3/2} \gg 1$ when $\varepsilon \ll 1$.

Consistency error estimate: sketch of proof

Use inverse estimates, to make the bound don't have $\varepsilon^{-3/2}$:

$$
\begin{aligned}
&F_N(u, w_N)\\
&= \varepsilon^2 \sum_{K\in\mathcal{T}_N} \sum_{i=1}^{2} \int_K \frac{\partial}{\partial x_i} \left[\left(\frac{\partial^2 u}{\partial n_K^2} - \pi_0 \frac{\partial^2 u}{\partial n_K^2} \right) \left(\frac{\partial w_N}{\partial x_i} - \pi_K \frac{\partial w_N}{\partial x_i} \right) \right] \mathrm{d}x_1\, \mathrm{d}x_2\\
&\le C\varepsilon^2 \sum_{K\in\mathcal{T}_N} \sum_{i=1}^{2} \sum_{j=1}^{2} h_j \left(\left\| \frac{\partial^3 u}{\partial x_i^3} \right\|_{0,K} \left\| \frac{\partial^2 w_N}{\partial x_i \partial x_j} \right\|_{0,K} + \left\| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right\|_{0,K} \left\| \frac{\partial^2 w_N}{\partial x_i^2} \right\|_{0,K} \right)\\
&\le C\varepsilon^2 \sum_{K\in\mathcal{T}_N} \left(\left\| \frac{\partial^3 u}{\partial x^3} \right\|_{0,K} \left\| \frac{\partial w_N}{\partial x} \right\|_{0,K} + \left\| \frac{\partial^3 u}{\partial y^3} \right\|_{0,K} \left\| \frac{\partial w_N}{\partial y} \right\|_{0,K} \right.\\
&\qquad \left. + h_1 \left\| \frac{\partial^3 u}{\partial y^2 \partial x} \right\|_{0,K} \left\| \frac{\partial^2 w_N}{\partial y^2} \right\|_{0,K} + h_2 \left\| \frac{\partial^3 u}{\partial x^2 \partial y} \right\|_{0,K} \left\| \frac{\partial^2 w_N}{\partial x^2} \right\|_{0,K} \right)
\end{aligned}
$$

Consistency error estimate: sketch of proof

Follow last slide, we have

$$\begin{aligned}\frac{|F_N(u,w_N)|}{\|w_N\|_{\varepsilon,N}} &\le C\varepsilon^2\left(\left\|\frac{\partial^3 u}{\partial x^3}\right\|_{0,\Omega}+\left\|\frac{\partial^3 u}{\partial y^3}\right\|_{0,\Omega}\right)\\ &\quad + C\varepsilon\left[\sum_{K\in\mathcal{T}_N}\left(h_1^2\left\|\frac{\partial^3 u}{\partial y^2\partial x}\right\|_{0,K}^2+h_2^2\left\|\frac{\partial^3 u}{\partial x^2\partial y}\right\|_{0,K}^2\right)\right]^{1/2}\\ &\le C\varepsilon^{1/2}.\end{aligned}$$

Above all, we have

$$\sup_{w_N\in V_{N0}}\frac{|F_N(u,w_N)|}{\|w_N\|_{\varepsilon,N}}\le C\min\left\{\varepsilon^{1/2},\varepsilon^{-3/2}N^{-2}\right\}.$$

Convergence result

Theorem

There exists a constant C, which is independent of ε and N, such that

$$\|u - u_N\|_{\varepsilon,N} \leq C\left[\varepsilon^{1/2}(N^{-1}\ln N)^2 + \varepsilon N^{1-\alpha} + \min\left\{\varepsilon^{1/2}, \varepsilon^{-3/2}N^{-2}\right\} + N^{-\alpha} + N^{-3}\right].$$

Assume that $\alpha \geq 3$. Then there exists a constant C, which is independent of ε and N, such that

$$\|u - u_N\|_{\varepsilon,N} \leq \begin{cases} C\left(\varepsilon^{1/2} + N^{-3}\right) & \text{if } \varepsilon \leq N^{-1}, \\ C\left[\varepsilon^{1/2}(N^{-1}\ln N)^2 + \varepsilon^{-3/2}N^{-2}\right] & \text{if } \varepsilon > N^{-1}. \end{cases}$$

Numerical example 1

Let the exact solution of (1.2) is $u(x,y) = g(x)p(y)$, where

$$g(x) = \frac{1}{2}\left[\sin(\pi x) + \frac{\pi\varepsilon}{1-e^{-1/\varepsilon}}\left(e^{-x/\varepsilon} + e^{(x-1)/\varepsilon} - 1 - e^{-1/\varepsilon}\right)\right]$$

$$p(y) = 2y(1-y^2)+\varepsilon\left[ld(1-2y) - 3\frac{q}{l} + \left(\frac{3}{l} - d\right)e^{-y/\varepsilon} + \left(\frac{3}{l} + d\right)e^{(y-1)/\varepsilon}\right]$$

with $l = 1 - e^{-1/\varepsilon}$, $q = 2 - l$ and $d = 1/(q - 2\varepsilon l)$.

Numerical example 1

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$
1.0e-01	4.62e-03	1.17e-03	2.95e-04	7.38e-05
	1.98	1.99	2.00	
1.0e-02	2.49e-02	9.37e-03	3.25.e-03	1.08e-03
	1.41	1.53	1.59	
1.0e-03	1.59e-02	8.49e-03	3.83e-03	1.40e-03
	0.90	1.15	1.45	
1.0e-04	5.77e-03	3.05e-03	1.55e-03	7.66e-04
	0.92	0.97	1.02	
1.0e-05	2.49e-03	1.01e-03	5.10e-04	2.55e-04
	1.30	0.99	1.00	
1.0e-06	1.83e-03	3.86e-04	1.64e-04	8.15e-05
	2.25	1.23	1.01	
1.0e-07	1.76e-03	2.47e-04	5.87e-05	2.60e-05
	2.83	2.07	1.17	
1.0e-08	1.75e-03	2.28e-04	3.30e-05	8.92e-06
	2.94	2.79	1.89	

Numerical example 2

Choose $f = 2\pi^2 [1 - \cos(2\pi x) \cos(2\pi y)]$. The exact solution of this problem is unknown (but it is known to contain boundary layers when ε is small), so to estimate errors and rates of convergence we use the double-mesh principle.

Numerical example 2

ε	$N = 16$	$N = 32$	$N = 64$
1.0e-01	4.37e-02	1.12e-02	2.81e-03
	1.97	1.99	
1.0e-02	1.40e-01	5.11e-02	1.78e-02
	1.46	1.52	
1.0e-03	1.02e-01	5.42e-02	2.35e-02
	0.92	1.20	
1.0e-04	4.03e-02	2.03e-02	1.04e-02
	0.99	0.97	
1.0e-05	2.20e-02	7.07e-03	3.45e-03
	1.64	1.04	
1.0e-06	1.91e-02	3.30e-03	1.14e-03
	2.53	1.53	
1.0e-07	1.88e-02	2.64e-03	4.83e-04
	2.83	2.45	
1.0e-08	1.88e-02	2.28e-03	3.55e-04
	2.87	2.85	

Reference

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Thank you!