Maximal parabolic regularity: an overview

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in tandem with Joachim Rehberg

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An abstract motivation: a quasilinear problem

$$y'(t) + \mathcal{A}(y(t))y(t) = f(t) \quad \text{in } X, \quad y(0) = y_0$$

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$$\big(\partial + \mathcal{A}(y), \gamma_0\big)y = (f, y_0)$$

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$$y_w = \mathcal{T}(w) := \big(\partial + \mathcal{A}(w), \gamma_0\big)^{-1}(f, y_0)$$

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- solution quasilinear equation $\longleftrightarrow$ fixed point of $\mathcal{T}$
- rigorous flexible practical framework

$\longrightarrow$ **maximal parabolic regularity (MPR)** for $\mathcal{A}(w)$

What to expect

1. **this talk**: abstract introduction and overview to MPR

2. **next talk** by Joachim Rehberg:
   MPR for div-grad operators for real-world problems

What to expect

1. **this talk**: abstract introduction and overview to MPR

autonomous MPR: $A \in \mathcal{L}(Y; X)$

$\downarrow$

nonautonomous MPR: $J \ni t \mapsto A(t) \in \mathcal{L}(Y(t); X)$

(important special case: $Y(t) \equiv Y$ constant)

$\downarrow$

MPR for quasilinear equations $(\mathcal{A}(w(t)) \sim A(t))$

2. **next talk** by Joachim Rehberg:
MPR for div-grad operators for real-world problems

Maximal parabolic regularity for A

- $A\colon X \supseteq_d Y \to X$ closed operator $\qquad (J = [0,T],\ 1 < s < \infty)$
- e.g. $A = -\Delta, \quad X = \mathrm{L}^2(\mathbb{R}^n), \quad Y = \mathrm{dom}_{\mathrm{L}^2(\mathbb{R}^n)}(-\Delta) = \mathrm{H}^2(\mathbb{R}^n)$

Maximal parabolic regularity for A

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A has/satisfies **maximal parabolic $\mathrm{L}^s(J;X)$-regularity**

$$\Longleftrightarrow$$

for *every* $f \in \mathrm{L}^s(J;X)$ exists *unique* $y \in \mathrm{W}^{1,s}(J;X) \cap \mathrm{L}^s(J;Y)$:

$$y'(t) + Ay(t) = f(t) \quad \text{in } X, \qquad y(0) = 0.$$

- equivalent formulation: (open mapping theorem)

$$\partial + A \in \mathscr{L}_{\mathrm{iso}}\big(\mathrm{W}_0^{1,s}(J;X) \cap \mathrm{L}^s(J;Y); \mathrm{L}^s(J;X)\big)$$

Maximal parabolic regularity: properties

- A satisfies maximal parabolic $\mathrm{L}^s(J;X)$ regularity ... ([Dor93])

  $\longrightarrow$ A satisfies MPR on $\mathrm{L}^r(J;X)$ for every $1 < r < \infty$,

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- abstract parabolic (e.g. $e^{-At}y_0 \in \operatorname{dom} A$, $\forall t > 0$)

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  . . . if X Hilbert space ([dS64])

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  ... iff X Hilbert space ([dS64, KL00])

Interpolation and perturbation

Interpolation:

- A satisfies MPR on X_i with domain Y_i, $\quad i = 0, 1$ ([HDR09])

$\longrightarrow \quad A$ satisfies MPR on $[X_1, X_0]_\theta$ with domain $[Y_1, Y_0]_\theta$

- e.g. $X_1 = \mathrm{W}^{-1,q}(\mathbb{R}^n)$, $X_0 = L^q(\mathbb{R}^n) \longrightarrow$ MPR on $\mathrm{H}^{-\theta,q}(\mathbb{R}^n)$

Perturbation: let A satisfy MPR on X

- if $\|(\partial + A)^{-1} B\| < 1$, or
- B arbitrarily ε-small relatively A-bounded: $\|Bx\|_X \leq \varepsilon \|Ax\|_X$

$\longrightarrow \quad A + B$ satisfies MPR on X ([KW01])

Interlude: nonzero initial value

- maximal regularity solutions continuous in interpolation space:

$$\mathbb{W}^{1,s}(J;X,Y) := \mathrm{W}^{1,s}(J;X) \cap \mathrm{L}^s(J;Y) \hookrightarrow \mathrm{C}\big(\overline{J};(Y,X)_{\frac{1}{s},s}\big),$$

- in fact: (factor space topology)

$$(Y,X)_{\frac{1}{s},s} := \gamma_0 \mathbb{W}^{1,s}(J;X,Y)$$

- example: $(\mathrm{H}^2(\mathbb{R}^n),\mathrm{L}^2(\mathbb{R}^n))_{\frac{1}{s},s} \doteq \mathrm{B}^{2/s'}_{2,s}(\mathbb{R}^n)$ (Besov space)

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- example: $(\mathrm{H}^2(\mathbb{R}^n), \mathrm{L}^2(\mathbb{R}^n))_{\frac{1}{s},s} \doteq \mathrm{B}^{2/s'}_{2,s}(\mathbb{R}^n)$ (Besov space)
- equivalent formulation for MPR: (lifting by e^{-At})

$$(\partial + A, \gamma_0) \in \mathscr{L}_{\mathrm{iso}}\big(\mathbb{W}^{1,s}(J; X, Y); \mathrm{L}^s(J; X) \times (Y, X)_{\frac{1}{s},s}\big)$$

Two ansatzes

How to **actually** obtain maximal parabolic regularity?

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1. operator sum ansatz: ([DG79, DV87])

$$\partial + A \quad \text{invertible on} \quad \mathrm{dom}_{\mathrm{L}^2(J;X)}(\partial) \cap \mathrm{dom}_{\mathrm{L}^2(J;X)}(A)\,?$$

   where

$$\mathrm{dom}_{\mathrm{L}^2(J;X)}(\partial) = \mathrm{H}^1_0(J;X), \qquad \mathrm{dom}_{\mathrm{L}^2(J;X)}(A) = \mathrm{L}^2(J;Y)$$

   $\longrightarrow$ functional calculus, bounded imaginary powers A^{it}

2. harmonic analysis ansatz ([Wei01])

Harmonic analysis ansatz

$$y'(t) + Ay(t) = f(t), \quad y(0) = 0$$

$$(\Leftarrow)\!\Rightarrow \quad y(t) = \int_0^t e^{-A(t-s)} f(s)\,\mathrm{d}s := \big(e^{-A\cdot} * f\big)(t)$$

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A has/satisfies **maximal parabolic** $\mathrm{L}^s(J;X)$**-regularity**

$$\Longleftrightarrow$$

$$f \mapsto Ae^{-A\,\cdot} * f \in \mathscr{L}\big(\mathrm{L}^s(J;X)\big)$$

- singular kernel: $\|Ae^{-At}\|_{\mathcal{L}(X)} \sim 1/t \quad \longrightarrow \quad$ harmonic analysis

Maximal parabolic regularity: sufficient conditions

Warning

Showing from scratch that a given operator satisfies maximal parabolic regularity is a **formidable** task ...

...but known for many "usual" operators, e.g. such which ...

- admit a bounded H^∞-calculus/BIP (if X UMD-space, [DV87])
- satisfy Gaussian bounds on L^p ([HP97])
- generate contractive positive semigroups on L^p ([Lam87])
- are $\mathcal{R}$-bounded (if X UMD-space, [Wei01])

$\longrightarrow$ in particular **general elliptic operators on** L^p **spaces**.

Nonautonomous maximal parabolic regularity

"Nonautonomous maximal parabolic regularity is not very well understood"

—common saying in papers about nonautonomous maximal parabolic regularity

Nonautonomous maximal parabolic regularity

- $A(t)\colon X \supseteq_d Y(t) \to X$ closed, $t \in J = [0, T]$ $\quad (1 < s < \infty)$

A has/satisfies **nonautonomous MPR on** $\mathrm{L}^s(J;X)$

$$\iff$$

for *every* $f \in \mathrm{L}^s(J;X)$ exists *unique* $y \in \mathrm{W}^{1,s}(J;X)$ such that $y(t) \in Y(t) = \mathrm{dom}_X(A(t))$ f.a.a. $t \in J$ and $Ay \in \mathrm{L}^s(J;X)$ and

$$y'(t) + A(t)y(t) = f(t) \quad \text{in } X, \qquad y(0) = 0.$$

- very little general properties to infer
- no unified "one-size-fits-all" theory available

A first theorem

Theorem ([AT87])

- $-A(t)$ *uniform generators of analytic semigroups on* X
- *some kind of Hölder regularity for* A^{-1}*:* *(*$0 \leq \gamma < \beta \leq 1$*)*

$$\left\| A(t)R(\lambda, A(t))\bigl(A(t)^{-1} - A(s)^{-1}\bigr)\right\|_{\mathcal{L}(X)} \lesssim \frac{|t-s|^\beta}{1+|\lambda|^{1-\gamma}}$$

Then A *satisfies nonautonomous MPR on* X*.*

- maximal regularity on $\mathrm{L}^s(J;X)$ **for all** $1 < s < \infty$ ([HM00])

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Then A *satisfies nonautonomous MPR on* X*.*

- maximal regularity on $\mathrm{L}^s(J;X)$ **for all** $1 < s < \infty$ ([HM00])
- nice in theory, hard to verify/use ([BN18] though)

Lions' theorem

Theorem ([DL92] Volume V)

- *Gelfand triple* $V \hookrightarrow H \hookrightarrow V'$, *family of forms* $\mathfrak{a}_t$ *on* $V \times V$
- $\mathfrak{a}_t$ *uniformly bounded and coercive,* $t \mapsto \mathfrak{a}_t(v, w)$ *measurable*

Then the operator induced by $v \mapsto \mathfrak{a}_t(v, \cdot)$ *satisfies maximal nonautonomous* $\mathrm{L}^2(J; V')$ *regularity.*

- remarkable: **no** time regularity assumed
- "hidden" constant domains: $v \mapsto \mathfrak{a}_t(v, \cdot) \in \mathcal{L}(V; V')$ f.a.a. t

Lions' theorem

Theorem ([DL92] Volume V+ [DtER17])

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Then the operator induced by $v \mapsto \mathfrak{a}_t(v, \cdot)$ *satisfies maximal nonautonomous* $\mathrm{L}^s(J; V')$ *regularity for all* $s \in \mathcal{I}$, $\mathcal{I} \supset \{2\}$ *open.*

- remarkable: **no** time regularity assumed
- "hidden" constant domains: $v \mapsto \mathfrak{a}_t(v, \cdot) \in \mathcal{L}(V; V')$ f.a.a. t
- $s > 2$: $\mathbb{W}^{1,s}(J; V', V) \hookrightarrow \mathrm{C}(\overline{J}; H)$ potentially very useful!

Constant domains, continuous dependence

Theorem ([ACFP07])

- *$t \mapsto A(t) \in \mathcal{L}(Y;X)$ strongly measurable*
- *$A(\tau)$ has MPR on X for every $\tau \in \overline{J}$*
- *A is relatively continuous: for all $t \in \overline{J}, \varepsilon > 0 \quad \exists \eta \geq 0, \delta > 0$:*

$$|t - s| < \delta \implies \|A(t)x - A(s)x\|_X \leq \varepsilon\|x\|_Y + \eta\|x\|_X$$

Then A satisfies nonautonomous $\mathrm{L}^s(J;X)$ MPR for all $1 < s < \infty$.

- if A accretive **or** $A \in \mathrm{C}(\overline{J};\mathcal{L}(Y;X))$: ([Ama04, PS01])

$\longrightarrow$ pointwise $A(\tau)$ MPR **equivalent** to nonautonomous

Quasilinear equations

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recall:

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Theorem ([Prü02])

- $\mathcal{A}\colon (Y,X)_{\frac{1}{s},s} \to \mathcal{L}(Y;X)$ *Lipschitz on bounded sets*
- $y_0 \in (Y,X)_{\frac{1}{s},s}, \quad f \in \mathrm{L}^s(J;X)$
- $\mathcal{A}(\bar{y})$ *MPR on* X, *domain* Y, *for all* $\bar{y} \in (Y,X)_{\frac{1}{s},s}$.

Then (QLE) *has a unique maximal solution* $y \in \mathbb{W}^{1,s}(0,T_\bullet;X,Y)$.

- proof by fixed point theorem, contraction $\sim$ local-in-time

Further remarks to quasilinear equations

- more general existence theorem available ([Ama05])

  $\longrightarrow$ requires nonautonomous MPR (constant domains)

  $\longrightarrow$ allows **nonlocal** dependence of $\mathcal{A}$ on y

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Warning

Establishing a self-consistent framework of constant domains is **nontrivial** and the main task!

- constant domains $\sim$ elliptic regularity results (uniform in data)
- further reading $\longrightarrow$ [MMR17, MR16, HMR16]

Several branches of maximal regularity

- discrete MPR (time-discretization) ([LV18])
- MPR with temporal weights ($\longrightarrow$ IV) ([MS12])
- maximal parabolic $\mathrm{C}^\alpha(J;X)$ regularity: ([Lun95])

$$(\partial + A)^{-1} \in \mathscr{L}_{\mathrm{iso}}\big(\mathrm{C}^\alpha(J;X); \mathrm{C}^{1+\alpha}(J;X) \cap \mathrm{C}^\alpha(J;Y)\big)$$

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Open Problem (Lions [AO19, ADF17, Fac17])

- *Gelfand triple* $V \hookrightarrow H \hookrightarrow V'$, *family of forms* $\mathfrak{a}_t$ *on* $V \times V$
- $\mathfrak{a}_t$ *uniformly bounded and coercive,* $t \mapsto \mathfrak{a}_t(v, w)$ *measurable*

Let $f \in \mathrm{L}^2(J; H)$. *When does the solution of*

$$y'(t) + \mathfrak{a}_t(y(t), \cdot) = f(t) \quad \text{in } V', \quad y(0) = 0$$

satisfy $y \in \mathrm{H}^1(J; H)$?

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—fin—

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