Maximal Parabolic Regularity
&
Consistent Space-Time Schemes

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FWF Der Wissenschaftsfonds.

Outline

- General Linear Parabolic Initial-Boundary Value Problems
- Special Problem Classes
- 3d Eddy Current Problems
- Space-Time Book

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Classical results: Space-Time VF in Sobolev spaces on Q

Let $Q = Q_T := \Omega \times (0,T)$ be the space-time cylinder, $\Omega \subset \mathbb{R}^d$ - b. & Lip, and $\Sigma := \partial\Omega \times (0,T)$, $\Sigma_0 := \overline{\Omega} \times \{0\}$ and $\Sigma_T := \overline{\Omega} \times \{T\}$.

[Ladyzhenskaya, 1954 / 1973] considered the parabolic IBVP:

$$\begin{aligned} \partial_t u - \partial_{x_i}(a_{ij}(x,t)\partial_{x_j} u + a_i(x.t)u) + b_i(x,t)\partial_{x_i} u + a(x,t)u & \\ =& f + \partial_{x_i} f_i \quad && \text{in } Q, \\ u(x,t) =& g_D := 0 && \text{on } \Sigma, \\ u(x,0) =& u_0 && \text{on } \overline{\Sigma}_0. \end{aligned}$$

and showed, that under the conditions

$$a_{ij} = a_{ji}, \quad \mu_1 \xi^2 \le a_{ij}(x,t)\xi_i\xi_j \le \mu_2 \xi^2, \tag{1}$$

$$\sum a_i^2,\ \sum b_i^2,\ |a|^2 \le \mu^2, \tag{2}$$

$$u_0 \in L_2(\Omega),\ f \in L_{2,1}(Q),\ f_i \in L_2(Q), \tag{3}$$

Classical results: Space-Time VF in Sobolev spaces on Q

the Space-Time VF: find $u \in H_0^{1,0}(Q) = \mathbf{W}_2^{1,0}(Q)$ such that

$$\int_Q (-u\partial_t v + a_{ij}\partial_{x_i} u \partial_{x_j} v + ...) = \int_Q (fv - f_i \partial_{x_i} v) + \int_{\Sigma_0} u_0 v \quad (4)$$

for all $v \in H_{0,\bar{0}}^{1,1}(Q)$, has a unique solution u in $H_0^{1,0}(Q)$ that even belongs to $V_{2,0}^{1,0}(Q) := V_2^{1,0}(Q) \cap H_0^{1,0}(Q)$, i.e. $\|u(\cdot,t)\|_{L_2(\Omega)}$ cont.

Moreover, if $u_0 \in H_0^1(\Omega)$, $f \in L_2(Q)$, $\partial_{x_i} f_i \in L_2(Q)$, $a_{ij} = \delta_{ij}$, and $a_i = b_i = a = 0$, then the generalized solution $u \in H_0^{1,0}(Q)$ belongs to the space

$$H_0^{\Delta,1}(Q) = \{v \in H_0^1(Q) : \Delta_x v \in L_2(Q)\},$$

and continuously depends on t in the norm of the space $H_0^1(\Omega)$.
$\Rightarrow$ Maximal Parabolic Regularity: $\partial_t u - \Delta_x u = f - \partial_{x_i} f_i$ in $L_2(Q)$

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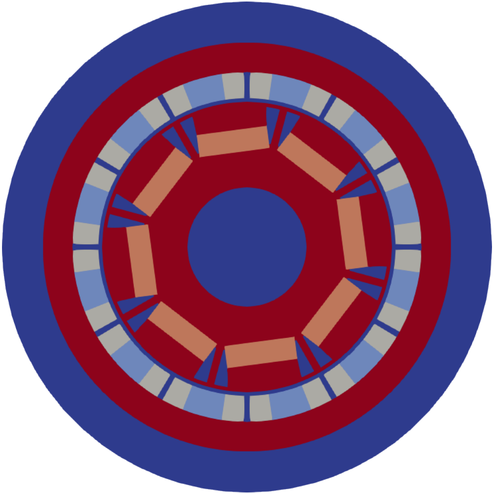
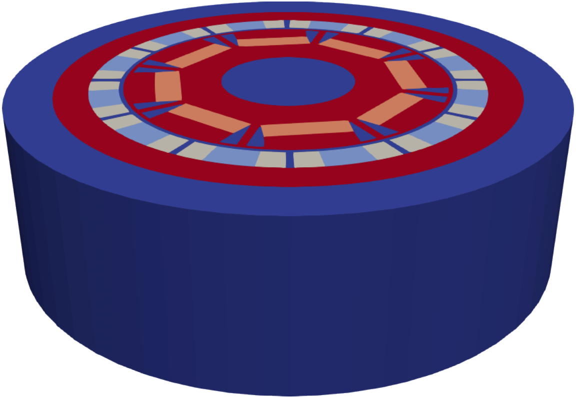
2D eddy current: Electric Motor (Cross Section)



2D eddy current: Electric Motor (Space-Time Cylinder)



A non-autonomous, non-linear parabolic model problem

Let $Q = Q_T := \Omega \times (0,T)$ be the space-time cylinder, $\Omega \subset \mathbb{R}^d$, and $\Sigma := \partial\Omega \times (0,T)$, $\Sigma_0 := \overline{\Omega} \times \{0\}$ and $\Sigma_T := \overline{\Omega} \times \{T\}$.

Given data f, $\mathbf{f} = (f_1, ..., f_d)^T$, g_D, σ, ν, and u_0, find u such that

$$\begin{aligned} \partial_t(\sigma(x,t)u) - \mathrm{div}_x(\nu(x,t,|\nabla_x u|)\nabla_x u) &= f + \mathrm{div}_x(\mathbf{f}) && \text{in } Q,\\ u(x,t) &= g_D && \text{on } \Sigma,\\ u(x,0) &= u_0 && \text{on } \overline{\Sigma}_0. \end{aligned}$$

where σ and ν are uniformly positive and bounded coefficients.

A non-autonomous parabolic model problem

Let $Q = Q_T := \Omega \times (0,T)$ be the space-time cylinder, $\Omega \subset \mathbb{R}^d$, and $\Sigma := \partial\Omega \times (0,T)$, $\Sigma_0 := \overline{\Omega} \times \{0\}$ and $\Sigma_T := \overline{\Omega} \times \{T\}$.

Given data f, $\mathbf{f} = (f_1, ..., f_d)^T$, g_D, σ, ν, and u_0, find u such that

$$
\begin{aligned}
\partial_t(\sigma(x,t)u) - \mathrm{div}_x(\nu(x,t)\nabla_x u) &= f + \mathrm{div}_x(\mathbf{f}) && \text{in } Q,\\
u(x,t) &= g_D && \text{on } \Sigma,\\
u(x,0) &= u_0 && \text{on } \overline{\Sigma}_0.
\end{aligned}
$$

where σ and ν are uniformly positive and bounded coefficients.

A simplified non-autonomous parabolic model problem

Let $Q = Q_T := \Omega \times (0,T)$ be the space-time cylinder, $\Omega \subset \mathbb{R}^d$, and $\Sigma := \partial\Omega \times (0,T)$, $\Sigma_0 := \overline{\Omega} \times \{0\}$ and $\Sigma_T := \overline{\Omega} \times \{T\}$.

Given $f \in L_2(Q)$, $\mathbf{f} = \mathbf{0}$, $g_D = 0$, $\sigma = 1$, ν, and $u_0 = 0$, find u:

$$\begin{aligned} \partial_t u - \mathrm{div}_x(\nu(x,t)\nabla_x u) &= f && \text{in } Q, \\ u(x,t) &= 0 && \text{on } \Sigma, \\ u(x,0) &= 0 && \text{on } \overline{\Sigma}_0, \end{aligned}$$

where $\nu(x,t)$ is a uniformly positive and bounded coefficient.

Under which additional assumptions on ν, we get mpr in $L_2(Q)$, i.e.

$$\partial_t u,\, \mathrm{div}_x(\nu(x,t)\nabla_x u) \in L_2(Q)$$

implying $\partial_t u - \mathrm{div}_x(\nu(x,t)\nabla_x u) = f$ in $L_2(Q)$ that is the starting point for getting consistent spacee-time schemes.

Distributional sources

Given data f, $\mathbf{f} = (f_1, ..., f_d)^T$, and ν, find u such that

$$\begin{aligned} \partial_t u - \operatorname{div}_x(\nu(x,t)\nabla_x u) &= f + \operatorname{div}_x(\mathbf{f}) && \text{in } Q, \\ u(x,t) &= g_D := 0 && \text{on } \Sigma, \\ u(x,0) &= u_0 := 0 && \text{on } \overline{\Sigma}_0. \end{aligned}$$

Assume that

- $\nu(x,t)$ is a uniformly positive and bounded coefficient,
- $f \in L_2(Q)$,
- $\mathbf{f} \in H^1(Q_i)$ for all Q_i: $\overline{Q} = \bigcup \overline{Q}_i$, ...

Distributional sources (cond.)

Taking in Ladyzhenskaya's space-time variational formulation, find $u \in H_0^{1,0}(Q)$ such that

$$a(u,v) = \langle F, v\rangle \quad \forall v \in H^{1,1}_{0,\bar{0}}(Q), \quad \text{with} \tag{5}$$

$$a(u,v) = -\int_Q u(x,t)\partial_t v(x,t) d(x,t) + \int_Q \nu(x,t)\nabla_x u \cdot \nabla_x v \, d(x,t),$$

$$\langle F, v\rangle = \int_Q f \, v \, d(x,t) - \int_Q \mathbf{f} \cdot \nabla_x v \, d(x,t),$$

test functions $v \in H^{1,1}_{0,\bar{0}}(Q)$ with compact support in Q_i, we get

$$\text{div}(-\nu(x,t)\nabla_x u, u) = f + \text{div}_x \mathbf{f} \ \text{ in } L_2(Q_i)$$

and

$$\sum \int_{\partial Q_i} (-\nu \nabla_x u, u) \cdot \mathbf{n} \, v \, ds(x,t) = -\sum \int_{\partial Q_i} \mathbf{f} \cdot \mathbf{n}_x \, v \, ds(x,t)$$

Distributional sources (cond.)

The equations

$$\operatorname{div}(-\nu(x,t)\nabla_x u, u) = f + \operatorname{div}_x \mathbf{f} \ \text{ in } L_2(Q_i)$$

and

$$\sum \int_{\partial Q_i} (-\nu \nabla_x u, u) \cdot \mathbf{n}\, v\, ds(x,t) = -\sum \int_{\partial Q_i} \mathbf{f} \cdot \mathbf{n}_x\, v\, ds(x,t)$$

can be used as starting point for constructing consistent, coercive, and bounded space-time scheme, see next talk by A. Schafelner.

But under which assumptions we have at least $\partial_t u$ and $\operatorname{div}_x(\nu(x,t)\nabla_x u)$ in $L_2(Q_i)$?

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A non-autonomous, non-linear parabolic model problem

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Given data f, $\mathbf{f} = (f_1, ..., f_d)^T$, g_D, σ, ν, and u_0, find u such that

$$\begin{aligned} \partial_t(\sigma(x,t)u) + \mathrm{curl}_x(\nu(x,t,|\mathrm{curl}_x u|)\mathrm{curl}_x u) &= f + \mathrm{curl}_x(\mathbf{f}) && \text{in } Q, \\ u(x,t) &= g_D && \text{on } \Sigma, \\ u(x,0) &= u_0 && \text{on } \overline{\Sigma}_0. \end{aligned}$$

where σ and ν are uniformly positive and bounded coefficients.

Question: maximal eddy current regularity ?

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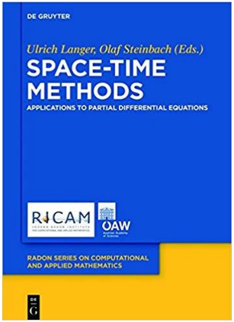
Space-Time Book



Dörfler, Findeisen, Wieners, Ziegler: Parallel adaptive dG in space and time for waves.
Dohr, Niino, Steinbach: Space-time BEM for the heat equation.
Ernesti, Wieners: A space-time dPG method for acoustic waves.
Gopalakrishnan, Sepulveda: Space-time dPG for the wave equation.
Langer, Matculevich, Repin: Adaptive space-time IgA for parabolic problems.
Steinbach, Yang: Overview paper.

Karabelas, Neumüller: Space-time meshes.