Maximal parabolic regularity and consistent space-time schemes

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ABSTRACT

Already O.A. Ladyženskaya showed that a weak solution u from $H_0^{1,0}(Q)$ of the Dirichlet initial-boundary value problem for the parabolic equation $\partial_t u - \Delta_x u = f$ in $Q = \Omega \times (0,T)$ belongs to $H_0^{\Delta,1}(Q)$ provided that the right-hand side $f \in L_2(Q)$ and the initial data $u_0 \in H_0^1(\Omega)$. Later this question of the so-called maximal parabolic regularity was posed by J.-L. Lions for the non-autonomous case and for non-linear parabolic equations.

In the case of maximal parabolic regularity, one can start from the parabolic PDE in $L_2(Q)$ to derive consistent coercive space-time finite element or isogeometric analysis schemes; see [1, 3] and [2]. In fact, one needs a local version of the maximal parabolic regularity, i.e. $\partial_t u \in L_2(Q_i)$ and $L_x u \in L_2(Q_i)$, where L_x is the elliptic partial differential operator replacing $(-\Delta_x)$ in the parabolic PDE above, and $\bar{Q} = \cup \bar{Q}_i$ is some non-overlapping domain decomposition of the space-time cylinder Q.

Moreover, we are interested in the elliptic-parabolic case generated by replacing $\partial_t u$ by $\partial_t(\sigma u)$, where σ can vanish in some parts of the computational cylinder Q as it is the case for the electrical conductivity in eddy current problems. 3d eddy current problems are also of interested in connection with local maximal parabolic regularity. Furthermore, the source term is not always a L_2 function or a L_2 vector-function, but can be a distribution of the form $\mathrm{div}_x(\mathbf{f})$ or $\mathrm{curl}_x(\mathbf{f})$ where the components of the vector-function $\mathbf{f}$ (permanent magnetization) are at least from $H^1(Q_i)$, i.e. piecewise smooth. Under which conditions one can show local regularity results such that the PDE or the PDE system make sense in $L_2(Q_i)$? We will also discuss typical non-linear eddy-current problems for which local parabolic regularity results would be desirable for deriving consistent space-time schemes.

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