Matrix-Free Multigrid for Phase-Field Fracture Problems

D. Jodlbauer [1], U. Langer [1], T. Wick [2]

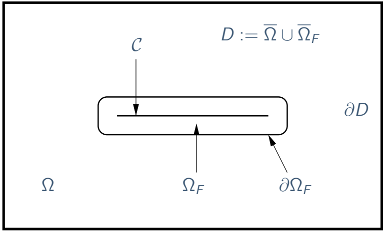
[1]RICAM, Austrian Academy of Sciences, Linz, Austria

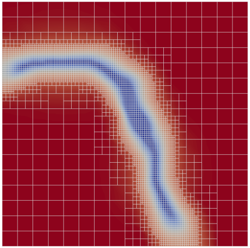
[2]Cluster of Excellence PhoenixD (Photonics, Optics, and Engineering - Innovation Across Disciplines), Leibniz Universität Hannover, Germany

July 1, 2019

Fractures

- Determine propagation of fractures[1]
- Phase-field model for fracture[2]





[1] Griffith and Taylor, "The phenomena of rupture and flow in solids", 1921.

[2] Francfort and Marigo, "Revisiting brittle fracture as an energy minimization problem", 1998.

Energy Minimization

Problem (Simplified Model Problem)

Find displacement u and fracture indicator φ such that

$$E(u,\varphi) = E_S(u,\varphi) + E_C(\varphi) \to min$$

subject to $\partial_t \varphi \leq 0$ (crack irreversibility) with

$$E_S(u,\varphi) = \frac{1}{2}(\varphi^2 \sigma(u), e(u))$$

$$E_C(\varphi) = G_c(\frac{1}{2\epsilon}\|1-\varphi\|^2 + \frac{\epsilon}{2}\|\nabla\varphi\|^2)$$

Technicalities

- Non-convexity: extrapolation[3] $\varphi^2 \rightsquigarrow \tilde{\varphi}^2$.
- Splitting[4]: tensile/compressive parts

$$E_S = E_S^+ + E_S^- \qquad \sigma = \sigma^+ + \sigma^-$$

  - Amor: $E^+ := \frac{1}{2} K \mathrm{tr}^+(e)^2 + \mu e^{dev} : e^{dev}, \quad e^{dev} := e - \mathrm{tr}(e)/d$
  - Miehe: $E^+ := \frac{1}{2} \lambda \mathrm{tr}^+(e)^2 + \mu e^+ : e^+, \quad e^+ := Q \Lambda^+ Q^{-1}$
  - No split: $E^+ = E_S$, $E^- = 0$

[3] Heister, Wheeler, and Wick, "A primal-dual active set method and predictor-corrector mesh adaptivity for computing fracture propagation using a phase-field approach", 2015.

[4] Ambati, Gerasimov, and Lorenzis, "A review on phase-field models of brittle fracture and a new fast hybrid formulation", 2015.

Problem

Find $\{u,\varphi\} \in \{u_D + V\} \times W_{in}$ *such that:*

$$A(u,\varphi)(w,\psi) \geq 0 \qquad \forall w \in V, \quad \psi \in W \cap L^\infty(D).$$

$$\begin{aligned} A(u,\varphi)(w,\psi) := &\left(g_\kappa(\varphi)\sigma^+(u), e(w)\right) + \left(\sigma^-(u), e(w)\right) + \left(\varphi^2 p, \operatorname{div} w\right) \\ &+ \frac{1}{2}\left(\partial_\varphi g_\kappa(\varphi) E_s^+(e(u)), \psi - \varphi\right) + 2(\varphi \; p \operatorname{div} \; u, \psi - \varphi) \\ &+ G_c\left(-\frac{1}{\varepsilon}(1-\varphi, \psi - \varphi) + \varepsilon\left(\nabla\varphi, \nabla(\psi - \varphi)\right)\right) \end{aligned}$$

- Additional terms included

Solution of the Variational Inequality

- Heister, T., M. F. Wheeler, and T. Wick. "A primal-dual active set method and predictor-corrector mesh adaptivity for computing fracture propagation using a phase-field approach". In: *Comp. Meth. Appl. Mech. Engrg.* 290.0 (2015), pp. 466–495.
- Ambati, M., T. Gerasimov, and L. De Lorenzis. "A review on phase-field models of brittle fracture and a new fast hybrid formulation". English. In: *Computational Mechanics* 55.2 (2015), pp. 383–405.
- Farrell, P. E. and C. Maurini. "Linear and nonlinear solvers for variational phase-field models of brittle fracture". In: *Int. J. Numer. Meth. Engrg.* 109 (2017), pp. 648–667.
- Kopanicakova, A. and R. Krause. "Recursive multilevel trust region method with application to fully monolithic phase-field models of brittle fracture". In: *arXiv* (2019).

Primal-Dual Active-Set Method[5]

- Assemble residual $R := -A(u_k, \varphi_k)$
- Determine active set $\mathcal{A}_k := \{i : (M^{-1}R)_i + c(\varphi_k - \varphi^{n-1})_i > 0\}$
- Enforce constraints $\varphi_k = \varphi^{n-1},\ \delta\varphi_k = 0$ on $\mathcal{A}_k$
- Compute Newton update $(\delta u, \delta\varphi) = (\nabla A)^{-1} r$ on $\mathcal{A}_k^C$
- Repeat until convergence ($\mathcal{A}_k = \mathcal{A}_{k-1}$ and Newton criterion)

M . . . mass matrix k . . . Newton step n . . . time step

[5] Heister, Wheeler, and Wick, "A primal-dual active set method and predictor-corrector mesh adaptivity for computing fracture propagation using a phase-field approach", 2015.

Linear Systems

Resulting linear system matrix

$$H(u,\varphi) := \nabla A(u,\varphi) = \begin{bmatrix} U & 0 \\ B & P \end{bmatrix}$$

0-block due to extrapolation

- FEM library deal.II[6]
- Q_1 elements on quadrilaterals/hexahedra

[6] www.dealii.org

Linear Systems

- Linear solver: GMRES
- Preconditioner: Geometric Multigrid (GMG)

Only requires matrix-vector multiplications

$\Rightarrow$ Suitable for matrix-free implementation (MF)

Primary idea: Instead of assembling H and compute $H \cdot v$, assemble Hv directly (i.e. element-wise matrix-vector product)

Matrix-Free Formulation

Example: $H \sim (D\nabla u, \nabla v)$

$$H \cdot v = \sum_C P_C^T B_R^T J_C^{-1} D_C J_C^{-T} B_R P_C v$$

- Readily included in FEM library deal.II[7]
- Utilizes tensor product structure, vectorization, parallelization

P_C ... global to local mapping

B_R ... gradient matrix on reference element

J_C ... element transformation (block-diagonal)

D_C ... diagonal coefficient matrix

[7] Kronbichler and Kormann, "A generic interface for parallel cell-based finite element operator application", 2012.

Matrix-Free Formulation

Pros:

- No need to store matrix H
- Automatic "rebuild" on every "multiplication"

Cons:

- Matrix entries not available

Matrix-Free Geometric Multigrid

System matrix:

$$H = \begin{bmatrix} U & 0 \\ B & P \end{bmatrix}$$

Several options:

- Block-diagonal GMG preconditioner:

$$\mathcal{P}_{diag} := \begin{bmatrix} MG(U) & 0 \\ 0 & MG(P) \end{bmatrix}$$

with Chebyshev smoothers for $MG(U)$ and $MG(P)$.

Matrix-Free Geometric Multigrid

System matrix:

$$H = \begin{bmatrix} U & 0 \\ B & P \end{bmatrix}$$

Several options:

- Block-diagonal GMG preconditioner
- Global (monolithic) GMG: $\mathcal{P}_{full} := MG(H)$

  utilize block-structure inside smoother

$$\mathcal{S}_{full} := \begin{bmatrix} \mathcal{S}(U) & 0 \\ 0 & \mathcal{S}(P) \end{bmatrix}.$$

Coarser Levels

Coarse linearization point:

- Fine-grid linearization on coarse levels is expensive
- Use restrictions: $\{\mathcal{R}^\ell u, \mathcal{R}^\ell \varphi\}$

Coarse active set: using restricted indicator function

$$a := (a_i) := \begin{cases} 1 & i \in \mathcal{A} \\ 0 & \text{else} \end{cases} \qquad a^\ell := \mathcal{R}^\ell a \qquad \mathcal{A}^\ell := \{i : a_i^\ell = 1\}.$$

L-Shaped Panel (Miehe-Splitting)

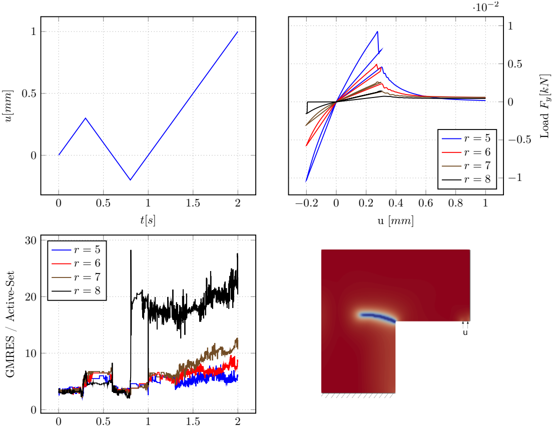
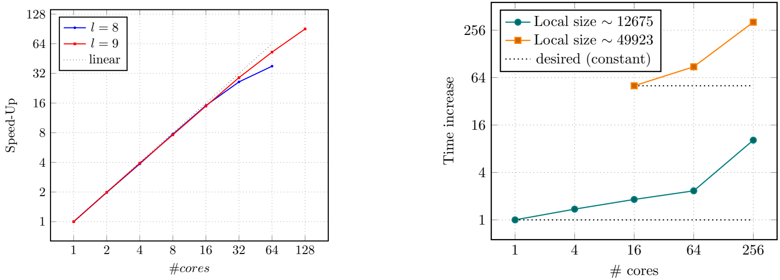


Figure: $10k$ — $640k$ dofs

Parallelization (Strong / Weak)



- Time for single linear solve (Multiple Fractures)
- # DoFs: $\ell = 7 : 198k$, $\ell = 8 : 789k$, $\ell = 9 : 3.15m$

Summary

- Parallel geometric multigrid suitable for active set
- $\sim 5 - 30$ iterations

✓ $2d/3d$

✓ Automatic differentiation

✓ Validation of parallel consistency

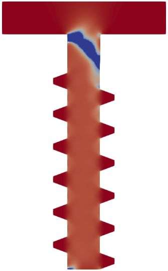
Preprint (RICAM-Report 2019-6 / arXiv): *D.J., U.Langer, T.Wick*

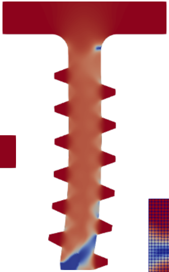
"Matrix-free multigrid solvers for phase-field fracture problems"

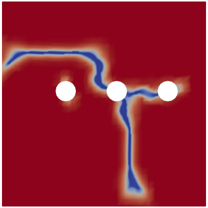
Future Work (In Progress)

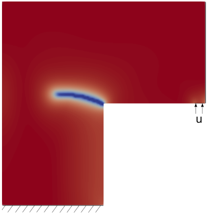
- ✓ Adaptivity
  - error-estimator
- General framework: apply to other block systems, e.g.
  - coupled to PFF
  - optimal control
- Nonlinear solver
- Higher-Order elements, hp-FEM

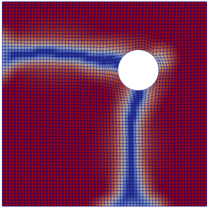
Gallery

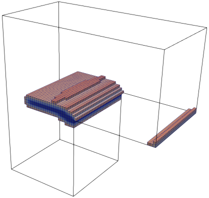












References I

Griffith, A.A. and G.I. Taylor. "The phenomena of rupture and flow in solids". In: *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character* 221.582-593 (1921), pp. 163–198.

Francfort, G.A. and J.-J. Marigo. "Revisiting brittle fracture as an energy minimization problem". In: *J. Mech. Phys. Solids* 46.8 (1998), pp. 1319–1342.

Heister, T., M. F. Wheeler, and T. Wick. "A primal-dual active set method and predictor-corrector mesh adaptivity for computing fracture propagation using a phase-field approach". In: *Comp. Meth. Appl. Mech. Engrg.* 290.0 (2015), pp. 466–495.

Ambati, M., T. Gerasimov, and L. De Lorenzis. "A review on phase-field models of brittle fracture and a new fast hybrid formulation". English. In: *Computational Mechanics* 55.2 (2015), pp. 383–405.

Farrell, P. E. and C. Maurini. "Linear and nonlinear solvers for variational phase-field models of brittle fracture". In: *Int. J. Numer. Meth. Engrg.* 109 (2017), pp. 648–667.

Kopanicakova, A. and R. Krause. "Recursive multilevel trust region method with application to fully monolithic phase-field models of brittle fracture". In: *arXiv* (2019).

References II

Kronbichler, M. and K. Kormann. "A generic interface for parallel cell-based finite element operator application". In: *Computers & Fluids* 63 (2012), pp. 135–147. ISSN: 0045-7930.

Miehe, C., M. Hofacker, and F. Welschinger. "A phase field model for rate-independent crack propagation: Robust algorithmic implementation based on operator splits". In: *Comput. Meth. Appl. Mech. Engrg.* 199 (2010), pp. 2765–2778.

Amor, H., J.-J. Marigo, and C. Maurini. "Regularized formulation of the variational brittle fracture with unilateral contact: Numerical experiments". In: *J. Mech. Phys. Solids* 57 (2009), pp. 1209–1229.

This work has been supported by the Austrian Science Fund (FWF) grant No. P-29181:
'Goal-Oriented Error Control for Phase-Field Fracture Coupled to Multiphysics Problems'.

Thank you.