Sharp spatial H^1-norm analysis of a finite element method for a time-fractional diffusion problem

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Outline

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1. Fractional PDE

2. The idea of analysis

3. H^1-norm analysis of the fully discrete FEM

4. Numerical experiments

Fractional PDE

Fractional-derivative PDE (initial-boundary value problem)

$$Lu := D_t^\alpha u - \Delta u = f(x,t) \qquad (1)$$

for $(x,t) \in Q := \Omega \times (0,T]$, with

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega,$$
$$u|_{\partial\Omega} = 0 \ \text{ for } 0 < t \le T,$$

where $\alpha \in (0,1)$, the functions f is continuous on $\overline{Q} = \overline{\Omega} \times [0,T]$, and $u_0 \in C(\overline{\Omega})$. Here the spatial domain $\Omega \subset \mathbb{R}^d$ (where $d \in \{1,2,3\}$) is bounded, with a Lipschitz continuous boundary $\partial\Omega$.

D_t^α denotes the Caputo fractional derivative defined by

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(x,s)}{\partial s}\, ds.$$

The previous works:

- L1 scheme
  - M. Stynes et al., SIAM J. Numer. Anal., 55(2) (2017) 1057-1079.

$$\left\|u(\cdot,t)\right\|_q \le C \quad \text{for } q \in \mathbb{N}_0,$$

$$\left\|\partial_t^l u(x,t)\right\|_q \le C(1 + t^{\alpha - l}) \quad \text{for } l = 0,1,2, \text{and } q \in \mathbb{N}_0, \qquad (2)$$

$$\left\|D_t^\alpha u(\cdot,t)\right\|_q \le C \quad \text{for} \quad q \in \mathbb{N}_0.$$

  - N. Kopteva, Math. Comput., Doi:10.1090/mcom/3410, 2018.
  - H. L. Liao et al., SIAM J. Numer. Anal., 56(2) (2018) 1112-1133.

- L2-1_σ scheme
  - H. L. Liao et al., arXiv:2018.
  - H. L. Liao et al., SIAM J. Numer. Anal., 57(1) (2019) 218-237.
  - H. Chen and M. Stynes, J. Sci. Comput., 79(1) (2019), 624-647.

By Poincare inequality, one has

$$\boxed{\|u^n - u_h^n\| \le C\|\nabla u^n - \nabla u_h^n\|.} \tag{3}$$

C. B. Huang and M. Stynes, Appl. Numer. Math, 135 (2019) 15-29.

N. Kopteva, Math. Comput, DOI:10.1090/mcom/3410, 2018.

$$\boxed{\|\nabla u^n - \nabla u_h^n\| \le C\tau_n^{-\alpha/2}(N^{-\min\{2-\alpha, r\alpha\}} + h^{k+1}).} \tag{4}$$

Note: The finite difference method for H^1 norm

J. C. Ren et al., arXiv:1811.08059.

Outline

Vidar Thomée, Galerkin finite element methods for parabolic problems, 2006.

The idea of analysis:

- ◇ Rewrite the fully discrete FEM into the discrete differential formulation.
- ◇ Multiply the discrete differential equation $D_N^\alpha \mu^n - \Delta_h \mu^n = P_h g^n$ by $-\Delta_h \mu^n$.(contrasts with the related classical technique for $H^1(\Omega)$-norm analysis of the semidiscrete problem, where the discrete differential equation is multiplied by $(\mu^n)_t$.)
- ◇ Applying the definition of the discrete Laplacian Δ_h yields

$$(D_N^\alpha \nabla \mu^n, \nabla \mu^n) + \|\Delta_h \mu^n\|^2 = (\nabla P_h g^n, \nabla \mu^n). \tag{5}$$

- ◇ By $\|\nabla P_h v\| \le K \|\nabla v\|$, one has

$$(D_N^\alpha \nabla \mu^n, \nabla \mu^n) \le K \|\nabla g^n\| \|\nabla \mu^n\|. \tag{6}$$

Three operators

Define the L^2 *projector* $P_h : L^2(\Omega) \to V_{0h}$ by

$$(P_h w, v_h) = (w, v_h) \ \ \forall\, v_h \in V_{0h}.$$

J. H. Bramble, J. E. Pasciak, and O. Steinbach, Math. Comput., 71(237):147-156, 2002.

$$\boxed{\|\nabla P_h v\| \le K\|\nabla v\| \ \text{ for all } v \in H_0^1(\Omega).} \tag{7}$$

Define the *Ritz projector* $R_h : H_0^1(\Omega) \to V_{0h}$ by

$$(\nabla R_h w, \nabla v_h) = (\nabla w, \nabla v_h) \ \ \forall\, v_h \in V_{0h}.$$

It is well known that

$$\|w - R_h w\| + h\|w - R_h w\|_1 \le Ch^{k+1}|w|_{k+1} \ \ \forall\, w \in H^{k+1}(\Omega) \cap H_0^1(\Omega). \tag{8}$$

Define the *discrete Laplacian* $\Delta_h : V_{0h} \to V_{0h}$ by

$$\boxed{(\Delta_h v, w) = -(\nabla v, \nabla w) \ \ \forall\ v, w \in V_{0h}}. \tag{9}$$

V. Thomée, Galerkin finite element methods for parabolic problems, 2006.

$$\boxed{\Delta_h R_h v = P_h \Delta v \quad \forall\ v \in H^2(\Omega)}. \tag{10}$$

FEM discretisation in space

Let M be a positive integer. Partition Ω by a quasiuniform mesh of M elements $\{K_m : m = 1, \dots, M\}$. Set

$$h_m = \mathrm{diam}(K_m) \text{ for each } m \text{ and } h = \max_{1 \le m \le M} \{h_m\}.$$

Weak formulation: Find $u(\cdot, t) \in H_0^1(\Omega)$ for each $t \in (0, T]$, such that

$$\begin{cases} (D_t^\alpha u, v) + (\nabla u, \nabla v) = (f, v) \quad \forall\ v \in H_0^1(\Omega), \\ (u(0, \cdot), v(\cdot)) = (u_0, v) \quad \forall\ v \in H_0^1(\Omega), \end{cases} \tag{11}$$

Define the finite element spaces on spatial mesh by

$$V_h = \left\{ v_h \in L^2(\Omega) : v_h\big|_{K_m} \in P^k(K_m),\ \ m = 1, 2, \cdots, M \right\},$$
$$V_{0h} = \{ v_h \in V_h : v_h|_{\partial\Omega} = 0 \},$$

where $P^k(K_m)$ denotes the space of polynomials on K_m with degree at most k.

The semi-discrete FEM: Find $u_h(\cdot, t) \in V_{0h}$ for each $t \in (0, T]$, such that

$$\begin{cases} (D_t^\alpha u_h, v_h) + (\nabla u_h, \nabla v_h) = (f, v_h) \ \ \forall\ v_h \in V_{0h}, \\ (u_h(0, \cdot), v_h(\cdot)) = (u_0, v_h) \ \ \forall\ v_h \in V_{0h}. \end{cases}$$

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Graded mesh in time

Let N be positive integer. Set

$$t_n := T(n/N)^r \text{ for } n = 0, 1, \cdots, N$$

with mesh grading $r \geq 1$ chosen by user.

L1 discretisation in time

The Caputo fractional derivative is approximated by L1 scheme (graded mesh in time)

$$D_N^\alpha u_m^n := \frac{1}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} \frac{u_m^{i+1} - u_m^i}{\tau_{i+1}} \left[(t_n - t_i)^{1-\alpha} - (t_n - t_{i+1})^{1-\alpha} \right]. \qquad (12)$$

The truncation error:

$$\left\| D_t^\alpha u(x, t_n) - D_N^\alpha u(x, t_n) \right\|_1 \leq C n^{-\min\{2-\alpha, r\alpha\}}.$$

The fully discrete L1 FEM:

$$\begin{cases} (D_N^\alpha u_h^n, v_h) + (\nabla u_h^n, \nabla v_h) = (f^n, v_h) \ \text{ for } n = 1, \dots, N \text{ and all } v_h \in V_{0h}, \\ (u_h^0, v_h) = (u_0, v_h) \ \ \forall \ v_h \in V_{0h}, \end{cases} \tag{13}$$

where $f^n(\cdot) := f(\cdot, t_n)$.

Applying (9), the L1 FEM (13) takes the form: find $u_h^n \in V_{0h}$ for $n = 0, 1, \dots, N$ such that

$$\begin{cases} (D_N^\alpha u_h^n, v_h) - (\Delta_h u_h^n, v_h) = (P_h f^n, v_h) \ \text{ for } n = 1, \dots, N \text{ and all } v_h \in V_{0h}, \\ (u_h^0, v_h) = (P_h u_0, v_h) \ \ \forall \ v_h \in V_{0h}, \end{cases} \tag{14}$$

The discrete differential equation:

$$\begin{cases} D_N^\alpha u_h^n - \Delta_h u_h^n = P_h f^n \quad \text{for } n = 1, \dots, N, \\ u_h^0 = P_h u_0. \end{cases} \tag{15}$$

Stability and convergence result

Stability result

Theorem 1 ($H^1(\Omega)$-stability of the L1 FEM)

Let u_h^n be the solution of (15). Then

$$\|\nabla u_h^n\| \leq \|\nabla u_h^0\| + \frac{KT^\alpha \Gamma(2-\alpha)}{1-\alpha} \max_{1\leq j\leq n} \|\nabla f^j\| \quad \text{for } n = 1, 2, \ldots, N.$$

Denote

$$\zeta^n := R_h u^n - u_h^n \text{ and } \rho^n := R_h u^n - u^n.$$

Error equation:

$$\begin{aligned} D_N^\alpha \zeta^n - \Delta_h \zeta^n &= (R_h D_N^\alpha u^n - \underbrace{\Delta_h R_h}_{P_h \Delta} u^n) - (\underbrace{D_N^\alpha u_h^n - \Delta_h u_h^n}_{P_h f^n}) \\ &= (R_h - P_h) D_N^\alpha u^n + P_h (D_N^\alpha u^n - \Delta u^n) - P_h f^n \\ &= P_h (R_h - I) D_N^\alpha u^n + P_h (f^n + \varphi^n) - P_h f^n \\ &= P_h (D_N^\alpha \rho^n + \varphi^n), \end{aligned} \tag{16}$$

where $\varphi^n := D_N^\alpha u^n - D_t^\alpha u^n$.

Convergent result:

Theorem 2 (Error estimate for the L1 FEM)

Let u^n and u_h^n be the solutions of (11) and (13), respectively. Then for $n = 1, 2, \ldots, N$, there exists a constant C such that

$$\|\nabla u^n - \nabla u_h^n\| \le C \left(N^{-\min\{2-\alpha, r\alpha\}} + h^k \right). \tag{17}$$

If $r \ge (2-\alpha)/\alpha$, then one has

$$\|u^n - u_h^n\|_{H^1(\Omega)} \le C \left(N^{-(2-\alpha)} + h^k \right) \quad \text{for } n = 0, 1, \ldots, N.$$

For $n = 0, \ldots, N-1$ and $0 \le \sigma \le 1$, set $t_{n+\sigma} = t_n + \sigma \tau_{n+1}$.

L2-1_σ discretisation in time

The Caputo fractional derivative is approximated by L2-1_σ scheme at $t_{n+\sigma}$ (graded mesh in time)

$$D_t^\alpha v(t_{n+\sigma}) \approx \delta_{t_{n+\sigma}}^\alpha v := g_{n,n} v^{n+1} - \sum_{j=0}^{n} (g_{n,j} - g_{n,j-1}) v^j \quad \text{for} \quad n = 0, \ldots, N-1. \tag{18}$$

Here $g_{0,0} = \tau_1^{-1} a_{0,0}$, $g_{n,-1} = 0$, and for $n \ge 1$ one has

$$g_{n,j} = \begin{cases} \tau_{j+1}^{-1}(a_{n,0} - b_{n,0}) & \text{if} \quad j = 0, \\ \tau_{j+1}^{-1}(a_{n,j} + b_{n,j-1} - b_{n,j}) & \text{if} \quad 1 \le j \le n-1, \\ \tau_{j+1}^{-1}(a_{n,n} + b_{n,n-1}) & \text{if} \quad j = n, \end{cases}$$

where

$$a_{n,n} = \frac{1}{\Gamma(1-\alpha)} \int_{t_n}^{t_{n+\sigma}} (t_{n+\sigma} - \eta)^{-\alpha} d\eta = \frac{\sigma^{1-\alpha}}{\Gamma(2-\alpha)} \tau_{n+1}^{1-\alpha} \quad \text{for} \quad n \geq 0,$$

$$a_{n,j} = \frac{1}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1}} (t_{n+\sigma} - \eta)^{-\alpha} d\eta \quad \text{for} \quad n \geq 1 \quad \text{and} \quad 0 \leq j \leq n-1,$$

$$b_{n,j} = \frac{1}{\Gamma(1-\alpha)} \frac{2}{t_{j+2} - t_j} \int_{t_j}^{t_{j+1}} \frac{\eta - t_{j+1/2}}{(t_{n+\sigma} - \eta)^{\alpha}} \, d\eta \quad \text{for} \quad n \geq 1 \quad \text{and} \quad 0 \leq j \leq n-1.$$

The fully discrete L2-1_σ FEM :

$$(\delta^{\alpha}_{t_{n+\sigma}} u_h, v_h) + (\nabla u_h^{n,\sigma}, \nabla v_h) = (f^{n+\sigma}, v_h) \quad \forall\ v_h \in V_{0h}, \tag{19}$$

where we set $u_h^0 = R_h u_0$ and $f^{n+\sigma} = f(\cdot, t_{n+\sigma})$ and $u^{n,\sigma} = \sigma u_h^{n+1} + (1-\sigma) u_h^n$ for $n = 0, 1, \ldots, N-1$.

The discrete differential form:

$$\delta^{\alpha}_{t_{n+\sigma}} u_h - \Delta_h u_h^{n,\sigma} = P_h f^{n+\sigma} \quad \text{for} \quad n = 0, \ldots, N-1, \tag{20}$$

with $u_h^0 = R_h u_0$.

Stability result

Theorem 3 ($H^1(\Omega)$-stability of the L2-1_σ FEM)

The L2-1_σ FEM solution u_h^n of (20) *satisfies*

$$\|\nabla u_h^n\|^2 \leq \|\nabla u_0\|^2 + \Gamma(1-\alpha)T^\alpha \max_{0\leq j\leq N-1} \|f^{j+\sigma}\|^2 \quad \textit{for } n = 0, 1, \ldots, N-1.$$

Error equation:

$$
\begin{aligned}
\delta^{\alpha}_{t_{n+\sigma}}\zeta - \Delta_h \zeta^{n,\sigma} &= (R_h \delta^{\alpha}_{t_{n+\sigma}} u - \Delta_h R_h u^{n,\sigma}) - (\delta^{\alpha}_{t_{n+\sigma}} u_h - \Delta_h u_h^{n,\sigma}) \\
&= (R_h - P_h)\delta^{\alpha}_{t_{n+\sigma}} u + P_h(\delta^{\alpha}_{t_{n+\sigma}} u - \Delta u^{n,\sigma}) - P_h f^{n+\sigma} \\
&= P_h(R_h - I)\delta^{\alpha}_{t_{n+\sigma}} u + P_h(\delta^{\alpha}_{t_{n+\sigma}} u - \Delta u^{n,\sigma}) - P_h(D^{\alpha}_t u^{n+\sigma} - \Delta u^{n+\sigma}) \\
&= P_h \delta^{\alpha}_{t_{n+\sigma}} \rho + P_h(\delta^{\alpha}_{t_{n+\sigma}} u - D^{\alpha}_t u^{n+\sigma}) + P_h(\underbrace{\Delta u^{n+\sigma} - \Delta u^{n,\sigma}}_{\neq 0}) \\
&= P_h\left(\delta^{\alpha}_{t_{n+\sigma}} \rho + \varphi^{n+\sigma} + R^{n+\sigma}\right), \qquad (21)
\end{aligned}
$$

where $\varphi^{n+\sigma} := \delta^{\alpha}_{t_{n+\sigma}} u - D^{\alpha}_t u^{n+\sigma}$ and $R^{n+\sigma} := \Delta u^{n+\sigma} - \Delta u^{n,\sigma}$.

Convergent result:

Theorem 4 (Error estimate for the L2-1_σ FEM)

Suppose $\sigma = 1 - \alpha/2$. Let u^n and u_h^n be the solutions of (11) and (20), respectively. Assume that $u \in L^\infty(0, T; H_0^1(\Omega) \cap H^{k+1}(\Omega))$, $D_t^\alpha u \in L^\infty(0, T; H_0^1(\Omega) \cap H^{k+1}(\Omega))$, and $\|\partial_t^l u\|_3 \lesssim 1 + t^{\alpha - l}$ for $l = 0, 1, 2, 3$. Then there exists a constant C such that

$$\max_{1 \le n \le N} \|\nabla u^n - \nabla u_h^n\| \le C \left(N^{-\min\{r\alpha, 2\}} + h^k \right). \tag{22}$$

If $r \ge 2/\alpha$, then one has

$$\max_{1 \le n \le N} \|u^n - u_h^n\|_{H^1(\Omega)} \le C \left(N^{-2} + h^k \right) \quad \text{for } n = 0, 1, \ldots, N.$$

Outline

Numerical experiments

Example 1

Consider the following problem with an exact analytical solution:

$$\begin{cases} D_t^\alpha u - \frac{\partial^2 u}{\partial x^2} = f(x,t) & \text{for } (x,t) \in (0,1) \times (0,1], \\ u(0,t) = u(1,t) = 0 & \text{for } t \in (0,1], \\ u(x,0) = (e^x - 1)(x-1) & \text{for } x \in [0,1]. \end{cases}$$

The function $f(x,t)$ in (23) is chosen such that the exact solution of the problem is $u(x,t) = (E_\alpha(-t^\alpha) + t^3)(e^x - 1)(x-1)$, where $E_\alpha(z) = \sum_{j=0}^{\infty} z^j/\Gamma(j\alpha + 1)$ is the Mittag-Leffler function. This solution u displays typical layer behaviour near $t = 0$.

Taking $r = (2-\alpha)/\alpha$ and $N = M$, the spatial error dominates the result. Predicted rate: $O(N^{-(2-\alpha)})$.

Table 1: $(L^\infty(H^1), N)$ errors and orders of convergence for L1 FEM

	M=N=64	M=N=128	M=N=256	$M = N = 512$
$\alpha = 0.4$	7.3119E-4	2.5834E-4	8.9420E-5	3.0529E-5
		1.5009	1.5306	1.5504
$\alpha = 0.6$	1.0224E-3	3.9823E-4	1.5356E-4	5.8847E-5
		1.3603	1.3747	1.3838
$\alpha = 0.8$	1.7345E-3	7.6222E-4	3.3358E-4	1.565E-4
		1.1862	1.1921	1.1955

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$$\Downarrow$$

$$\boxed{O(N^{-(2-\alpha)})}$$

Taking $r = 2/\alpha$ and $N = M$, the spatial error dominates the result. Predicted rate: $O(N^{-2})$.

Table 2: $(L^\infty(H^1), N)$ errors and orders of convergence for L2-1_σ FEM

	M=N=64	M=N=128	M=N=256	$M = N = 512$
$\alpha = 0.4$	1.9215E-3	4.9315E-4	1.2481E-4	3.1382E-5
		1.9621	1.9822	1.9917
$\alpha = 0.6$	1.1549E-3	2.9153E-4	7.3179E-5	1.8321E-5
		1.9860	1.9942	1.9979
$\alpha = 0.8$	7.6079E-4	1.9064E-4	4.7674E-5	1.1908E-5
		1.9966	1.9996	2.0011

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$\Downarrow$

$O(N^{-2})$

Thank You