Space-Time Variational Formulations for Maxwell's Equations

Julia I.M. Hauser, **Olaf Steinbach**

Institute for Applied Mathematics

Content

Content

The Differential Equations

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain, $T > 0$ and $Q := \Omega \times (0, T)$. Let $\epsilon, \mu \in [L^\infty(\Omega)]^3$ be symmetric and positive definite. Consider

$$
\begin{aligned}
\epsilon \partial_{tt} E + \nabla \times (\mu^{-1} \nabla \times E) &= -\partial_t J && \text{in } Q, \\
n \times E &= 0 && \text{on } \Sigma, \\
\epsilon \partial_t E(0, x) &= -J(0, x) + \nabla \times H_0(x) && \text{for } x \in \Omega, \\
E(0, x) &= E_0(x) && \text{for } x \in \Omega.
\end{aligned}
$$

Sobolev Spaces

We define

$$H^{curl;1}_{0;}(Q) := L^2(0, T; H_0(curl; \Omega)) \cap H^1(0, T; [L^2(\Omega)]^3),$$
$$H^{curl;1}_{0;,0}(Q) := L^2(0, T; H_0(curl; \Omega)) \cap H^1_{.,0}(0, T; [L^2(\Omega)]^3).$$

Sobolev Spaces - Properties

One can show that the embedding

$$H^1(0,T;[L^2(\Omega)]^3) \subset C([0,T],[L^2(\Omega)]^3)$$

is continuous, see e.g. [Zeidler, 1990].

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The space

$$C^\infty(0,T)\hat{\otimes}[C^\infty(\overline{\Omega})]^3$$

is dense in

$$H^1(0,T)\hat{\otimes}[L^2(\Omega)]^3 \text{ and } L^2(0,T)\hat{\otimes}H(curl;\Omega)$$

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See [Weidmann, 2000], Ch 1.6, and [Aubin, 2000], Ch 12.7, 12.6.

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$$H_{0;}^{curl;1}(Q) := L^2(0,T;H_0(curl;\Omega)) \cap H^1(0,T;[L^2(\Omega)]^3),$$
$$H_{0;,0}^{curl;1}(Q) := L^2(0,T;H_0(curl;\Omega)) \cap H^1_{\cdot,0}(0,T;[L^2(\Omega)]^3).$$

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Content

The Differential Equation

Consider the Maxwell's equations as the equation

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\begin{aligned}
\epsilon\partial_{tt}E + \nabla\times(\mu^{-1}\nabla\times E) &= -\partial_t J && \text{in } Q,\\
\epsilon\partial_t E(0,x) &= -J(0,x) + \nabla\times H_0(x) && \text{for } x\in\Omega,\\
E(0,x) &= E_0(x) && \text{for } x\in\Omega,\\
n\times E &= 0 && \text{on } \Sigma.
\end{aligned}
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The Differential Equation

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$$\begin{aligned} \epsilon\partial_{tt}u + \nabla\times(\mu^{-1}\nabla\times u) &= -\partial_t J && \text{in } Q,\\ \partial_t u(0,x) &= \psi(x) && \text{for } x\in\Omega,\\ u(0,x) &= \phi(x) && \text{for } x\in\Omega,\\ n\times u &= 0 && \text{on } \Sigma = \partial\Omega\times(0,T) \end{aligned}$$

for $\phi \in H_0(curl;\Omega)$, $\psi \in [L^2(\Omega)]^3$.

Definition

We define

$$
\begin{aligned}
\|u\|^2_{L^2(\Omega,\epsilon)} &:= (\epsilon u, u)_{L^2(\Omega)}, \\
\|u\|^2_{H(curl;\Omega,\epsilon,\mu)} &:= (\epsilon u, u)_{L^2(\Omega)} + (\mu^{-1}\nabla \times u, \nabla \times u)_{L^2(\Omega)}, \\
|u|^2_{H^{curl;1}(Q)} &:= (\epsilon \partial_t u, \partial_t u)_{L^2(Q)} + (\mu^{-1}\nabla \times u, \nabla \times u)_{L^2(Q)}.
\end{aligned}
$$

Existence and Uniqueness

Theorem

Let $\partial_t J \in L^1(0, T; [L^2(\Omega)]^3)$, $\phi \in H_0(curl; \Omega)$ *and* $\psi \in [L^2(\Omega)]^3$.

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Theorem

Let $\partial_t J \in L^1(0,T;[L^2(\Omega)]^3)$, $\phi \in H_0(curl;\Omega)$ *and* $\psi \in [L^2(\Omega)]^3$.

Then there exists a unique solution to the variational formulation:

Find $u \in H^{curl;1}_{0;}(Q)$, *with* $u(x,0) = \phi$, *such that*

$$-(\epsilon\partial_t u, \partial_t v)_Q + \left(\mu^{-1}\nabla \times u, \nabla \times v\right)_Q = -(\partial_t J, v)_Q - (\epsilon\psi, v(0,.))_\Omega$$

for all $v \in H^{curl;1}_{0;,0}(Q)$.

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for all $v \in H^{curl;1}_{0;,0}(Q)$. *If additionally* $\partial_t J \in [L^2(Q)]^3$ *then the following inequality holds true*

$$|u|^2_{H^{curl;1}(Q)} \leq 3T\|\psi\|^2_{L^2(\Omega,\epsilon)} + 3T\|\phi\|^2_{H(curl;\Omega,\epsilon,\mu)} + 3T^2\|\partial_t J\|^2_{[L^2(Q)]^3}.$$

The ODE corresponding to Maxwell's Equations

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and the Ansatzfunction

$$u(t,x) = \sum_i U_i(t) e_i(x),$$

where $\{e_i\}_{i \in \mathbb{N}}$ is a fundamental system in $H_0(curl; \Omega)$ which has the following property:

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for all $\phi \in H_0(curl;\Omega)$. [Monk, 2003]

The ODE corresponding to Maxwell's Equations

By rewriting the variational formulation

Find $u \in H^{curl;1}_{0;}(Q)$, with $u(x,0) = \phi$, such that

$$-(\epsilon \partial_t u, \partial_t v)_Q + \left(\mu^{-1} \nabla \times u, \nabla \times v\right)_Q = F(v)$$

for all $v \in H^{curl;1}_{0;,0}(Q)$.

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we get

Find $U_j \in H^1(0,T)$, with $U_j(0) = \phi$, such that either

$$-(\partial_t U_j, \partial_t V)_{(0,T)} + \kappa_j (U_j, V)_{(0,T)} = F(V e_j)$$

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or

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for all $V \in H^1_{,0}(0,T)$.

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$$J = \sigma E + J_a.$$

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Theorem

Let $\partial_t J_a \in L^1(0, T; [L^2(\Omega)]^3)$, $\phi \in H_0(curl; \Omega)$, $\psi \in [L^2(\Omega)]^3$ *and* $\sigma \in L^\infty(\Omega)$ *strictly positive a.e.*

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Definitions

Define

$$D^{Q_-}_{tt,cc} u := \epsilon \partial_{tt} u + \nabla \times (\mu^{-1} \nabla \times u)$$

as an operator in the distributional sense for distributions, i.e. $D^{Q_-}_{tt,cc} : \mathcal{D}'(Q_-) \to \mathcal{D}'(Q_-)$ and

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Consider

$$\mathcal{H}^{curl,1}(Q) := \{u|_Q \mid u \in L^2(Q_-), u|_{(-\infty,0)\times\Omega} = 0, D^{Q_-}_{tt,cc} T_u \in [H^{curl;1}_{0;,0}(Q)]'\}$$

with

$$Q_- := (-\infty, T) \times \Omega,$$

$$T_u(\phi) := \int_Q u\phi \, dx \, dt \qquad \forall \phi \in \mathcal{D}(Q),$$

$$\|u\|^2_{\mathcal{H}^{curl,1}(Q)} := \|u\|^2_{L^2(Q)} + \|D^{Q_-}_{tt,cc} T_u\|^2_{[H^{curl;1}_{0;,0}(Q)]'}.$$

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$$\mathcal{H}^{curl,1}_{0,}(Q) = \overline{H^{curl;1}_{0;0,}(Q)}^{\|\cdot\|_{\mathcal{H}^{curl,1}(Q)}}.$$

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We define

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If $u \in H^{curl;1}_{0;}(Q)$, with $\partial u(0,.) = 0$, and $v \in H^{curl;1}_{0;,0}(Q)$, then

$$\left\langle D^{Q_-}_{tt,cc} T_u, v \right\rangle_Q = -\left(\epsilon \partial_t u, \partial_t v\right)_{L^2(Q)} + \left(\mu^{-1} \nabla \times u, \nabla \times v\right)_{L^2(Q)}.$$

Further Generalization

Theorem

Let $f \in [H^{curl;1}_{0;,0}(Q)]'$ *then there exists a unique* $u \in \mathcal{H}^{curl,1}_{0,}(Q)$ *such that*

$$\left\langle D^{Q_-}_{tt,cc} T_u, v \right\rangle_Q = \langle f, v \rangle_Q \qquad \forall v \in H^{curl;1}_{0;,0}(Q)$$

and

$$\mathcal{L} : \mathcal{H}^{curl,1}_{0,}(Q) \to [H^{curl;1}_{0;,0}(Q)]'$$

with $\mathcal{L}u := f$ *is an isomorphism.*

Content

Example

We take a look at the following example over $Q = (0,1)^2 \times (0,1)$ with $\epsilon, \mu \equiv 1$

$$u(x,t) = t^2(x - x^2)(y - y^2)\begin{pmatrix} y \\ -x \end{pmatrix}.$$

J.I.M. Hauser, Institute for Applied Mathematics
2019-06-18

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Consider $u_h(t,x) = \sum_{i=1}^{N}\sum_{j=1}^{M} u_{i,j}\phi_i(t)\psi_j(x)$, where $\{\phi_i\}_i \subset S^1_{h_t}(\mathcal{T}^t_N)$ and $\{\vec{\psi}_j\}_j \subset N^I_\alpha(\mathcal{T}^x_M)$.

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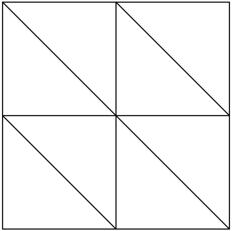
hx\ht	1/8	1/16	1/32	1/64	1/128	1/256
1/4	188.10	4897.40	18.3346	3.4329	0.01086	0.011
1/8	1148.722	31.69	10.2872	4.2366	0.01082	0.011
1/16	2.271	0.0099	0.0105	0.01088	0.011	0.011

CFL-Condition



For a form regular triangulation with isosceles rectangular triangles and $\mathcal{N}_0^I$ elements in space the CFL-condition is given by

$$h_t < \sqrt{\frac{12\pi}{18c_F^2}} h_x \approx 0,299725\ h_x.$$

Summary

- A variational formulation for the 2nd order PDE and $\partial_t J \in L^1(0, T; [L^2(\Omega)]^3)$

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J.I.M. Hauser, Institute for Applied Mathematics

Summary

- A variational formulation for the 2nd order PDE and $\partial_t J \in L^1(0, T; [L^2(\Omega)]^3)$
- A variational formulation for linear dependencies in the right hand side
- A variational formulation for the 2nd order PDE and $\partial_t J \in [H^{1;curl}_{,0;0}(Q)]'$
- One numerical example

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Number of Dofs

hx\ ht	0.125000	0.062500	0.031250
0.250000	909	1717	3333
0.125000	3393	6409	12441
0.062500	12897	24361	47289

hx\ ht	0.015625	0.007812	0.003906
0.250000	6565	13029	25957
0.125000	24505	48633	96889
0.062500	93145	184857	368281

The Variational Formulation for the System

Consider the operator $L : H^{curl;1}_{0;0,} \times H^{curl;1}_{;0,} \to [L^2(Q)]^3 \times [L^2(Q)]^3$ with

$$L\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}.$$

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Define

$$V := \overline{H^{curl;1}_{;0,} \times H^{curl;1}_{;0,}}^{\|\cdot\|_V}$$

with

$$\|v\|_V^2 := \|v\|^2_{[L^2(Q)]^3 \times [L^2(Q)]^3} + \|Lv\|^2_{[L^2(Q)]^3 \times [L^2(Q)]^3}.$$

The Variational Formulation for the System

Theorem

Let $(f, g) \in [L^2(Q)]^3 \times [L^2(Q)]^3$*, then there exists a unique pair* $(E, H) \in V$*, s.t.*

$$(L(E, H), (\phi, \nu))_W = ((f, g), (\phi, \nu))_W,$$

for all $(\phi, \nu) \in [L^2(Q)]^3 \times [L^2(Q)]^3$*.*

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for all $(\phi, \nu) \in [L^2(Q)]^3 \times [L^2(Q)]^3$. *Furthermore*

$$\|(E,H)\|_V \leq (1 + T^2 c(\epsilon,\mu)^2)^{1/2} \|(f,g)\|_W.$$

Generalization-Definitions

Define

$$D^{Q_-}_{tt,t,cc} u := \epsilon \partial_{tt} u - \sigma \partial_t u + \nabla \times (\mu^{-1} \nabla \times u)$$

as an operator in the distributional sense for distributions, i.e. $D^{Q_-}_{tt,t,cc} : \mathcal{D}'(Q_-) \to \mathcal{D}'(Q_-)$ and

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Consider

$$\hat{\mathcal{H}}^{curl,1}(Q) := \{u|_Q \mid u \in L^2(Q_-), u|_{(-\infty,0)\times\Omega} = 0, D^{Q_-}_{tt,t,cc} T_u \in [H^{curl;1}_{0;,0}(Q)]'\}$$

with

$$Q_- := (-\infty, T) \times \Omega,$$

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$$\hat{\mathcal{H}}^{curl,1}_{0,}(Q) = \overline{H^{curl;1}_{0;0,}(Q)}^{\|\cdot\|_{\hat{\mathcal{H}}^{curl,1}(Q)}}.$$

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It holds true that

- The space $(\hat{\mathcal{H}}^{curl,1}(Q), (.,.)_{\hat{\mathcal{H}}^{curl,1}(Q)})$ is a Hilbert space.

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- If $u, v \in H^{curl;1}(Q)$, then

$$\left(D^{Q_-}_{tt,t,cc}u, v\right)_Q = -(\epsilon\partial_t u, \partial_t v)_{L^2(Q)} + (\sigma\partial_t u, v)_{L^2(Q)} + \left(\mu^{-1}\nabla\times u, \nabla\times v\right)_{L^2(Q)}.$$

Further Generalization

Theorem

Let $f \in [H^{curl;1}_{0;,0}(Q)]'$ *then there exists a unique* $u \in \hat{\mathcal{H}}^{curl,1}_{0,}(Q)$ *such that*

$$\left(D^{Q_-}_{tt,t,cc} u, v\right)_Q = (f, v)_Q \qquad \forall v \in H^{curl;1}_{0;,0}(Q)$$

and

$$\mathcal{L} : \hat{\mathcal{H}}_{0,}(Q) \to [H^{curl;1}_{0;,0}(Q)]'$$

with $\mathcal{L}u := f$ *is an isomorphism.*

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and the variational formulation

$$-(\epsilon \partial_t \vec{u}, \partial_t \vec{v})_Q + (\mu^{-1} \nabla \times \vec{u}, \nabla \times \vec{v})_Q = \left(\vec{f}, \vec{v}\right)_Q .$$