Simulating a Heart Valve using a Varying Permeability Approach

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GOAL

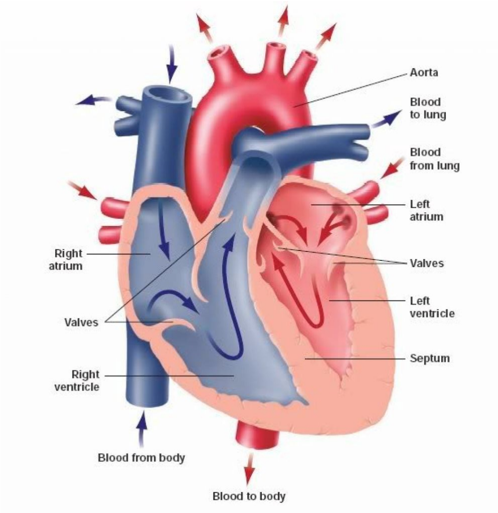
Patient-specific model of total heart function

Physics involved:

- Electrophysiology
- Mechanics
- Fluid Dynamics

Code basis:

Cardiac Arrhythmia Research Package



http://getdrawings.com/

Objective

GOAL

Model the impact of a valve in the human heart

ISSUES

- complex multiphysics problem
- interface conditions
- projecting boundary data from one domain into the other
- costly in terms of computation time

IDEA

Penalize the Navier-Stokes equations to model obstacles in the fluid domain (fictitious domain approach[1])

[1] Khodor Khadra et al. "Fictitious domain approach for numerical modelling of Navier–Stokes equations". In: *International Journal for Numerical Methods in Fluids* 34.8 (2000), pp. 651–684.

BENEFITS

- flexible and robust computational technique
- straight forward implementation
- saving computation time
- different media are accounted for by their physical characteristics assigned to the grid points
- time-dependent geometry and material properties can easily be taken into account

Domain and Boundary:

$$\begin{aligned} \Omega &= \Omega_f \cup \Omega_p \cup \Omega_s \\ \Gamma &= \Gamma_e \cup \Gamma_n \end{aligned} \tag{1}$$

Navier-Stokes-Brinkmann Model:

$$\begin{cases} \rho\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u},p)\right) + \frac{\mu}{K}\boldsymbol{u} = \rho\boldsymbol{f} & \text{in } \mathbb{R}^+ \times \Omega \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \mathbb{R}^+ \times \Omega \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{g}, & \text{on } \Gamma_e \\ \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{h}, & \text{on } \Gamma_n \end{cases} \tag{2}$$

Stress Tensor $\boldsymbol{\sigma}$:

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u},p) &= -p\mathbb{1} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \boldsymbol{\varepsilon}(\boldsymbol{u}) &= \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathsf{T}}) \end{aligned} \tag{3}$$

$$K(t,\boldsymbol{x}) = \begin{cases} K_f \to +\infty, & \forall \boldsymbol{x} \in \Omega_f \\ K_p, & \forall \boldsymbol{x} \in \Omega_p \\ K_s \to 0^+, & \forall \boldsymbol{x} \in \Omega_s \end{cases} \tag{4}$$

- Ω_f: $\frac{\mu}{K}\boldsymbol{u} \to 0$ classical Navier–Stokes equations
- Ω_p: full Navier-Stokes-Brinkmann equations
- Ω_s: $\frac{\mu}{K}\boldsymbol{u}$ forces $\boldsymbol{u} \to 0$ Darcy's law (5)

$$\nabla p + \frac{\mu}{K}\boldsymbol{u} = \rho \boldsymbol{f} \tag{5}$$

Note: No-slip condition on $\partial\Omega_s$ is asymptotically satisfied

Variational Arbitrary Lagrangian Eulerian (ALE) Formulation

The Navier-Stokes-Brinkman Model for a moving domain:

Find $\boldsymbol{u} \in \mathcal{S}_u$ and $p \in \mathcal{S}_p$, such that $\forall \boldsymbol{w} \in \mathcal{V}_u$ and $q \in \mathcal{V}_p$:

$$\int_\Omega \boldsymbol{w} \cdot \rho \left(\partial_t \boldsymbol{u} + \left(\boldsymbol{u} - \hat{\boldsymbol{u}} \right) \cdot \nabla \boldsymbol{u} - \boldsymbol{f} + \frac{\nu}{K} \boldsymbol{u} \right) d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{w}) : \boldsymbol{\sigma}(\boldsymbol{u}, p)\, d\Omega - \int_{\Gamma_n} \boldsymbol{w} \cdot \boldsymbol{h}\, d\Gamma + \int_\Omega q \nabla \cdot \boldsymbol{u}\, d\Omega = 0 \tag{6}$$

$\boldsymbol{u}$ . . . fluid velocity

$\hat{\boldsymbol{u}}$ . . . fluid domain velocity respective to a reference domain

RBVMS[2] formulation of the ALE NSB equations:

Decomposition into coarse (^{h}) and fine ($'$) scale subspaces:

$$\mathcal{S}_u = \mathcal{S}_u^h \oplus \mathcal{S}_u' \qquad \mathcal{S}_p = \mathcal{S}_p^h \oplus \mathcal{S}_p' \qquad (7)$$
$$\mathcal{V}_u = \mathcal{V}_u^h \oplus \mathcal{V}_u' \qquad \mathcal{V}_p = \mathcal{V}_p^h \oplus \mathcal{V}_p'$$

Model the fine scale velocity and the fine scale pressure:

$$\boldsymbol{u}' = -\frac{\tau_{SUPS}}{\rho}\, \boldsymbol{r}^M(\boldsymbol{u}^h, p^h) \qquad (8)$$
$$p' = -\rho\, \nu_{LSIC}\, r^C(\boldsymbol{u}^h)$$

Residuals $\boldsymbol{r}^M$ and r^C:

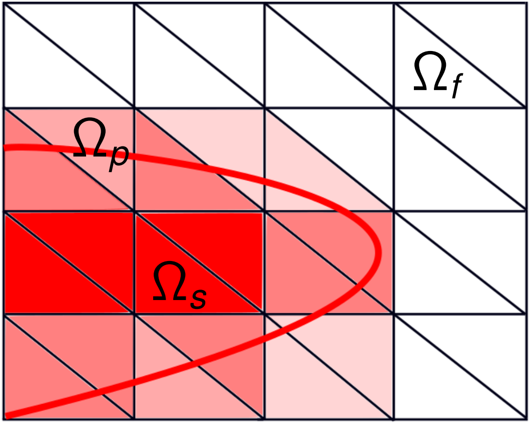
$$\boldsymbol{r}^M(\boldsymbol{u}^h, p^h) = \rho\left(\partial_t \boldsymbol{u}^h + \left(\boldsymbol{u} - \hat{\boldsymbol{u}}\right)\cdot\nabla\boldsymbol{u} - \boldsymbol{f}^h + \frac{\nu}{K}\boldsymbol{u}^h\right) - \nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}^h, p^h) \qquad (9)$$
$$r^C(\boldsymbol{u}^h) = \nabla\cdot\boldsymbol{u}^h$$

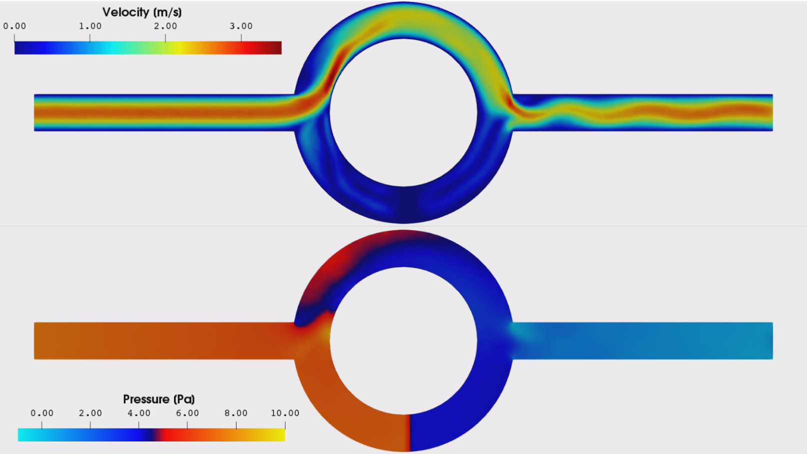
[2] Yuri Bazilevs, Kenji Takizawa, and Tayfun E. Tezduyar. *Computational Fluid-Structure Interaction: Methods and Applications*. English. John Wiley and Sons, 2013.

Find $\boldsymbol{u}^h \in \mathcal{S}_u^h$ and $p^h \in \mathcal{S}_p^h$, such that $\forall \boldsymbol{w}^h \in \mathcal{V}_u^h$ and $q^h \in \mathcal{V}_p^h$:

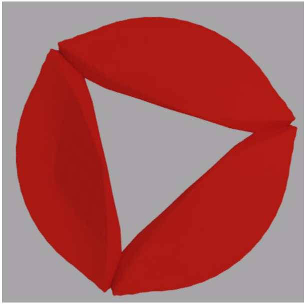
$$
\begin{aligned}
&\int_\Omega \boldsymbol{w}^h \cdot \rho \left(\partial_t \boldsymbol{u}^h + \left(\boldsymbol{u}^h - \hat{\boldsymbol{u}}^h\right) \cdot \nabla \boldsymbol{u}^h - \boldsymbol{f}^h + \frac{\nu}{K}\boldsymbol{u}^h\right) d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{w}^h) : \boldsymbol{\sigma}(\boldsymbol{u}^h, p^h)\, d\Omega \\
&- \int_{\Gamma_n} \boldsymbol{w}^h \cdot \boldsymbol{h}^h d\Gamma + \int_{\Omega_t} q^h \nabla \cdot \boldsymbol{u}^h\, d\Omega \\
&+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \boxed{\tau_{SUPS}} \left(\left(\boldsymbol{u}^h - \hat{\boldsymbol{u}}^h\right) \cdot \nabla \boldsymbol{w}^h + \frac{\nabla q^h}{\rho} - \frac{\nu}{K}\boldsymbol{w}^h\right) \cdot \boxed{\boldsymbol{r}^M(\boldsymbol{u}^h, p^h)}\, d\Omega \\
&+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \boxed{\rho \nu_{LSIC}}\, \nabla \cdot \boldsymbol{w}^h\, \boxed{r^C(\boldsymbol{u}^h)}\, d\Omega \\
&- \sum_{e=1}^{n_{el}} \int_{\Omega_e} \boxed{\tau_{SUPS}}\, \boldsymbol{w}^h \cdot \left(\boxed{\boldsymbol{r}^M(\boldsymbol{u}^h, p^h)} \cdot \nabla \boldsymbol{u}^h\right) d\Omega \\
&- \sum_{e=1}^{n_{el}} \int_{\Omega_e} \frac{\nabla \boldsymbol{w}^h}{\rho} : \left(\boxed{\tau_{SUPS}\, \boldsymbol{r}^M(\boldsymbol{u}^h, p^h)}\right) \otimes \left(\boxed{\tau_{SUPS}\left(\boldsymbol{r}^M(\boldsymbol{u}^h, p^h)\right)}\right) d\Omega = 0
\end{aligned}
\tag{10}
$$

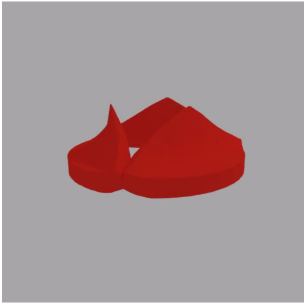
How to choose the permeability parameters to model the valve?




Velocity (m/s)
0.00
1.00
2.00
3.00
Pressure (Pa)
0.00
2.00
4.00
6.00
8.00
10.00

1st STEP: Moving Valve



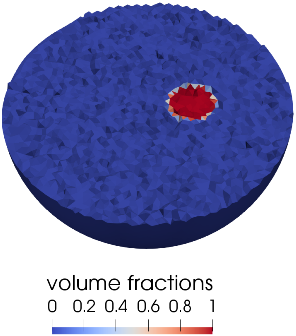


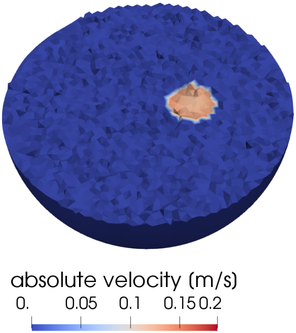
Adapt the penalization term to fulfill no-slip condition[3] :

$$\frac{\mu}{K}\boldsymbol{u} \rightarrow \frac{\mu}{K}(\boldsymbol{u} - \boldsymbol{u}_S) \qquad (11)$$

[3] Dmitry Kolomenskiy and Kai Schneider. "A Fourier spectral method for the Navier-Stokes equations with volume penalization for moving solid obstacles". In: *J. Comput. Physics* 228 (2009), pp. 5687–5709.

2nd STEP: Moving Valve + Moving Mesh





PRO

- suitable to model the impact of the valve
- straight forward implementation
- robust and fast

CONTRA

- not physical in the direct vicinity of the valve

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