Mesh adaptivity and error estimates for optimal control problems and multiple quantities of interest

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What is the quantity of interest ?

- Y, U, V Banach spaces
- $I, J : Y \times U \mapsto \mathbb{R}$
- $A : Y \times U \mapsto V^*$

The Goal

Find $I(y, u) \in \mathbb{R}$ such that $(y, u) \in Y \times U$ is a minimizer of

$$\min_{(y,u)\in Y\times U} J(y, u),$$

$$A(y, u) = 0.$$

Example for A: regularized p-Laplace

- The PDE (in strong form)

$$-\operatorname{div}\left((1+|\nabla y|^2)^{\frac{p-2}{2}}\nabla y\right) = f+u \qquad \text{in } L^2(\Omega),$$
$$y = 0 \qquad \text{on } \partial\Omega.$$

- The corresponding operator A is given by the form

$$A(y,u)(v) := \int_\Omega \left[(f+u)v - (\varepsilon^2+|\nabla y|^2)^{\frac{p-2}{2}}\nabla y\cdot\nabla v\right]\ \mathrm{d}x,$$

for all $y \in Y$ and $v \in V$.

Example for J

$$J(y,u) := \frac{1}{2}\|y - y^d\|_X^2 + \frac{\alpha}{2}\|u - u^d\|_Y^2$$

Examples for I

- $I(y,u) = \int_{\Omega_1 \subset \Omega} y(x)\ \mathrm{d}x$
- $I(y,u) = \int_{\Omega_2 \subset \Omega} u(x)\ \mathrm{d}x$
- $I(y,u) := \frac{1}{2}\|y - y^d\|_X^2$
- $I(y,u) := \frac{1}{2}\|u - u^d\|_Y^2$
- $I(y,u) := \frac{1}{2}\int_\Omega y(x)^2 u(x)^2\ \mathrm{d}x$

Solution operator

Assumption

Let us assume there exists a unique bijective operator $S : U \mapsto Y$, such that

$$A(S(u), u) = 0, \qquad \forall u \in U.$$

Reformulation of the Goal in Reduced Form

The Goal

Find $I(y,u) \in \mathbb{R}$ such that $(y,u) \in Y \times U$ is a minimizer of

$$\min_{(y,u)\in Y\times U} J(y,u),$$

$$A(y,u) = 0.$$

The Goal in Reduced Form

Find $i(u) := I(S(u),u) \in \mathbb{R}$ such that $u \in U$ is a minimizer of

$$\min_{u\in U} j(u) := J(S(u),u).$$

Discrete Solution operator

- $Y_h \subset Y, V_h \subset V, U_h \subset U$ finite dimensional subspaces

Assumption

Let us assume there exists a unique bijective operator $S_h : U_h \mapsto Y_h$, such that

$$A(S_h(u_h), u_h) = 0, \qquad \forall u_h \in U_h.$$

The finite dimensional problem

Find $i_h(u_h) := I(S_h(u_h), u_h) \in \mathbb{R}$ such that $u_h \in U_h$ minimizes

$$\min_{u_h \in U_h} j_h(u_h) := J(S_h(u_h), u_h).$$

We wish that

- the error $i(u) - i_h(\tilde{u}_h)$ is small,
- low computational cost.

Our Solution:

- adaptive refinement for our goal functional i.
  This means
  solve $\rightarrow$ *estimate* $\rightarrow$ *mark* $\rightarrow$ *refine.*

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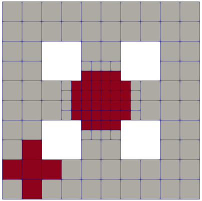
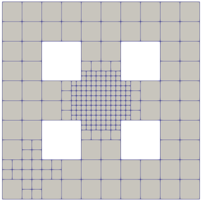


$\rightarrow$



We employ the dual weighted residual method [Becker & Rannacher (2001)]:

Adjoint problem for the reduced Problem

Find $v \in U$ such that

$$j''(u)(\mathfrak{u}, v) = i'(u)(\mathfrak{u}) \qquad \forall \mathfrak{u} \in U,$$

for $u \in U$ minimizing $j(u)$.

Error Representation

- split in discretization and linearization error

Theorem (error representation); see [Rannacher & Vihharev 2013]

Let $\tilde{u} \in U$, $\tilde{v} \in U$ and $j \in C^4$. Then it holds:

$$
\begin{aligned}
i(u) - i(\tilde{u}) =& \frac{1}{2}\left[-j'(\tilde{u})(v-\tilde{v}) - i'(\tilde{u})(u-\tilde{u}) + j''(\tilde{u})(u-\tilde{u},\tilde{v})\right] \\
&+ j'(\tilde{u})(\tilde{v}) + \mathcal{O}(e^3),
\end{aligned}
$$

where

$$
e := \max(\|u-\tilde{u}\|_U, \|v-\tilde{v}\|_U).
$$

Error Estimates

For approximations $\tilde{u}_h, \tilde{v}_h$ of u, v, it holds:

$$i(u) - i(\tilde{u}_h) \approx \underbrace{\frac{1}{2}\left[-j'(\tilde{u}_h)(v - \tilde{v}_h) - i'(\tilde{u}_h)(u - \tilde{u}_h) + j''(\tilde{u}_h)(u - \tilde{u}_h, \tilde{v}_h)\right]}_{\eta_h^S}$$

$$+ \underbrace{j'(\tilde{u}_h)(\tilde{v}_h)}_{\eta_k^S}.$$

It turns out that

- η_h^S is related to the discretization error $|i(u) - i(u_h)|$.
- η_k^S is related to the linearization error $|i(u_h) - i(\tilde{u}_h)|$.
- $|\eta_k^S| \leq \gamma |\eta_h^S|$ with $\gamma \in (0, 1]$ can be used as stopping rule for the nonlinear solver.

solve $\rightarrow$ *estimate* $\rightarrow$ *mark* $\rightarrow$ *refine.*

solve → *estimate* → *mark* → *refine.*

Localization

- $j(u)$ and $i(u)$ depend on S.
- in general S is not localizeable (practically).

solve → *estimate* → *mark* → *refine.*

solve $\rightarrow$ *estimate* $\not\rightarrow$ *mark* $\rightarrow$ *refine.*

solve $\rightarrow$ *estimate* $\rightarrow$ *mark* $\rightarrow$ *refine.*

The Goal

Find $I(y,u) \in \mathbb{R}$ such that $(y,u) \in Y \times U$ is a minimizer of

$$\min_{(y,u)\in Y\times U} J(y,u),$$
$$A(y,u) = 0.$$

We define the Lagrangian as

$$\mathcal{L}(y,u,\lambda) := J(y,u) - A(y,u)(\lambda).$$

Abbreviation for some operator B:

$$B'_\zeta := \frac{\partial}{\partial \zeta} B$$

We employ the dual weighted residual method [Becker & Rannacher (2001)]:

Adjoint problem

Find $(z, v_2, \gamma) \in Y \times U \times V$ such that

$$\begin{pmatrix} \mathcal{L}''_{yy} & \mathcal{L}''_{yu} & \mathcal{L}''_{y\lambda} \\ \mathcal{L}''_{uy} & \mathcal{L}''_{uu} & \mathcal{L}''_{u\lambda} \\ \mathcal{L}''_{\lambda y} & \mathcal{L}''_{\lambda u} & 0 \end{pmatrix} \begin{pmatrix} z \\ v_2 \\ \gamma \end{pmatrix} = - \begin{pmatrix} I'_y \\ I'_u \\ 0 \end{pmatrix},$$

for $(y, u, \lambda) \in Y \times U \times V$ solving $\mathcal{L}'(y, u, \lambda) = 0$.

It turn out that $v = v_2$.

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for $(y, u, \lambda) \in Y \times U \times V$ solving $\mathcal{L}'(y, u, \lambda) = 0$.

Theorem (Error Representation (see [Vexler,Wollner 2008]))

Let us assume that $\mathcal{L} \in \mathcal{C}^4$ *and* $I \in \mathcal{C}^3$*. Then it holds for arbitrary fixed* $\tilde{\xi}_h := (\tilde{y}_h, \tilde{u}_h, \tilde{\lambda}_h)$*,* $\tilde{\xi}_h^* := (\tilde{z}_h, \tilde{v}_h, \tilde{\gamma}_h)$

$$\begin{aligned} I(y,u) - I(\tilde{y}_h, \tilde{u}_h) = &\frac{1}{2}[\rho_y(\tilde{\xi}_h, \tilde{\xi}_h^*)(\gamma - \tilde{\gamma}_h) + \rho_\lambda(\tilde{\xi}_h, \tilde{\xi}_h^*)(z - \tilde{z}_h) + \rho_u(\tilde{\xi}_h, \tilde{\xi}_h^*)(v - \tilde{v}_h) \\ &+\rho_z(\tilde{\xi}_h, \tilde{\xi}_h^*)(\lambda - \tilde{\lambda}_h) + \rho_\gamma(\tilde{\xi}_h, \tilde{\xi}_h^*)(y - \tilde{y}_h) + \rho_p(\tilde{\xi}_h, \tilde{\xi}_h^*)(u - \tilde{u}_h)] - \mathcal{L}"(\tilde{\xi}_h)(\tilde{\xi}_h, \tilde{\xi}_h^*) \\ &+\mathcal{R}^{(3)}, \end{aligned}$$

where

$$\begin{aligned} \rho_y(\tilde{\xi}_h, \tilde{\xi}_h^*)(\cdot) :=& \mathcal{L}'_\lambda(\tilde{\xi}_h)(\cdot), \\ \rho_u(\tilde{\xi}_h, \tilde{\xi}_h^*)(\cdot) :=& \mathcal{L}'_u(\tilde{\xi}_h)(\cdot), \\ \rho_\lambda(\tilde{\xi}_h, \tilde{\xi}_h^*)(\cdot) :=& \mathcal{L}'_y(\tilde{\xi}_h)(\cdot), \\ \rho_z(\tilde{\xi}_h, \tilde{\xi}_h^*)(\cdot) :=& \mathcal{L}''_{\lambda y}(\tilde{\xi}_h)(\cdot, \tilde{z}_h) + \mathcal{L}''_{\lambda u}(\tilde{\xi}_h)(\cdot, \tilde{v}_h), \\ \rho_v(\tilde{\xi}_h, \tilde{\xi}_h^*)(\cdot) :=& I'_u(\tilde{y}_h, \tilde{u}_h)(\cdot) + \mathcal{L}''_{yu}(\tilde{\xi}_h)(\tilde{z}_h, \cdot) + \mathcal{L}''_{uu}(\tilde{\xi}_h)(\tilde{v}_h, \cdot) + \mathcal{L}''_{\lambda u}(\tilde{\xi}_h)(\tilde{\gamma}_h, \cdot), \\ \rho_\gamma(\tilde{\xi}_h, \tilde{\xi}_h^*)(\cdot) :=& I'_y(\tilde{y}_h, \tilde{u}_h)(\cdot) + \mathcal{L}''_{yy}(\tilde{\xi}_h)(\tilde{z}_h, \cdot) + \mathcal{L}''_{uy}(\tilde{\xi}_h)(\tilde{v}_h, \cdot) + \mathcal{L}''_{\lambda y}(\tilde{\xi}_h)(\tilde{\gamma}_h, \cdot), \end{aligned}$$

with $\tilde{\xi}_h := (\tilde{y}_h, \tilde{u}_h, \tilde{\lambda}_h)$*,* $\tilde{\xi}_h^* := (\tilde{z}_h, \tilde{v}_h, \tilde{\gamma}_h)$*.*

Error Estimates

For approximations $\tilde{\xi}_h, \tilde{\xi}_h^*$ of $(y, u, \lambda), (z, v, \gamma)$, it holds:

$$I(y,u) - I(\tilde{y}_h, \tilde{u}_h) = \frac{1}{2}\Big[\underbrace{\rho_y(\tilde{\xi}_h, \tilde{\xi}_h^*)(\gamma - \tilde{\gamma}_h) + \rho_\lambda(\tilde{\xi}_h, \tilde{\xi}_h^*)(z - \tilde{z}_h) + \rho_u(\tilde{\xi}_h, \tilde{\xi}_h^*)(v - \tilde{v}_h)}_{\eta_{h,p}}$$

$$+ \underbrace{\rho_z(\tilde{\xi}_h, \tilde{\xi}_h^*)(\lambda - \tilde{\lambda}_h) + \rho_\gamma(\tilde{\xi}_h, \tilde{\xi}_h^*)(y - \tilde{y}_h) + \rho_v(\tilde{\xi}_h, \tilde{\xi}_h^*)(u - \tilde{u}_h)}_{\eta_{h,a}}\Big]$$

$$- \underbrace{\mathcal{L}''(\tilde{\xi}_h)(\tilde{\xi}_h, \tilde{\xi}_h^*)}_{\eta_k} + \mathcal{R}^{(3)}$$

It turns out that

- $\eta_h := \frac{1}{2}(\eta_{h,p} + \eta_{h,a})$ is related to the discretization error $I(y,u) - I(y_h, u_h)$.
- η_k is related to the iteration error $I(y_h, u_h) - I(\tilde{y}_h, \tilde{u}_h)$.
- $|\eta_k| \le \gamma |\eta_h|$ with $\gamma \in (0,1]$ can be used as stopping rule for the nonlinear sol zer.

Error Estimates

Since

$$I(y,u) - I(\tilde{y}_h,\tilde{u}_h) \approx \frac{1}{2}\Big[\underbrace{\rho_y(\tilde{\xi}_h,\tilde{\xi}_h^*)(\gamma-\tilde{\gamma}_h) + \rho_\lambda(\tilde{\xi}_h,\tilde{\xi}_h^*)(z-\tilde{z}_h) + \rho_u(\tilde{\xi}_h,\tilde{\xi}_h^*)(v-\tilde{v}_h)}_{\eta_{h,p}}$$

$$+ \underbrace{\rho_z(\tilde{\xi}_h,\tilde{\xi}_h^*)(\lambda-\tilde{\lambda}_h) + \rho_\gamma(\tilde{\xi}_h,\tilde{\xi}_h^*)(y-\tilde{y}_h) + \rho_v(\tilde{\xi}_h,\tilde{\xi}_h^*)(u-\tilde{u}_h)}_{\eta_{h,a}}\Big]$$

$$- \underbrace{\mathcal{L}''(\tilde{\xi}_h)(\tilde{\xi}_h,\tilde{\xi}_h^*)}_{\eta_k} + \mathcal{R}^{(3)},$$

holds for all approximations $\tilde{\xi}_h, \tilde{\xi}_h^*$, it also holds for $\tilde{y}_h = S_h(\tilde{u}_h)$.

$$i_h(\tilde{u}_h) := I(S_h(\tilde{u}_h),\tilde{u}_h)$$

$$j_h(\tilde{u}_h) := J(S_h(\tilde{u}_h),\tilde{u}_h)$$

Error Estimates

Since

$$
\begin{aligned}
i(u) - i_h(\tilde{u}_h) \approx \frac{1}{2}\Big[& \underbrace{\rho_y(\tilde{\xi}_h, \tilde{\xi}_h^*)(\gamma - \tilde{\gamma}_h) + \rho_\lambda(\tilde{\xi}_h, \tilde{\xi}_h^*)(z - \tilde{z}_h) + \rho_u(\tilde{\xi}_h, \tilde{\xi}_h^*)(v - \tilde{v}_h)}_{\eta_{h,p}} \\
& + \underbrace{\rho_z(\tilde{\xi}_h, \tilde{\xi}_h^*)(\lambda - \tilde{\lambda}_h) + \rho_\gamma(\tilde{\xi}_h, \tilde{\xi}_h^*)(y - \tilde{y}_h) + \rho_v(\tilde{\xi}_h, \tilde{\xi}_h^*)(u - \tilde{u}_h)}_{\eta_{h,a}} \Big] \\
& - \underbrace{{j_h}'(\tilde{u}_h)(\tilde{v}_h)}_{\eta_k} + \mathcal{R}^{(3)},
\end{aligned}
$$

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$$i_h(\tilde{u}_h) := I(S_h(\tilde{u}_h), \tilde{u}_h)$$

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Error Estimates

For approximations $\tilde{\xi}_h, \tilde{\xi}_h^*$ of $(y, u, \lambda), (z, v, \gamma)$, it holds:

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- $\eta_h := \frac{1}{2}(\eta_{h,p} + \eta_{h,a})$ is related to the discretization error $i(u) - i_h(u_h)$.
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- $|\eta_k| \leq \gamma|\eta_h|$ with $\gamma \in (0, 1]$ can be used as stopping rule for the nonlinear solver.

Error Estimates

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Error Estimates using Higher Order Approximation

For approximations $\tilde{\xi}_h, \tilde{\xi}_h^*$ of $(y, u, \lambda), (z, v, \gamma)$, it holds:

$$i(u) - i_h(\tilde{u}_h) \approx \frac{1}{2}\Big[\underbrace{\rho_y(\tilde{\xi}_h, \tilde{\xi}_h^*)(\gamma_h^{(2)} - \tilde{\gamma}_h) + \rho_\lambda(\tilde{\xi}_h, \tilde{\xi}_h^*)(z_h^{(2)} - \tilde{z}_h) + \rho_u(\tilde{\xi}_h, \tilde{\xi}_h^*)(v_h^{(2)} - \tilde{v}_h)}_{\eta_{h,p}^{(2)}}$$

$$+ \underbrace{\rho_z(\tilde{\xi}_h, \tilde{\xi}_h^*)(\lambda_h^{(2)} - \tilde{\lambda}_h) + \rho_\gamma(\tilde{\xi}_h, \tilde{\xi}_h^*)(y_h^{(2)} - \tilde{y}_h) + \rho_v(\tilde{\xi}_h, \tilde{\xi}_h^*)(u_h^{(2)} - \tilde{u}_h)}_{\eta_{h,a}^{(2)}}\Big]$$

$$- \underbrace{{j_h}'(\tilde{u}_h)(\tilde{v}_h)}_{\eta_k}$$

It turns out that

- $\eta_h^{(2)} := \frac{1}{2}\left(\eta_{h,p}^{(2)} + \eta_{h,a}^{(2)}\right)$ is related to the discretization error $i(u) - i_h(u_h)$.
- η_k is related to the iteration error $i_h(u_h) - i_h(\tilde{u}_h)$.
- $|\eta_k| \leq \gamma|\eta_h|$ with $\gamma \in (0, 1]$ can be used as stopping rule for the nonlinear solver.

Strengthened Saturation Assumption

Strengthened Saturation Assumption for the Goal Functional

Let $u_h^{(2)}$ be a minimizer on an enriched discrete space, and let $\tilde{u}_h$ be some approximation. Then we assume that the inequality

$$|i(u) - i_h^{(2)}(u_h^{(2)})| + |\mathcal{R}^{(3)} - \mathcal{R}^{(3)(2)}| + |\eta_k| + |\mathcal{R}^{(3)}| < b_{h,\gamma}|i(u) - i_h(\tilde{u}_h)|$$

holds true for some $b_{h,\gamma} < b_{0,\gamma}$ with some fixed $b_{0,\gamma} \in (0, 1)$.

Efficency and Reliability

Theorem (see [Endtmayer, Langer & Wick 2018])

Let the Strengthened Saturation Assumption be fulfilled, then $\eta_h^{(2)}$ *satisfies the efficiency and reliability estimates*

$$\underline{c}_\gamma |\eta_h^{(2)}| \leq |i(u) - i_h(\tilde{u}_h)| \leq \overline{c}_\gamma |\eta_h^{(2)}|,$$

with the positive constants $\underline{c}_\gamma := 1/(1 + b_{0,\gamma})$, $\overline{c}_\gamma := 1/(1 - b_{0,\gamma})$.

solve $\rightarrow$ *estimate* $\rightarrow$ *mark* $\rightarrow$ *refine.*

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Localization and Marking

- localizeable using integration per parts as in residual type error estimates
- localizeable using partition of unity
- use an arbitrary marking strategy (like Dörfler marking)

solve $\rightarrow$ *estimate* $\rightarrow$ *mark* $\rightarrow$ *refine.*

Multiple Goal Approach

- up to now we considered only one functional $I(.)$
- now we consider N functionals $I_1, I_2, \ldots, I_N$
- refine until $|I_\ell(y,u) - I_\ell(\tilde{y}_h, \tilde{u}_h)| < TOL_\ell$

$\rightarrow$ but that means we have to solve **N** linear systems !

$\rightarrow$ try to combine the functionals

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Example: A simple Combination

$$I_c = I_1 + I_2 + I_3$$

- $I_1(y,u) - I_1(\tilde{y}_h, \tilde{u}_h) = 1$
- $I_2(y,u) - I_2(\tilde{y}_h, \tilde{u}_h) = -1$
- $I_3(y,u) - I_3(\tilde{y}_h, \tilde{u}_h) = 0$

$\Longrightarrow \quad I_c(y,u) - I_c(\tilde{y}_h, \tilde{u}_h) = 0$

Definition (error-weighting function [Endtmayer, Langer & Wick 2018])

We say that $\mathfrak{E} : (\mathbb{R}_0^+)^N \times M \subseteq \mathbb{R}^N \mapsto \mathbb{R}_0^+$ is an **error-weighting function** if

- $\mathfrak{E}(\cdot, m) \in C^3((\mathbb{R}_0^+)^N, \mathbb{R}_0^+)$
- $\mathfrak{E}(\cdot, m)$ strictly monotonically increasing in each component
- $\mathfrak{E}(0, m) = 0$ for all $m \in M$.

- $\vec{I}(\mathfrak{y}, \mathfrak{u}) := (I_1(\mathfrak{y}, \mathfrak{u}), I_2(\mathfrak{y}, \mathfrak{u}), \ldots, I_N(\mathfrak{y}, \mathfrak{u}))$
- $|x|_N := (|x_1|, |x_2|, \ldots, |x_N|)$

This allows us to define the **error functional** as follows

$$\tilde{I}_{\mathfrak{E}}(\mathfrak{y}, \mathfrak{u}) := \mathfrak{E}(|\vec{I}(y, u) - \vec{I}(\mathfrak{y}, \mathfrak{u})|_N, \vec{I}(\tilde{y}_h, \tilde{u}_h)) \qquad \forall (\mathfrak{y}, \mathfrak{u}) \in \bigcap_{\ell=1}^{N} \mathcal{D}(I_\ell).$$

- This generalizes the approach of [Hartmann & Housten (2003)].

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- $|x|_N := (|x_1|, |x_2|, \ldots, |x_N|)$

This allows us to define the **error functional** as follows

$$\tilde{I}_{\mathfrak{E}}(\mathfrak{y}, \mathfrak{u}) := \mathfrak{E}(|\vec{I}(y, u) - \vec{I}(\mathfrak{y}, \mathfrak{u})|_N, \vec{I}(\tilde{y}_h, \tilde{u}_h)) \qquad \forall (\mathfrak{y}, \mathfrak{u}) \in \bigcap_{\ell=1}^{N} \mathcal{D}(I_\ell).$$

- This generalizes the approach of [Hartmann & Housten (2003)].

Definition (error-weighting function)

We say that $\mathfrak{E} : (\mathbb{R}_0^+)^N \times M \subseteq \mathbb{R}^N \mapsto \mathbb{R}_0^+$ is an **error-weighting function** if

- $\mathfrak{E}(\cdot, m) \in C^3((\mathbb{R}_0^+)^N, \mathbb{R}_0^+)$
- $\mathfrak{E}(\cdot, m)$ strictly monotonically increasing in each component
- $\mathfrak{E}(0, m) = 0$ for all $m \in M$.

- $\vec{I}(\mathfrak{y}, \mathfrak{u}) := (I_1(\mathfrak{y}, \mathfrak{u}), I_2(\mathfrak{y}, \mathfrak{u}), \ldots, I_N(\mathfrak{y}, \mathfrak{u}))$
- $|x|_N := (|x_1|, |x_2|, \ldots, |x_N|)$

This allows us to define the **error functional** as follows

$$I_{\mathfrak{E}}(\mathfrak{y}, \mathfrak{u}) := \mathfrak{E}(|\vec{I}(y_h^{(2)}, u_h^{(2)}) - \vec{I}(\mathfrak{y}, \mathfrak{u})|_N, \vec{I}(\tilde{y}_h, \tilde{u}_h)) \qquad \forall (\mathfrak{y}, \mathfrak{u}) \in \bigcap_{\ell=1}^{N} \mathcal{D}(I_\ell).$$

- This generalizes the approach of [Hartmann & Housten (2003)].

Numerical Results

The Model Problem

Find $(y, u) \in W_0^{k,p}(\Omega) \times L^2(\Omega)$ such that it minimizes

$$\min_{(y,u)\in W_0^{k,p}(\Omega)\times L^2(\Omega)} J(y, u) := \frac{1}{2}\|y - y^d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)},$$

under the constraints

$$\begin{aligned} -\operatorname{div}\left((1 + |\nabla y|^2)^{\frac{p-2}{2}}\nabla y\right) &= f + u \qquad \text{in } \Omega, \\ y &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

with $p = 10$, $\alpha = 10^{-5}$, $f = 0$.

Functionals of interest

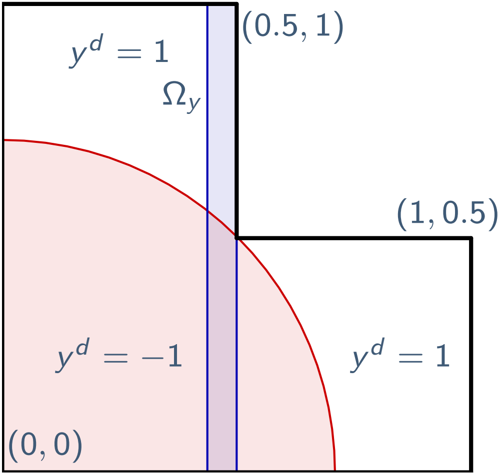
- $I_1(y,u) := \frac{1}{2}\|y - y^d\|^2_{L^2(\Omega)}$,
- $I_2(y,u) := \frac{1}{2}\|u\|^2_{L^2(\Omega)}$,
- $I_3(y,u) := \int_{\Omega_y} y(x) \, \mathbf{d}x$,
- $I_4(y,u) := \|u\|_{L^1(\Omega)}$,
- $I_5(y,u) := \|yu\|^2_{L^2(\Omega)}$.

We used

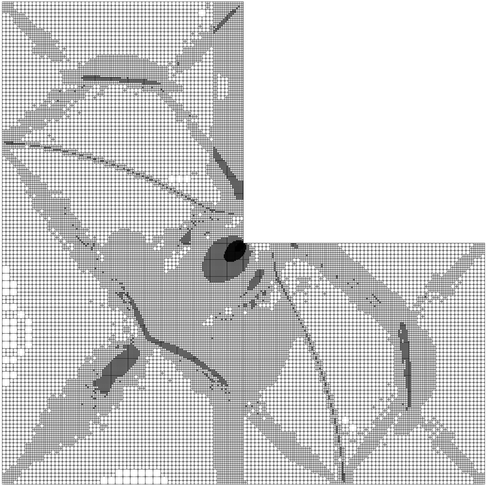
$$I_{\mathfrak{E}}(\tilde{y}_h, \tilde{u}_h) := \sum_{\ell=1}^{5} \frac{|I_\ell(y_h^{(2)}, u_h^{(2)}) - I_\ell(\tilde{y}_h, \tilde{u}_h)|}{|I_\ell(\tilde{y}_h, \tilde{u}_h)|},$$

following the idea in [Hartmann & Housten (2003)].

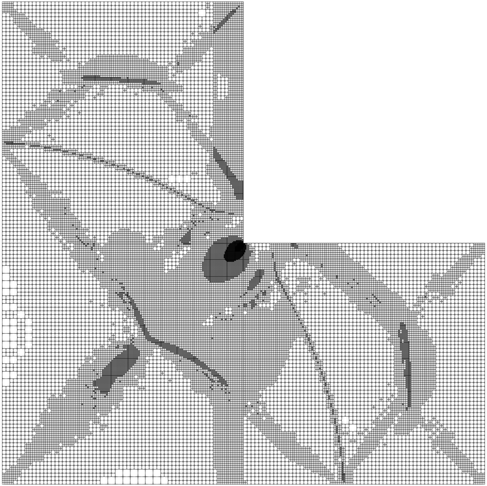
The domain



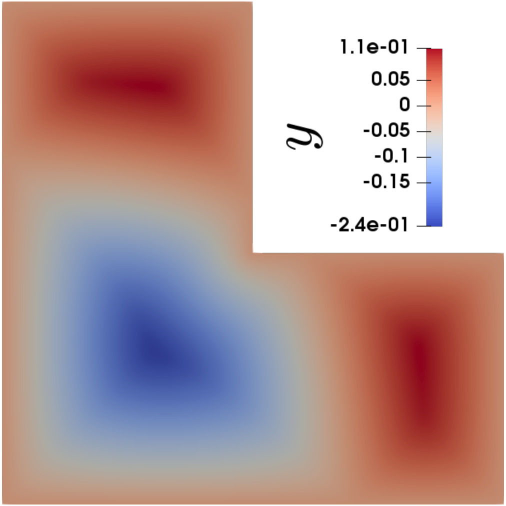
Resulting Mesh



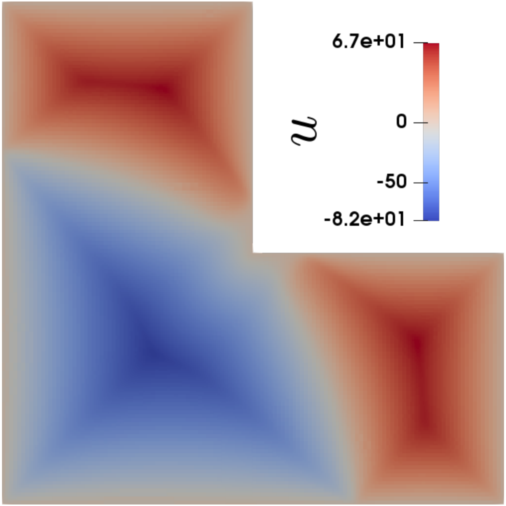
Resulting Mesh

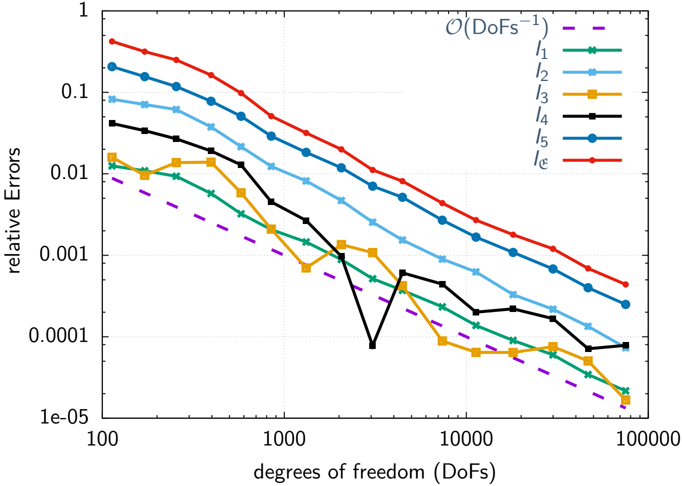


Solution: State

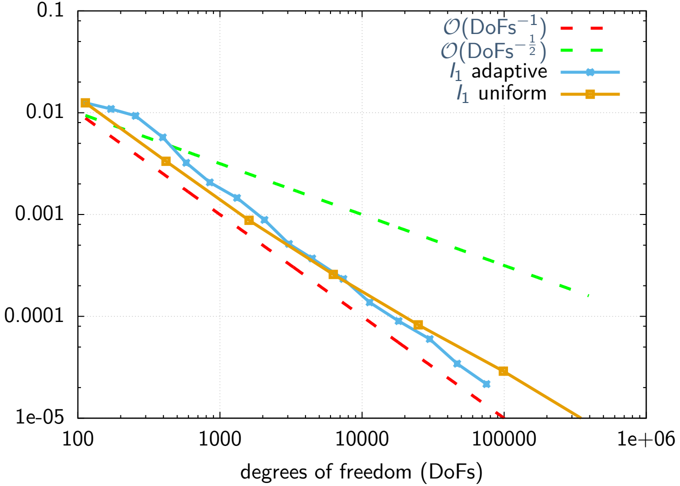


Solution: Control





1
0.1
0.01
0.001
0.0001
1e-05
100
1000
10000
100000
relative Errors
degrees of freedom (DoFs)
$\mathcal{O}(\text{DoFs}^{-1})$
I_1
I_2
I_3
I_4
I_5
$I_{\mathfrak{E}}$


0.1
0.01
0.001
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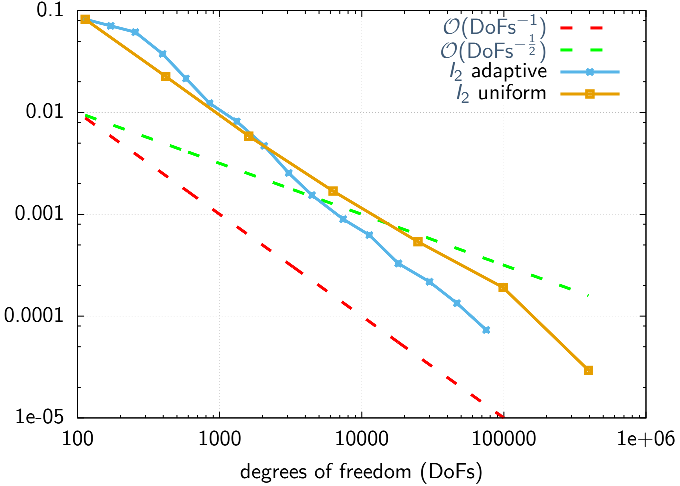
1e-05
100
1000
10000
100000
1e+06
degrees of freedom (DoFs)
$\mathcal{O}(\mathrm{DoFs}^{-1})$
$\mathcal{O}(\mathrm{DoFs}^{-\frac{1}{2}})$
I_1 adaptive
I_1 uniform


0.1
0.01
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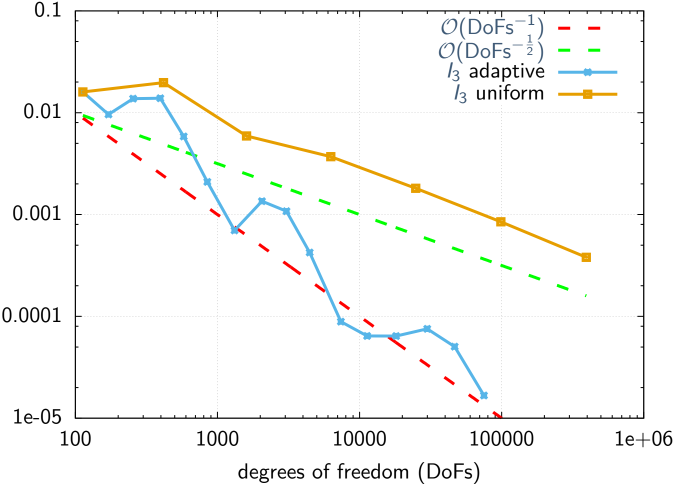
1e+06
degrees of freedom (DoFs)
$\mathcal{O}(\text{DoFs}^{-1})$
$\mathcal{O}(\text{DoFs}^{-\frac{1}{2}})$
l_2 adaptive
l_2 uniform


0.1
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100
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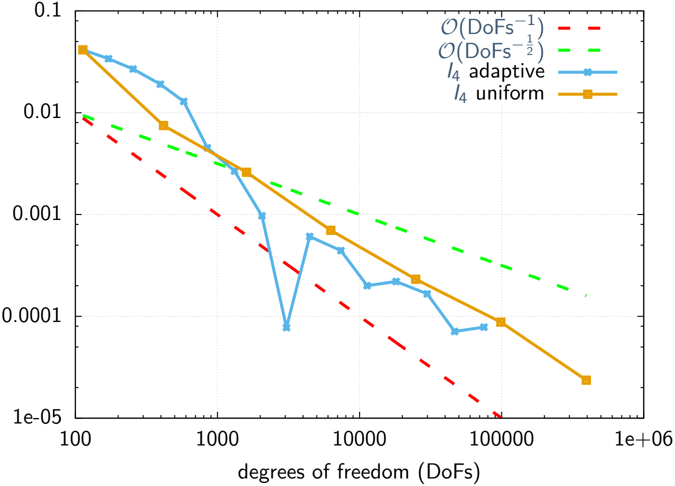
10000
100000
1e+06
degrees of freedom (DoFs)
$\mathcal{O}(\mathrm{DoFs}^{-1})$
$\mathcal{O}(\mathrm{DoFs}^{-\frac{1}{2}})$
I_3 adaptive
I_3 uniform


0.1
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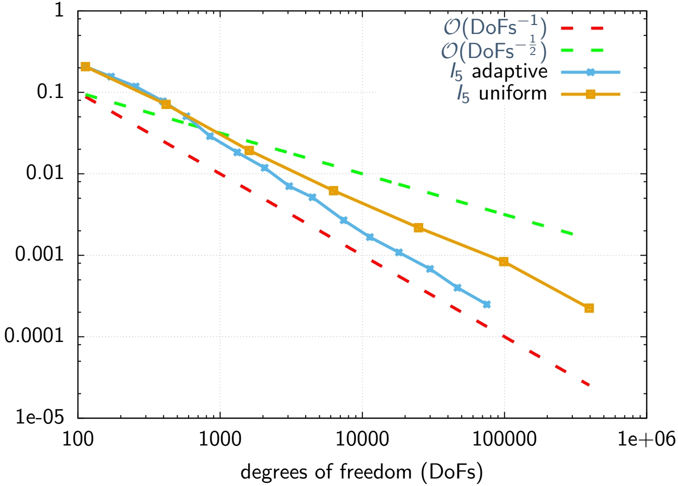
1000
10000
100000
1e+06
degrees of freedom (DoFs)
$\mathcal{O}(\mathrm{DoFs}^{-1})$
$\mathcal{O}(\mathrm{DoFs}^{-\frac{1}{2}})$
I_4 adaptive
I_4 uniform


1
0.1
0.01
0.001
0.0001
1e-05
100
1000
10000
100000
1e+06
degrees of freedom (DoFs)
$\mathcal{O}(\text{DoFs}^{-1})$
$\mathcal{O}(\text{DoFs}^{-\frac{1}{2}})$
I_5 adaptive
I_5 uniform

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RICAM-Report No. 2019-07.

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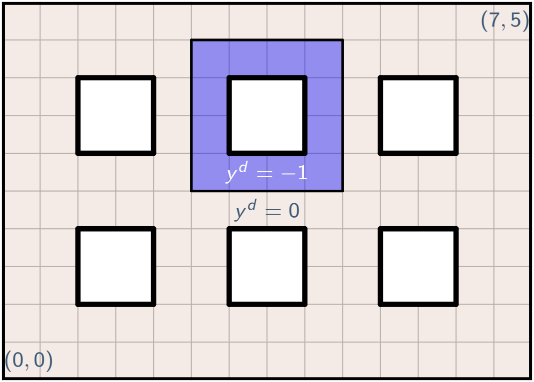
and
DFG SPP 1962
'P17: Optimizing Fracture Propagation Using a Phase-Field Approach'.

DFG Deutsche Forschungsgemeinschaft
SPP 1962
FWF

The Domain and y^d



Functionals of interest

- $I_1(y,u) = \frac{1}{2}\int_\Omega (y-y^d)^2 \,\mathrm{d}x \approx 1.15760$
- $I_2(y,u) = \frac{1}{2}\int_\Omega (u-u^d)^2 \,\mathrm{d}x \approx 21.3305$
- $I_3(y,u) = \int_{([4,5]\times\mathbb{R})\cap\Omega} y \,\mathrm{d}x \approx -0.236288$
- $I_4(y,u) = \int_{[1,\frac{25}{4}]\times[2,\frac{5}{2}]} u \,\mathrm{d}x \approx 0.328042$
- $I_5(y,u) = \frac{1}{2}\int_\Omega y^2u^2 \,\mathrm{d}x \approx 0.231615$

We used

$$I_{\mathfrak{E}}(\tilde{y}_h,\tilde{u}_h) := \sum_{\ell=1}^{5} \frac{|I_\ell(y_h^{(2)},u_h^{(2)}) - I_\ell(\tilde{y}_h,\tilde{u}_h)|}{|I_\ell(\tilde{y}_h,\tilde{u}_h)|},$$

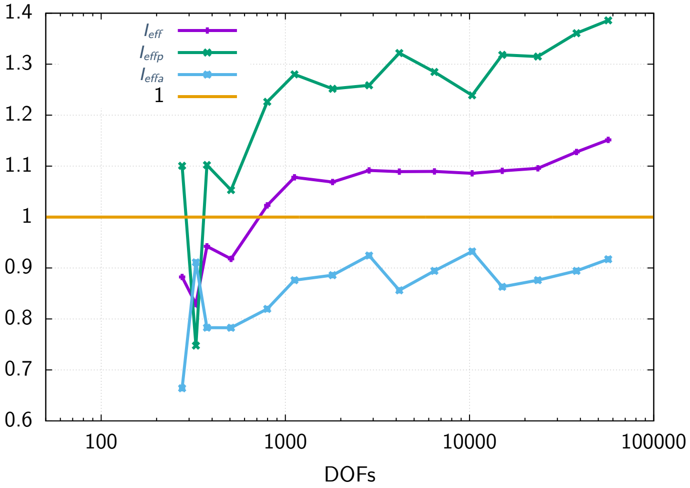
following the idea in [Hartmann & Housten (2003)].

$$I_{eff} := \frac{\eta_h^{(2)}}{I(u,y) - I(\tilde{u}_h, \tilde{y}_h)}$$

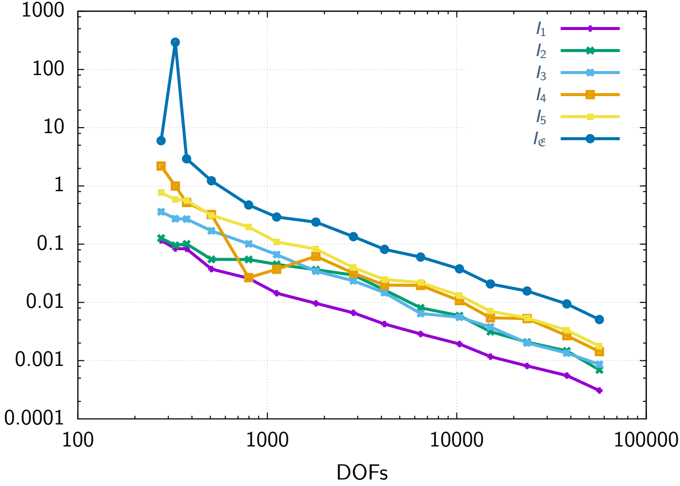
$$I_{effp} := \frac{\eta_{h,p}^{(2)}}{I(u,y) - I(\tilde{u}_h, \tilde{y}_h)}$$

$$I_{effa} := \frac{\eta_{h,a}^{(2)}}{I(u,y) - I(\tilde{u}_h, \tilde{y}_h)}$$

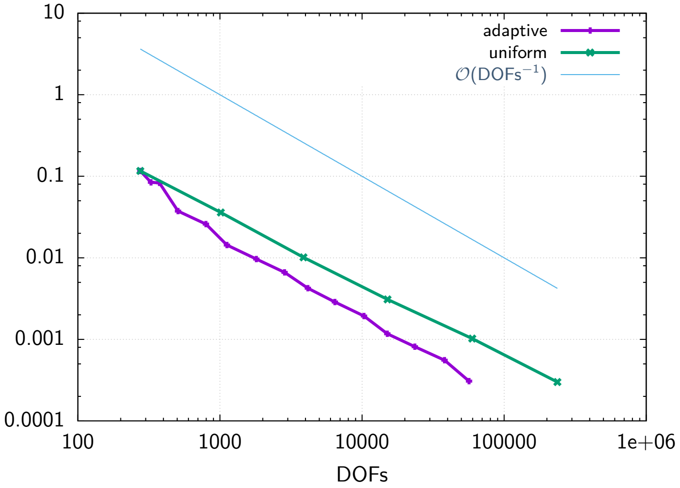
I_{eff} for $I_{\mathfrak{E}}$



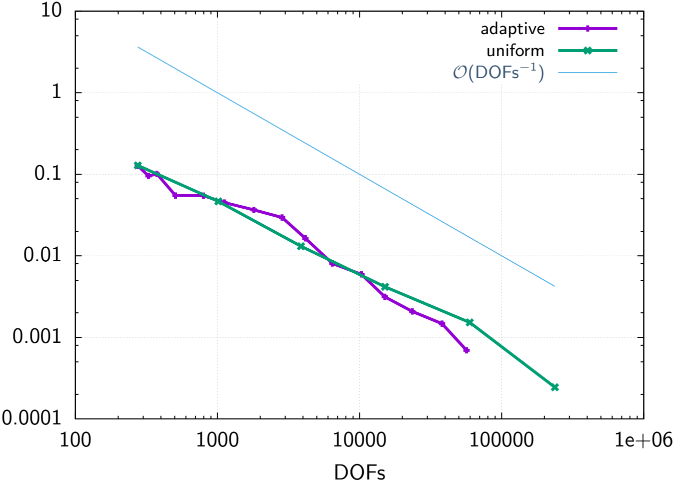
Errors in the Different Goal Functionals



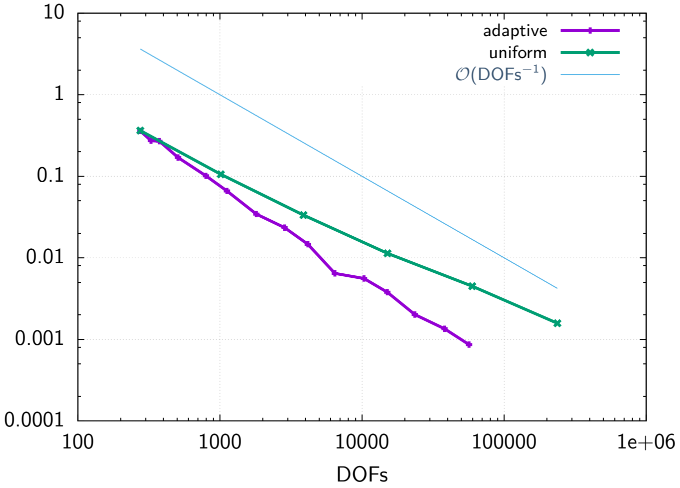
Errors in $I_1(y,u) = \frac{1}{2}\int_\Omega (y - y^d)^2 \, dx$



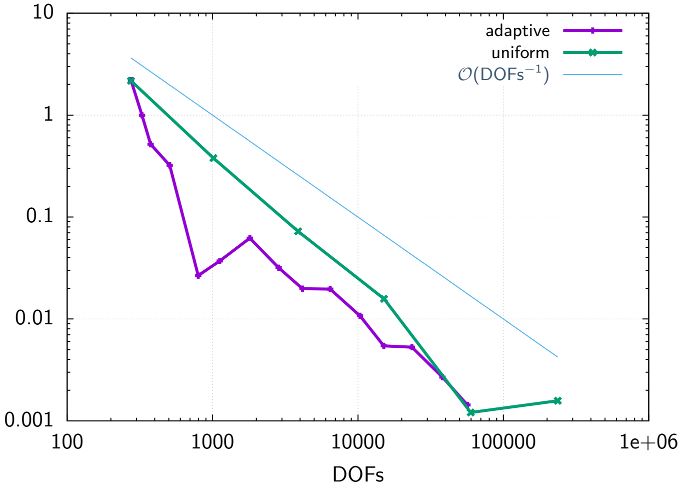
Errors in $I_2(y,u) = \frac{1}{2}\int_\Omega (u - u^d)^2 \, dx$



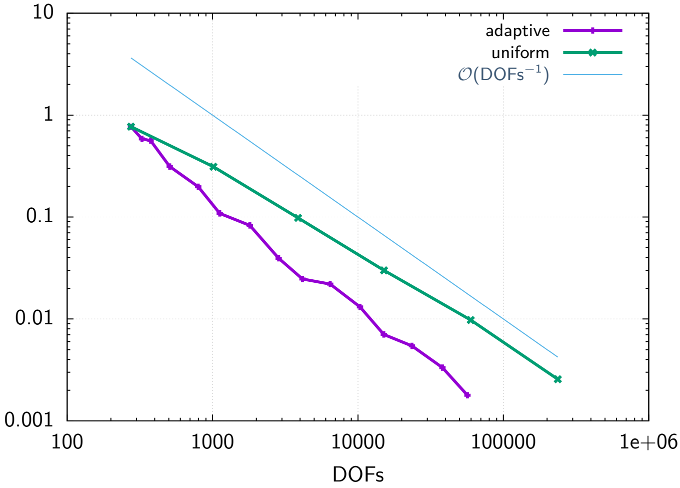
Errors in $I_3(y,u) = \int_{((4,5)\times\mathbb{R})\cap\Omega} y \, dx$



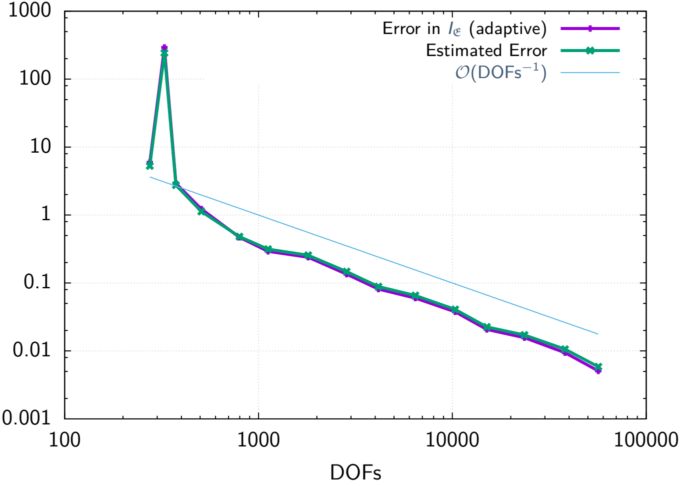
Errors in $I_4(y,u) = \int_{(1,6.25)\times(2,2.5)} u \, dx$



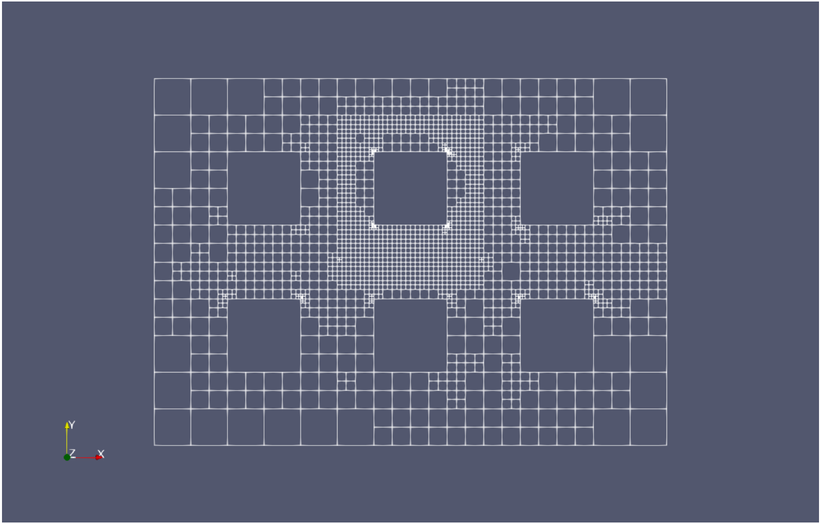
Errors in $I_5(y,u) = \frac{1}{2}\int_\Omega y^2 u^2 \,\mathrm{d}x$

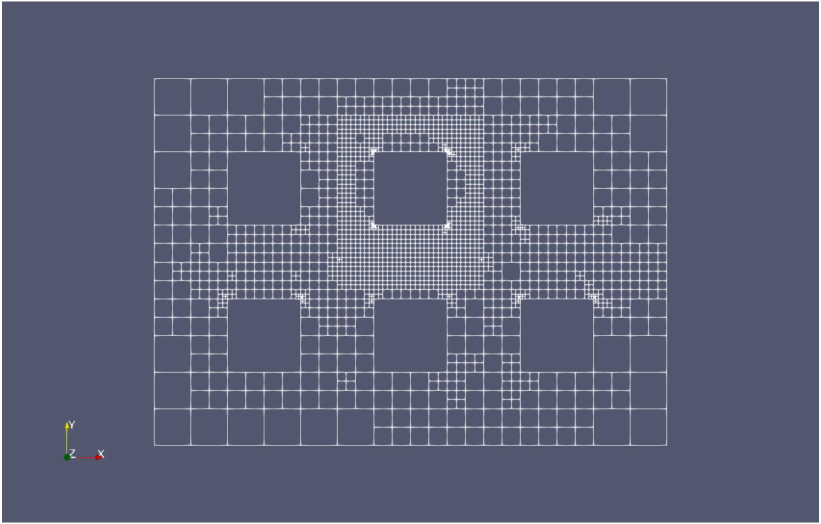


Real and Estimated Error



l	NIF	NIA	$\lvert\mathcal{T}_h\rvert_F$	$\lvert\mathcal{T}_h\rvert_A$	$I_{eff,FN}$	$I_{eff,AN}$	$I_{eff+it,FN}$	$I_{eff+it,AN}$	$J_{\mathfrak{E},FN}$	$J_{\mathfrak{E},AN}$
0	4	3	116	116	0.882	0.882	0.882	0.882	5.98E+00	5.98E+00
1	5	3	137	137	0.829	0.830	0.829	0.830	2.93E+02	2.75E+02
2	5	1	158	158	0.942	0.933	0.942	0.941	2.92E+00	2.98E+00
3	6	2	215	215	0.918	0.912	0.918	0.918	1.23E+00	1.24E+00
4	6	2	347	347	1.023	1.019	1.023	1.022	4.70E-01	4.73E-01
5	6	2	494	494	1.078	1.068	1.078	1.076	2.93E-01	2.96E-01
6	7	2	800	800	1.069	1.067	1.069	1.069	2.40E-01	2.40E-01
7	12	2	1283	1283	1.091	1.087	1.091	1.090	1.35E-01	1.36E-01
8	17	2	1898	1895	1.089	1.093	1.089	1.090	8.13E-02	8.09E-02
9	6	2	2966	2957	1.089	1.090	1.090	1.090	6.00E-02	6.00E-02
10	9	2	4802	4790	1.085	1.088	1.085	1.083	3.77E-02	3.77E-02
11	5	2	7097	7091	1.089	1.077	1.089	1.093	2.07E-02	2.11E-02
12	4	2	11153	11099	1.096	1.088	1.097	1.095	1.57E-02	1.58E-02
13	2	2	18206	18140	1.128	1.129	1.122	1.123	9.41E-03	9.45E-03
14	2	2	27341	27269	1.151	1.152	1.136	1.137	5.09E-03	5.11E-03


Y
Z
X


Y
Z
X