T-coercivity for solving variational formulations

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Abstract setting

- Let
  - V, W be two Hilbert spaces ;
  - $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times W$;
  - f be an element of W', the dual space of W.

Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = \langle f, w\rangle.$$

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- [Ladyzhenskaya-Babuska-Brezzi] Recall the *inf-sup condition*

$$(isc) \quad \exists \alpha > 0,\ \forall v \in V,\ \sup_{w \in W \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha \|v\|_V.$$

- The form $a(\cdot,\cdot)$ is $\mathtt{T}$-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V,W) \text{ bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha} \|v\|_V^2.$$

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$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = \langle f, w \rangle.$$

- `Theorem (Well-posedness)`

  The three assertions below are equivalent:

  (i) the Problem (VF) is well-posed ;

  (ii) the form $a(\cdot,\cdot)$ satisfies (isc) and $\{w \in W \,|\, \forall v \in V,\ a(v,w) = 0\} = \{0\}$;

  (iii) the form $a(\cdot,\cdot)$ is `T`-coercive.

The operator `T` realizes the inf-sup condition (isc) explicitly.

Abstract setting – hermitian case

- Let
  - V be a Hilbert space ;
  - $a(\cdot,\cdot)$ be a continuous sesquilinear, *hermitian* form on $V \times V$;
  - f be an element of V', the dual space of V.

  Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w \rangle.$$

- [Ladyzhenskaya-Babuska-Brezzi] The *inf-sup condition* writes

$$(isc) \quad \exists \alpha > 0,\ \forall v \in V,\ \sup_{w \in V \setminus \{0\}} \frac{|a(v,w)|}{\|w\|_V} \geq \alpha \, \|v\|_V.$$

- The form $a(\cdot,\cdot)$ is $\mathtt{T}$-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V),\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha} \, \|v\|_V^2.$$

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$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w \rangle.$$

- `Theorem (Well-posedness, hermitian case)`

  The three assertions below are equivalent:

  (i) the Problem (VF) is well-posed ;

  (ii) the form $a(\cdot,\cdot)$ satisfies (isc) ;

  (iii) the form $a(\cdot,\cdot)$ is T-coercive.

The operator T realizes the inf-sup condition (isc) explicitly.

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Abstract setting – hermitian case

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  - V be a Hilbert space ;
  - $a(\cdot,\cdot)$ be a continuous sesquilinear, *hermitian* form on $V \times V$;
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  Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w\rangle.$$

- Introduce the *weak inf-sup condition*

$$(wisc) \quad \exists \mathtt{C} \in \mathcal{K}(V),\ \alpha, \beta > 0,\ \forall v \in V,\ \sup_{w \in V\setminus\{0\}} \frac{|a(v,w)|}{\|w\|_V} \geq \alpha \|v\|_V - \beta \|\mathtt{C}v\|_V .$$

- The form $a(\cdot,\cdot)$ is *weakly* $\mathtt{T}$*-coercive* if

$$\exists \mathtt{C} \in \mathcal{K}(V),\ \mathtt{T} \in \mathcal{L}(V) \text{ bijective},\ \exists \underline{\alpha}, \underline{\beta} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha} \|v\|_V^2 - \underline{\beta} \|\mathtt{C}v\|_V^2 .$$

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Abstract setting – hermitian case

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  - V be a Hilbert space ;
  - $a(\cdot,\cdot)$ be a continuous sesquilinear, *hermitian* form on $V \times V$;
  - f be an element of V', the dual space of V.

  Aim: solve the Variational Formulation

$$(VF) \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = \langle f, w\rangle.$$

- `Theorem (Well-posedness in Fredholm sense, hermitian case)`
  The three assertions below are equivalent:
  (i) the Problem (VF) is well-posed in the Fredholm sense ;
  (ii) the form $a(\cdot,\cdot)$ satisfies $(wisc)$;
  (iii) the form $a(\cdot,\cdot)$ is weakly $\mathtt{T}$-coercive.

Application: neutron diffusion problem

- Let Ω be a domain of $\mathbb{R}^d$, $d = 1, 2, 3$.
- Goal: given $f \in L^2(\Omega)$, solve the diffusion problem
  Find $u \in H_0^1(\Omega)$ such that:

$$-\mathrm{div}\ (A\,\nabla u) + b\,u = f \text{ in } \Omega.$$

- "Physical" assumptions on the set of parameters:

(Hyp) $$\begin{cases} A \in \mathbb{L}^\infty_{sym}(\Omega), \exists A_{min} > 0,\ \forall \mathbf{z} \in \mathbb{R}^d,\ A\mathbf{z}\cdot\mathbf{z} \geq A_{min}|\mathbf{z}|^2 \text{ a.e. in } \Omega\,; \\ b \in L^\infty(\Omega), \exists b_{min} > 0,\ b \geq b_{min} \text{ a.e. in } \Omega. \end{cases}$$

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- Primal form:

  Find $u \in H_0^1(\Omega)$ such that:

$$\forall v \in H_0^1(\Omega), \quad (A\,\nabla u, \nabla v)_0 + (bu, v)_0 = (f, v)_0.$$

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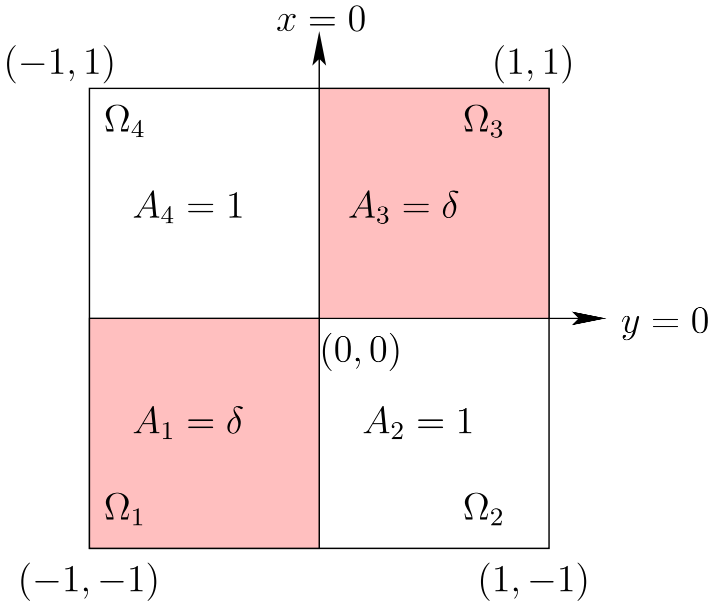
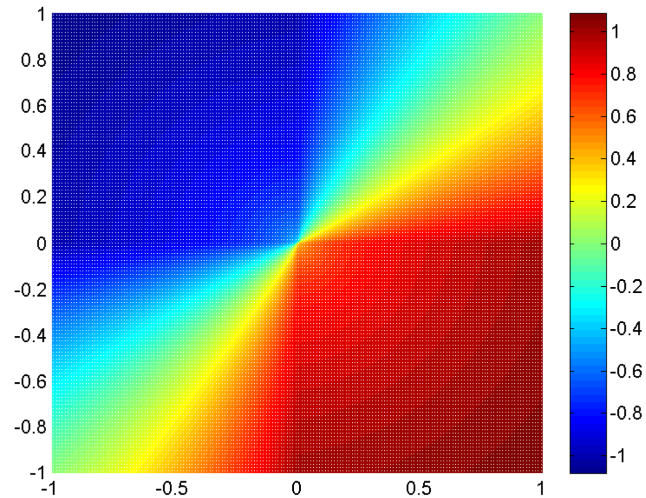
- Mixed form: introduce the auxiliary unknown $\mathbf{p} = -A\,\nabla u$.
  Find $(\mathbf{p}, u) \in \boldsymbol{H}(\mathrm{div}\,, \Omega) \times L^2(\Omega)$ such that:

$$\forall(\mathbf{q}, v) \in \boldsymbol{H}(\mathrm{div}\,, \Omega) \times L^2(\Omega),\ -(A^{-1}\mathbf{p}, \mathbf{q})_0 + (\mathrm{div}\,\mathbf{q}, u)_0 + (\mathrm{div}\,\mathbf{p}, v)_0 + (bu, v)_0 = (f, v)_0.$$

Application: neutron diffusion problem





RT_0 FE ; DDM+L^2-jumps on *non-nested* grids ($h_1 = h_3 = 2/3\,h$, $h_2 = h_4 = h$):

	$r_{max} = 0.45$ ($\delta \approx 7.35$)		$r_{max} = 0.20$ ($\delta \approx 39.9$)	
$1/h$	$\|\|u - u_h\|\|_0$	$\|\|\mathbf{p} - \mathbf{p}_h\|\|_0$	$\|\|u - u_h\|\|_0$	$\|\|\mathbf{p} - \mathbf{p}_h\|\|_0$
48	$5.11\,e^{-3}$	$9.63\,e^{-2}$	$3.04\,e^{-2}$	$4.39\,e^{-1}$
96	$2.71\,e^{-3}$	$7.02\,e^{-2}$	$2.34\,e^{-2}$	$3.80\,e^{-1}$
192	$1.44\,e^{-3}$	$5.12\,e^{-2}$	$1.80\,e^{-2}$	$3.29\,e^{-1}$
rate	$h^{0.92}$	$h^{0.46}$	$h^{0.38}$	$h^{0.20}$

Application: Helmholtz problem

- Let Ω be a domain of $\mathbb{R}^d$, $d = 1, 2, 3$.
- Goal: given $f \in L^2(\Omega)$, solve the Helmholtz problem
  Find $u \in H_0^1(\Omega)$ such that:

$$-\mathrm{div}\ (A\,\nabla u) + b\,u = f \text{ in } \Omega.$$

- Assumptions on the set of parameters:

$$\text{(Hyp)} \qquad \begin{cases} A \in \mathbb{L}^\infty_{sym}(\Omega), \exists A_{min} > 0,\ \forall \mathbf{z} \in \mathbb{R}^d,\ A\mathbf{z}\cdot\mathbf{z} \geq A_{min}|\mathbf{z}|^2 \text{ a.e. in } \Omega\,; \\ b < 0 \text{ s.t. } |b| \text{ is not an eigenvalue of corresponding eigenproblem.} \end{cases}$$

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- Primal form:

  Find $u \in H_0^1(\Omega)$ such that:

$$\forall v \in H_0^1(\Omega), \quad (A\,\nabla u, \nabla v)_0 - |b|\,(u, v)_0 = (f, v)_0.$$

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Application: Helmholtz problem

- Let Ω be a domain of $\mathbb{R}^3$, and $\omega > 0$.
- Goal: given $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, solve the Helmholtz Maxwell problem

  Find $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that:

$$\forall \boldsymbol{F} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega), \quad (\mu^{-1}\ \mathbf{curl}\, \boldsymbol{E}, \mathbf{curl}\, \boldsymbol{F})_0 - \omega^2(\varepsilon \boldsymbol{E}, \boldsymbol{F})_0 = \imath\omega\, (\boldsymbol{J}, \boldsymbol{F})_0.$$

- Assumptions on the set of parameters:

$$\text{(Hyp)} \qquad \begin{cases} \varepsilon > 0\,; \\ \mu > 0. \end{cases}$$

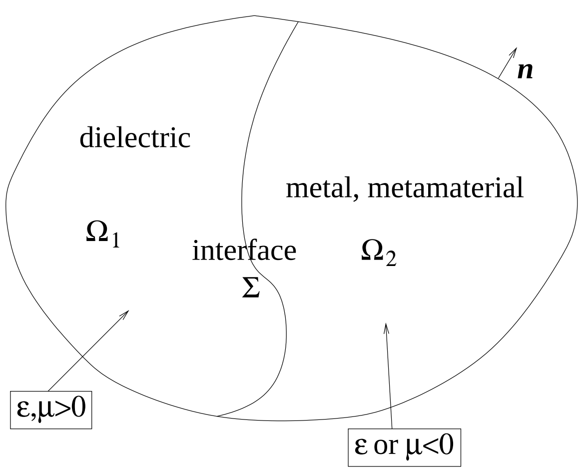
Application: sign-changing interface pb

- Let Ω be a domain of $\mathbb{R}^d$, $d = 1, 2, 3$.
- <u>Goal</u>: given $f \in L^2(\Omega)$, solve the transmission problem
  Find $u \in H^1_0(\Omega)$ such that:

$$-\mathrm{div}\ (A\,\nabla u) + b\,u = f \text{ in } \Omega.$$

- “Physical” assumption on the parameters:

$$\text{(Hyp)} \qquad \begin{cases} A \text{ piecewise constant}: \ A_1 := A_{|\Omega_1} > 0,\ A_2 := A_{|\Omega_2} < 0\,; \\ b \text{ piecewise constant.} \end{cases}$$



NB. The case of an *inclusion* with $\Sigma \cap \partial\Omega = \emptyset$ is also possible.

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- Primal form:

  Find $u \in H_0^1(\Omega)$ such that:

$$\forall v \in H_0^1(\Omega), \quad A_1\,(\nabla u_1, \nabla v_1)_{0,\Omega_1} - |A_2|\,(\nabla u_2, \nabla v_2)_{0,\Omega_2} + (bu, v)_0 = (f, v)_0.$$

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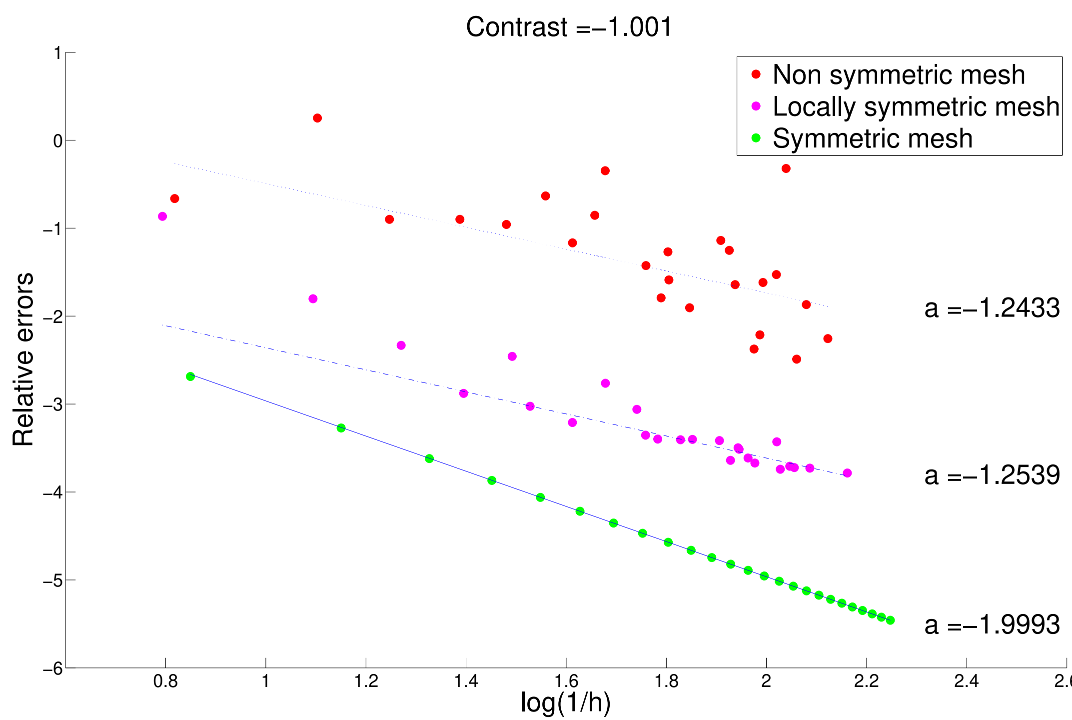
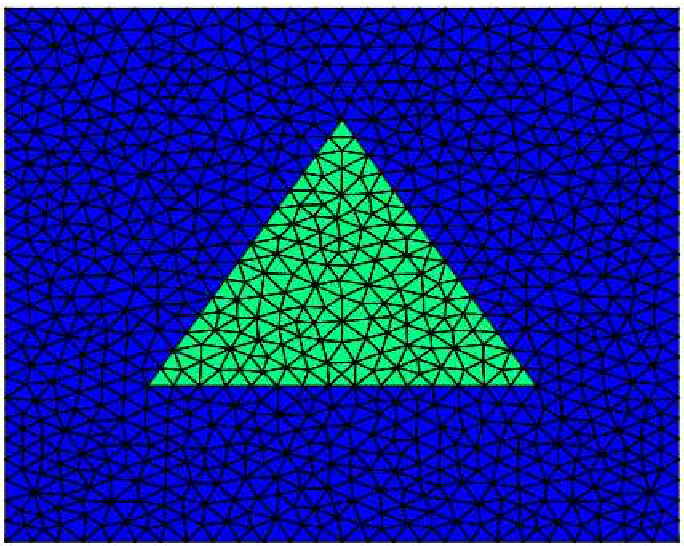
Application: sign-changing interface pb

- Case of a *symmetric domain*: $\Omega_1 =]-1,0[\times]0,1[$, $\Omega_2 =]0,1[\times]0,1[$.
- Contrast $A_2/A_1 = -1.001$, and $b = 0$.
- Discretization using P_1 Lagrange finite elements.
- Influence of the meshes? (relative errors in L^2-norm ; $O(h^2)$ is expected).

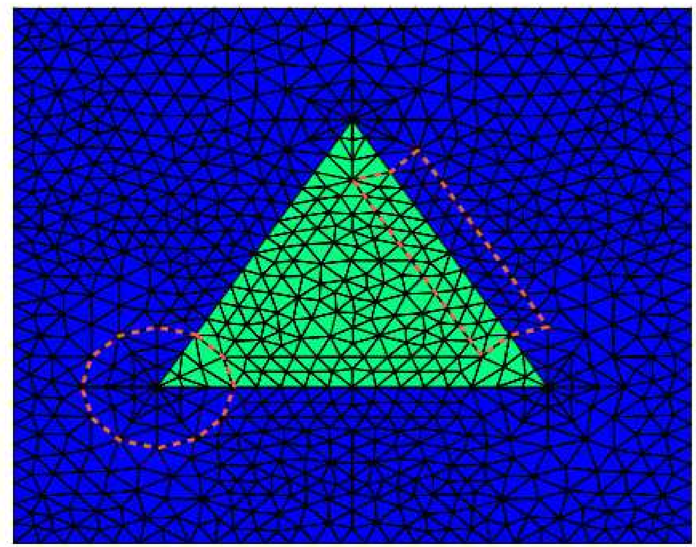


Application: sign-changing interface pb

- Contrast $A_2/A_1 = -5.2$ (critical interval $I_\Sigma = [-5, -1/5]$), and $b = 0$.
- Discretization using P_3 Lagrange finite elements.



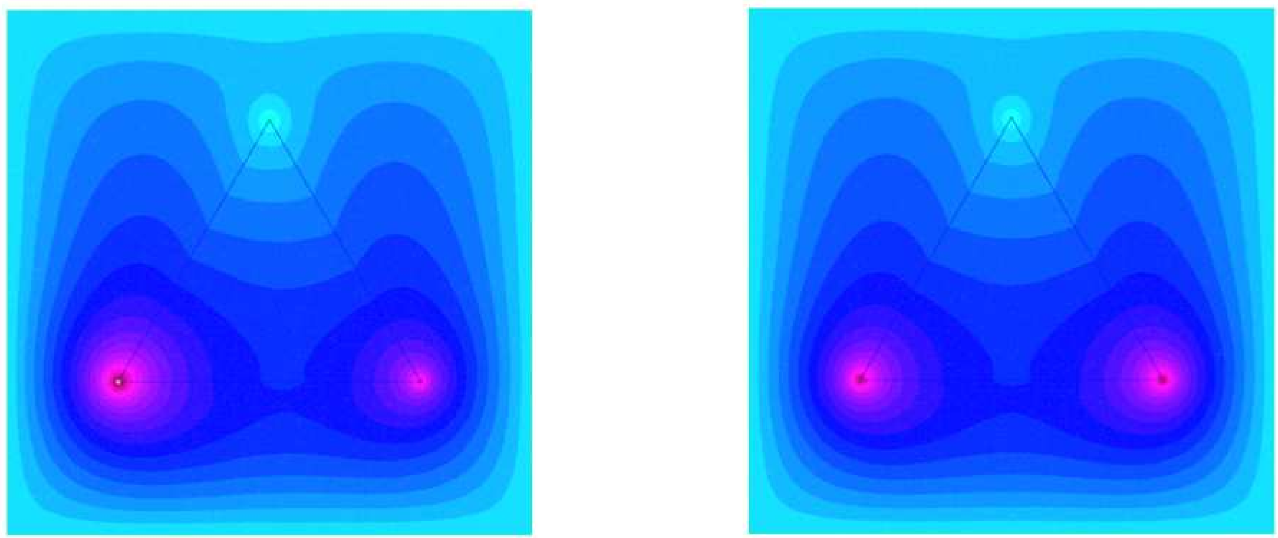
standard mesh



T-conform mesh

Application: sign-changing interface pb

- Contrast $A_2/A_1 = -5.2$ (critical interval $I_\Sigma = [-5, -1/5]$), and $b = 0$.
- Discretization using P_3 Lagrange finite elements.
- Comparison of the computed solutions ($\approx 10^5$ dof).



standard mesh　　　　T-conform mesh

Meshes must/should be carefully designed: T-conform meshes

To go further...

- Neutron diffusion problems:
  - Jamelot-PC *Journal of Computational Physics*, **241** (2013) ;
  - PC-Jamelot-Kpadonou *Computers and Mathematics with Applications*, **74** (2017) ;
  - PC-Giret-Jamelot-Kpadonou *ESAIM: M2AN*, **52** (2018) ;
  - PhD of Léandre Giret, Paris-Saclay University (2018).
- Helmholtz-type problems:
  - Hiptmair *Acta Numerica* (2002) ;
  - PC *Computers and Mathematics with Applications*, **64** (2012).
- Sign-changing interface problems:
  - BonnetBenDhia-Chesnel-PC *ESAIM: M2AN*, **46** (2012) ;
  - Chesnel-PC *Numerische Mathematik*, **124** (2013) ;
  - Carvalho-Chesnel-PC *C. R. Acad. Sci. Paris,Ser. I*, **355** (2017) ;
  - BonnetBenDhia-Carvalho-PC *Numerische Mathematik*, **138** (2018) ;
  - PhD of Lucas Chesnel, Ecole Polytechnique (2012).
  - PhD of Camille Carvalho, Ecole Polytechnique (2015).

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