An analysis of the Grünwald–Letnikov scheme for initial-value problems with weakly singular solutions

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The Grünwald-Letnikov scheme

A fractional initial-value problem

Error analysis of the GL approximation away from $t = 0$

Numerical results

Outline

Grünwald-Letnikov (GL) fractional derivative

The Grünwald-Letnikov (GL) fractional derivative of order α of v at each point $t > 0$ is defined by

$$D_{GL}^{\alpha} v(t) = \lim_{M \to \infty} \frac{1}{\tau_M^{\alpha}} \sum_{k=0}^{M} (-1)^k \binom{\alpha}{k} v(t - k\tau_M), \qquad (1)$$

where $\tau_M = t/M$.
The Riemann-Liouville integral operator I^{β} is defined for each $\beta > 0$ by

$$I^{\beta} w(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s)\, \mathrm{d}s.$$

If $v \in C^{\lceil \alpha \rceil}[0, T]$, then one has $D_{GL}^{\alpha} v(t) = D_{RL}^{\alpha} v(t)$, the Riemann-Liouville derivative of v, which is defined by $D_{RL}^{\alpha} w(t) := \frac{d}{dt} \left(I^{1-\alpha} w \right)(t)$.

GL finite difference operator L_t^α

Let M be a positive integer. Set $\tau = T/M$ and $t_m = m\tau$ for $m = 0, 1, \ldots, M$, so the mesh $\{t_m : m = 0, 1, \ldots, M\}$ is uniform. Then for any mesh function $\{V_j\}_{j=0}^M$, set

$$L_t^\alpha V_m := \frac{1}{\tau^\alpha} \sum_{k=0}^{m} \omega_k^{(\alpha)} V_{m-k} \quad \text{for } m = 1, \ldots M, \qquad (2)$$

where $\omega_k^{(\alpha)} := (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$.

Set

$$d_k^{(\alpha)} = \frac{\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k)} \quad \text{for } k = 1, 2, \ldots \qquad (3)$$

Gautschi's inequality [J. Math. Phys. 1960] applied to (3) yields

$$\frac{k^{-\alpha}}{\Gamma(1-\alpha)} < d_k^{(\alpha)} < \frac{(k-1)^{-\alpha}}{\Gamma(1-\alpha)} \quad \text{for } k = 1, 2, \ldots \qquad (4)$$

$d_k^{(\alpha)} > d_{k+1}^{(\alpha)}$ for $k = 1, 2, \ldots$. $0 < \alpha < 1$

Lemma

One has $\omega_k^{(\alpha)} = d_{k+1}^{(\alpha)} - d_k^{(\alpha)}$ *for* $k = 1, 2, \ldots$

Proof.

$$\begin{aligned} d_{k+1}^{(\alpha)} - d_k^{(\alpha)} &= \frac{\Gamma(k+1-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)} - \frac{\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k)} \\ &= \frac{(k-\alpha)\Gamma(k-\alpha)}{-\alpha\Gamma(-\alpha)\Gamma(k+1)} - \frac{k\Gamma(k-\alpha)}{-\alpha\Gamma(-\alpha)\Gamma(k+1)} \\ &= \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} = \omega_k^{(\alpha)} \end{aligned}$$

Another form of GL operator L_t^α

$$L_t^\alpha V_m := \frac{1}{\tau^\alpha} \sum_{k=0}^{m} \omega_k^{(\alpha)} V_{m-k} \quad \text{for } m = 1, \dots M, \tag{5}$$

$\omega_k^{(\alpha)} = d_{k+1}^{(\alpha)} - d_k^{(\alpha)}$ and $\omega_0^{(\alpha)} = 1$ enable us to rewrite the definition (2) of L_t^α as

$$L_t^\alpha V_m = \frac{1}{\tau^\alpha} \left[V_m - \sum_{k=1}^{m} (d_k^{(\alpha)} - d_{k+1}^{(\alpha)}) V_{m-k} \right] \quad \text{for } m = 1, \dots M, \tag{6}$$

where we remind the reader that $d_k^{(\alpha)} - d_{k+1}^{(\alpha)} > 0$ for each k.

A stability result for the operator L_t^α

Lemma

For any mesh function $\{V_j\}_{j=0}^M$ with $V_0 = 0$, one has

$$|V_k| \le \Gamma(1-\alpha) \max_{j=1,\ldots,k} \{t_j^\alpha L_t^\alpha |V_j|\} \quad \textit{for} \quad k = 1, \ldots, M.$$

Proof.

Fix $k \in \{1, 2, \ldots, M\}$. Suppose $\max_{j=1,\ldots,k} |V_j| = |V_n|$ for some $n \in \{1, \ldots, k\}$. Since $V_0 = 0$, the formula (6) becomes

$$\begin{aligned} L_t^\alpha |V_n| &= \frac{1}{\tau^\alpha}\left[|V_n| - \sum_{k=1}^{n-1}(d_k^{(\alpha)} - d_{k+1}^{(\alpha)})|V_{n-k}|\right] \\ &\ge \frac{1}{\tau^\alpha}\left[|V_n| - \sum_{k=1}^{n-1}(d_k^{(\alpha)} - d_{k+1}^{(\alpha)})|V_n|\right] \\ &= \frac{1}{\tau^\alpha} d_n^{(\alpha)} |V_n| > \frac{1}{\tau^\alpha} \frac{n^{-\alpha}}{\Gamma(1-\alpha)} |V_n|. \end{aligned}$$

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A fractional initial-value problem

Consider the fractional initial-value problem

$$D_{RL}^{\alpha} u(t) + c(t)u(t) = f(t) \quad \text{for } 0 < t \leq T, \tag{7a}$$

$$u(0) = 0, \tag{7b}$$

If $c \in C^2[0, T]$ and $f \in C^2[0, T]$, then its solution

$$u(t) = \sum_{(j,k)_\alpha} \gamma_{j,k} t^{j+k\alpha} + Y_2(t;\alpha) \quad \text{for } 0 \leq t \leq T, \tag{8}$$

where $(j,k)_\alpha := \{(j,k) : j,k \in \mathbb{N}_0, j + k\alpha < 2\}$, the coefficients $\gamma_{j,k}$ are some constants, and the function $Y_2(\cdot\,;\alpha) \in C^2[0, T]$; furthermore, one has

$$0 = Y_2(0;\alpha) = \left.\frac{dY_2(t;\alpha)}{dt}\right|_{t=0}.$$

Discretization of the IVP

$$L_t^\alpha U_m + c_m U_m = f_m \quad \text{for } m = 1, \dots M, \tag{9a}$$

$$U_0 = 0, \tag{9b}$$

Lemma (ZZK17,Lemma 2.1)

Let $v(t) = t^\sigma$ where $\sigma \geq 0$ is a constant. Then

$$D_{RL}^\alpha v(t_m) = L_t^\alpha v(t_m) + \tau \frac{\alpha \Gamma(\sigma+1)}{2\Gamma(\sigma-\alpha)} t_m^{\sigma-1-\alpha} + \tau^2 R^{m,\alpha,\sigma}, \tag{10}$$

where $|R^{m,\alpha,\sigma}| \leq C t_m^{\sigma-2-\alpha}$ for some constant C that is independent of m and τ.

Truncation error of the GL approximation L_t^α

This result enables us to give a truncation error bound for the non-smooth terms $\sum_{j,k} \gamma_{j,k} t^{j+k\alpha}$ in (8).

Lemma

Set $z(t) = \sum_{(j,k)_\alpha} \gamma_{j,k} t^{j+k\alpha}$ *for* $t \in [0, T]$. *Set* $\gamma = \min\{1, 2\alpha\}$. *Then*

$$|D_{RL}^\alpha z(t_m) - L_t^\alpha z(t_m)| \lesssim m^{-2} + \tau^{\gamma-\alpha} m^{-(1+\alpha-\gamma)} \quad \textit{for } m = 1, 2, \ldots, M.$$

Lemma

One has

$$|D_{RL}^\alpha Y_2(t_m) - L_t^\alpha Y_2(t_m)| \lesssim \tau \quad \textit{for } m = 1, 2, \ldots, M.$$

Global convergence result for the GL scheme

Theorem

Let u and $\{U_m\}_{m=0}^{M}$ be the solutions of (7) and (9), respectively. Then

$$|u(t_m) - U_m| \lesssim \tau^{\alpha} \quad \text{for } m = 1, \ldots, M.$$

Proof.

Set $e_m = u(t_m) - U_m$ for $m = 0, \ldots, M$. Subtraction of (9a) from (7a) gives

$$L_t^{\alpha} e_m + c(t_m) e_m = L_t^{\alpha} u(t_m) - D_{RL}^{\alpha} u(t_m) =: r_m. \qquad (11)$$

we get $L_t^{\alpha}|e_m| \le |r_m|$. But Lemmas 4 and 5 show that $|r_m| \lesssim m^{-2} + \tau^{\gamma-\alpha} m^{\gamma-\alpha-1} + \tau$, where $\gamma = \min\{1, 2\alpha\}$. From Lemma 2 it then follows that $|e_m| \lesssim \max_{j=1,\ldots,m}\{t_j^{\alpha}|r_j|\} \lesssim \tau^{\alpha}$. □

Outline

A sequence of stability multipliers

Define a sequence of stability multipliers $\{\sigma_n\}$ associated with the GL scheme by the recurrence relation

$$\sigma_0 := 1, \quad \sigma_n := \sum_{k=1}^{n} (d_k^{(\alpha)} - d_{k+1}^{(\alpha)})\sigma_{n-k} \;\; \text{for } n = 1, 2, \ldots \tag{12}$$

Lemma

The stability multipliers σ_n defined by (12) are given explicitly by

$$\sigma_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} \quad \textit{for } n = 0, 1, 2, \ldots \tag{13}$$

Corollary

One has

$$\frac{(n+1)^{\alpha-1}}{\Gamma(\alpha)} < \sigma_n < \frac{n^{\alpha-1}}{\Gamma(\alpha)} \quad \textit{for } n = 1, 2, \ldots \tag{14}$$

Stability of the scheme

Lemma

Let $\{U_m\}_{m=0}^{M}$ *be the solution of* (9). *Then*

$$|U_m| \le \tau^{\alpha} \sum_{j=1}^{m} \sigma_{m-j} |f_j| \quad \textit{for} \quad m = 1, \dots M. \tag{15}$$

Lemma

Let $m \in \{1, 2, \dots, M\}$. *Then*

$$\sum_{j=1}^{m} j^{-\beta} \sigma_{m-j} \lesssim \begin{cases} m^{\alpha-1} & \textit{if } \beta > 1, \\ m^{\alpha-1}(\ln m + 1) & \textit{if } \beta = 1, \\ m^{\alpha-\beta} & \textit{if } 0 \le \beta < 1. \end{cases}$$

Convergence of the scheme

Theorem

Let u and $\{U_m\}_{m=0}^M$ be the solutions of (7) and (9), respectively. Then

$$|u(t_m) - U_m| \lesssim \tau t_m^{\alpha-1} \quad \text{for } m = 1, \dots, M.$$

Proof.

Set $e_m = u(t_m) - U_m$ for $m = 0, \dots, M$. Subtracting (9a) from (7a), we get

$$L_t^\alpha e_m + c(t_m) e_m = L_t^\alpha u(t_m) - D_{RL}^\alpha u(t_m) =: r^m.$$

one has $|e_m| \le \tau^\alpha \sum_{j=1}^m \sigma_{m-j} |r^j|$. And $|r_j| \lesssim j^{-2} + \tau^{\gamma-\alpha} j^{-(\alpha+1-\gamma)} + \tau$ with $\gamma = \min\{1, 2\alpha\}$.

$$|e_m| \lesssim \tau^\alpha m^{\alpha-1} + \tau^\gamma m^{\gamma-1} + \tau^{1+\alpha} m^\alpha = \tau t_m^{\alpha-1} + \tau t_m^{\gamma-1} + \tau t_m^\alpha \lesssim \tau t_m^{\alpha-1}$$

using $t_m = m\tau$. □

Outline

Numerical example

We test the GL scheme (9) on an example of (7) whose solution is composed of the leading terms from (8).

Example

Take $c = 2$ and $T = 1$ in (7). Choose f such that $u(t) = t^\alpha + t^{2\alpha} + t^{1+\alpha}$ is the solution of (7).

Set $E1 := \max_{1 \le m \le M} |U_m - u(t_m)|$ and $E2 := |U_M - u(t_M)|$,

Numerical results Predicted rate: $E1\ O(\tau^{\alpha})$; $E2\ O(\tau)$

Table: Global errors and convergence rates

τ	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	$E1$	$Rate$	$E1$	$Rate$	$E1$	$Rate$
1/100	1.70e-02	0.20	8.14e-03	0.35	3.51e-03	0.91
1/200	1.49e-02	0.22	6.38e-03	0.40	1.86e-03	0.59
1/400	1.28e-02	0.23	4.84e-03	0.43	1.24e-03	0.64
1/800	1.09e-02	0.24	3.60e-03	0.45	7.96e-04	0.66
1/1600	9.23e-03	0.25	2.63e-03	0.47	5.03e-04	0.68
1/3200	7.76e-03		1.90e-03		3.14e-04	

Table: Errors and convergence rates at $t = 1$

τ	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	$E2$	$Rate$	$E2$	$Rate$	$E2$	$Rate$
1/100	6.44e-04	0.96	1.72e-03	0.99	3.51e-03	1.00
1/200	3.30e-04	0.97	8.66e-04	0.99	1.76e-03	1.00
1/400	1.68e-04	0.98	4.35e-04	1.00	8.80e-04	1.00
1/800	8.53e-05	0.98	2.18e-04	1.00	4.40e-04	1.00
1/1600	4.32e-05	0.99	1.09e-04	1.00	2.20e-04	1.00
1/3200	2.18e-05		5.47e-05		1.10e-04	

Thank you for your attention

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