AANMPDE 12

July 2, 2019

Optimal convergence rates in L^2 for a first order system least squares finite element method

Maximilian Bernkopf

joint work with

Jens Markus Melenk

TU Wien
Institute for Analysis and Scientific Computing

First order formulation

For $f \in L^2(\Omega)$ and $\gamma > 0$ consider

$$\begin{aligned} -\Delta u + \gamma u &= f \quad \text{in } \Omega, \\ \partial_n u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

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First order least squares approach

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Minimize a sum of squares!

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$$\|\nabla \cdot \boldsymbol{\varphi} + \gamma u - f\|_{L^2(\Omega)}^2 + \|\nabla u + \boldsymbol{\varphi}\|_{L^2(\Omega)}^2 \to \min$$

for $\boldsymbol{\varphi} \in \boldsymbol{H}_0(\Omega, \mathrm{div})$ *and* $u \in H^1(\Omega)$.

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Least squares formulation:

Find $(\boldsymbol{\varphi}, u) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ such that

$$b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) = F((\boldsymbol{\psi}, v)) \qquad \forall (\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega),$$

$$\begin{aligned} b((\boldsymbol{\varphi}, u), (\boldsymbol{\psi}, v)) &:= (\nabla \cdot \boldsymbol{\varphi} + \gamma u, \nabla \cdot \boldsymbol{\psi} + \gamma v)_\Omega + (\nabla u + \boldsymbol{\varphi}, \nabla v + \boldsymbol{\psi})_\Omega, \\ F((\boldsymbol{\psi}, v)) &:= (f, \nabla \cdot \boldsymbol{\psi} + \gamma v)_\Omega. \end{aligned}$$

Unique solvability can be shown!

Numerical discretization

Let $\mathcal{T}_h$ be a quasi-uniform regular triangulation of Ω. For discretization employ the following spaces:

- $\mathbf{V}^0_{p_v}(\mathcal{T}_h) \subset \boldsymbol{H}_0(\Omega, \mathrm{div})$, either $\mathbf{RT}^0_{p_v-1}(\mathcal{T}_h)$ or $\mathbf{BDM}^0_{p_v}(\mathcal{T}_h)$
- $S_{p_s}(\mathcal{T}_h) \subset H^1(\Omega)$.

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The FOSLS method is to find $(\boldsymbol{\varphi}_h, u_h) \in \mathbf{V}^0_{p_v}(\mathcal{T}_h) \times S_{p_s}(\mathcal{T}_h)$ such that

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Problem: This method is optimal in $\|\cdot\|_b$. This norm is however not really tractable, how about error estimates in the L^2 norm?

Duality argument

Lemma (B. & Melenk, 2018)

Let Γ *be smooth. For any* $(\boldsymbol{\varphi}, w) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ *there exists* $(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0(\Omega, \mathrm{div}) \times H^1(\Omega)$ *such that* $\|w\|_{L^2(\Omega)}^2 = b((\boldsymbol{\varphi}, w), (\boldsymbol{\psi}, v))$*. Furthermore,* $\boldsymbol{\psi} \in \boldsymbol{H}^2(\Omega)$*,* $\nabla \cdot \boldsymbol{\psi} \in H^2(\Omega)$ *and* $v \in H^2(\Omega)$*, with*

$$\|v\|_{H^2(\Omega)} + \|\boldsymbol{\psi}\|_{H^2(\Omega)} + \|\nabla \cdot \boldsymbol{\psi}\|_{H^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)} .$$

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- Similar ideas in [Bochev & Gunzburger, 2005].
- Application with $w = e^u$ and $\boldsymbol{\varphi} = \boldsymbol{e}^{\boldsymbol{\varphi}}$ gives

$$\|e^u\|^2_{L^2(\Omega)} = b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi}, v)).$$

First $L^2(\Omega)$ estimate

Lemma

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Let Γ be smooth and $(\boldsymbol{\varphi}_h, u_h)$ be the least squares approximation of $(\boldsymbol{\varphi}, u)$. Then for $e^u = u - u_h$ and $\boldsymbol{e}^{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \boldsymbol{\varphi}_h$:

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$$\begin{aligned}\|e^u\|_{L^2(\Omega)} &\lesssim \frac{h}{p}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b \\ &\lesssim \frac{h}{p}\left[\|u - \tilde{u}_h\|_{H^1(\Omega)} + \|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}\right]\end{aligned}$$

for any $\tilde{u}_h \in S_{p_s}(\mathcal{T}_h)$, $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}^0_{p_v}(\mathcal{T}_h)$.

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- Well known result for the h-version, see [Bochev & Gunzburger, 2005]

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- Well known result for the h-version, see [Bochev & Gunzburger, 2005]
- However, suboptimal...

Proof of $\|e^u\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b$

Duality argument gives

$$\|e^u\|^2_{L^2(\Omega)} = b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi}, v))$$

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\|e^u\|_{L^2(\Omega)}^2 &= b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi}, v)) \\
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Regularity of $\boldsymbol{\psi}$ not fully exploited!

$\|e^u\|_{L^2(\Omega)} \lesssim \frac{h}{p} \|(\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u)\|_b$ is suboptimal!

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$f \in L^2(\Omega)$

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$$f \in L^2(\Omega) \to u \in H^2(\Omega) \to \boldsymbol{\varphi} = -\nabla u \in H^1(\Omega)$$

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$f \in L^2(\Omega) \to u \in H^2(\Omega) \to \boldsymbol{\varphi} = -\nabla u \in H^1(\Omega) \to \nabla \cdot \boldsymbol{\varphi} \in L^2(\Omega)$
No further rate...

Suboptimal in terms of data regularity!

Towards optimality!

Reminder: $\boldsymbol{\psi} \in \boldsymbol{H}^2(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^2(\Omega)$ and $v \in H^2(\Omega)$

$$\|e^u\|_{L^2(\Omega)}^2 = b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi}, v))$$

Towards optimality!

Reminder: $\boldsymbol{\psi} \in \boldsymbol{H}^2(\Omega)$, $\nabla \cdot \boldsymbol{\psi} \in H^2(\Omega)$ and $v \in H^2(\Omega)$

$$
\begin{aligned}
\|e^u\|^2_{L^2(\Omega)} &= b((\boldsymbol{e}^{\boldsymbol{\varphi}}, e^u), (\boldsymbol{\psi}, v)) \\
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- limiting term $(\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h), \nabla \cdot \boldsymbol{e^\psi})_\Omega$

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- limiting term $(\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h), \nabla \cdot \boldsymbol{e}^{\boldsymbol{\psi}})_\Omega$
- still can choose special $\tilde{\boldsymbol{\varphi}}_h = \boldsymbol{I}^0_h \boldsymbol{\varphi}$ such that:
- Orthogonality on $\nabla \cdot \mathbf{V}^0_{p_v}(\mathcal{T}_h)$ and small $L^2(\Omega)$ error

$$\boldsymbol{I}_h^0 \boldsymbol{\varphi} = \underset{\boldsymbol{\varphi}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h)}{\operatorname{argmin}} \frac{1}{2} \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{L^2(\Omega)}^2$$

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s.t. $$(\nabla \cdot (\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}), \nabla \cdot \boldsymbol{\chi}_h)_{L^2(\Omega)} = 0, \qquad \forall \boldsymbol{\chi}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h).$$

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Lemma (B. & Melenk, 2018)

The operator $\boldsymbol{I}_h^0$ satisfies the following estimates for any $\tilde{\boldsymbol{\varphi}}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h)$

$$\left\|\boldsymbol{\varphi} - \boldsymbol{I}_h^0\boldsymbol{\varphi}\right\|_{L^2(\Omega)} \lesssim \left\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\right\|_{L^2(\Omega)} + \frac{h}{p}\left\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\right\|_{L^2(\Omega)}$$

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With $\boldsymbol{e} = \boldsymbol{\varphi} - \boldsymbol{I}_h^0 \boldsymbol{\varphi}$ we have

$$\|\boldsymbol{e}\|_{L^2(\Omega)}^2 = (\boldsymbol{e}, \boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)_{L^2(\Omega)} + (\boldsymbol{e}, \tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0 \boldsymbol{\varphi})_{L^2(\Omega)}.$$

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Helmholtz decompose $\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi}$ on discrete and continuous level:

$$\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi} = \nabla \times \boldsymbol{\mu} + \boldsymbol{r},$$
$$\tilde{\boldsymbol{\varphi}}_h - \boldsymbol{I}_h^0\boldsymbol{\varphi} = \nabla \times \boldsymbol{\mu}_h + \boldsymbol{r}_h$$

for $\boldsymbol{\mu} \in \boldsymbol{H}_0(\Omega, \mathrm{curl})$, $\boldsymbol{r} \in \boldsymbol{H}_0(\Omega, \mathrm{div})$, $\boldsymbol{\mu}_h \in \mathbf{N}_{p_v}^0(\mathcal{T}_h)$ and $\boldsymbol{r}_h \in \mathbf{V}_{p_v}^0(\mathcal{T}_h)$.

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With $e = \varphi - I_h^0\varphi$ we have

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$$(e, \tilde{\varphi}_h - I_h^0\varphi)_{L^2(\Omega)} = (e, \nabla \times \mu_h + r_h)_{L^2(\Omega)}$$

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$$\begin{aligned}(e, \tilde{\varphi}_h - I_h^0\varphi)_{L^2(\Omega)} &= (e, \nabla\times\mu_h + r_h)_{L^2(\Omega)} = (e, r_h)_{L^2(\Omega)}\\ &= (e, r_h - r)_{L^2(\Omega)} + (e, r)_{L^2(\Omega)}\end{aligned}$$

Optimal rate!

Theorem (B. & Melenk, 2018)

Let Γ *be smooth and* $(\boldsymbol{\varphi}_h, u_h)$ *be the least squares approximation of* $(\boldsymbol{\varphi}, u)$. *Then for* $e^u = u - u_h$

$$\|e^u\|_{L^2(\Omega)} \lesssim \frac{h}{p}\|u - \tilde{u}_h\|_{H^1(\Omega)} + \frac{h^2}{p^2}\|\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h\|_{L^2(\Omega)} + \frac{h^2}{p^2}\|\nabla \cdot (\boldsymbol{\varphi} - \tilde{\boldsymbol{\varphi}}_h)\|_{L^2(\Omega)}$$

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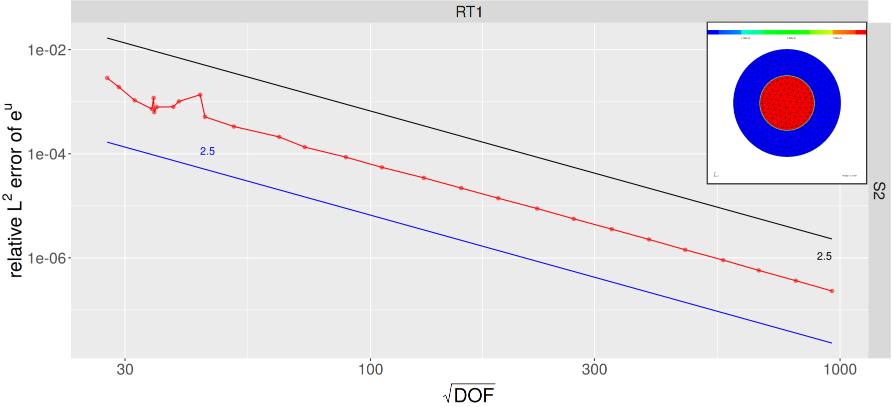
For $f \in L^2(\Omega)$ the rate h^2 is in fact achieved!

2d with $f = \chi_{(0,1/2)}(r) \in H^{0.5-\varepsilon}(\Omega)$



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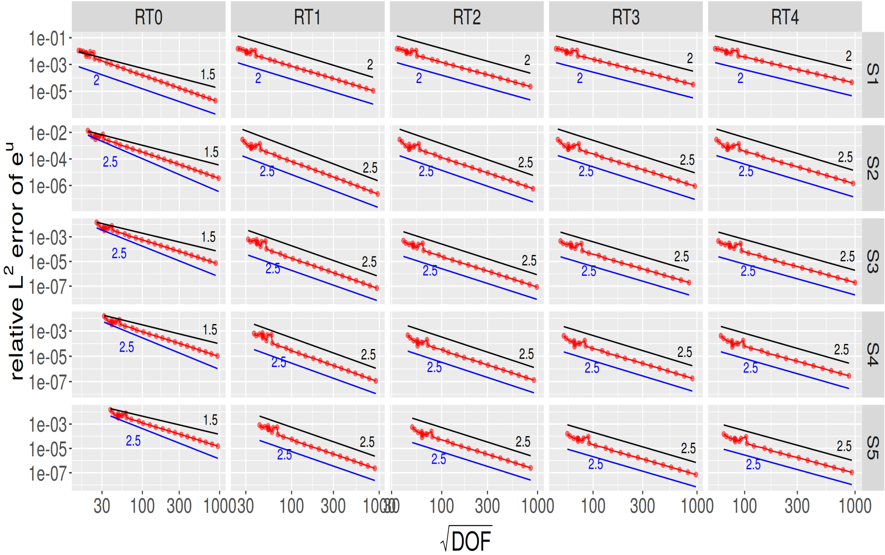
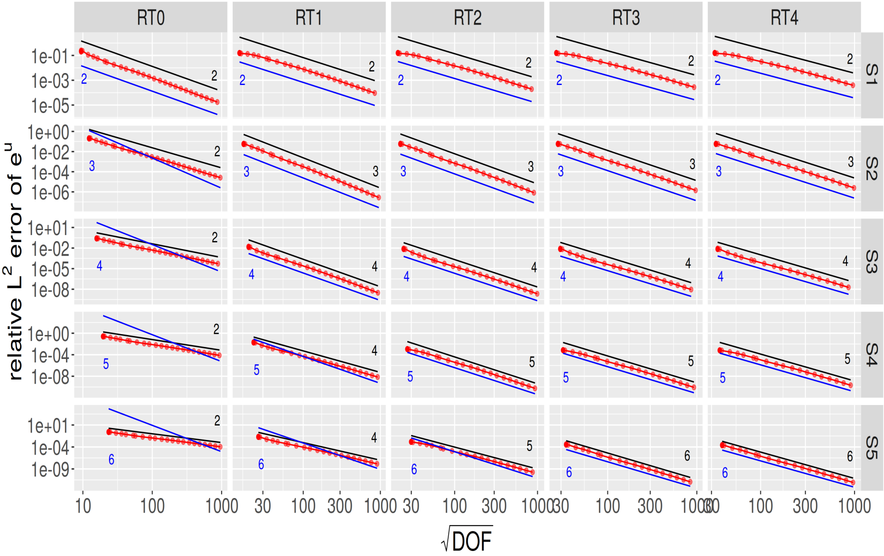


2d with u smooth



Analogous results & extensions

Analogous results hold for:

- Duality arguments and optimality results for ∇e^u and $\boldsymbol{e}^{\boldsymbol{\varphi}}$

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Main difficulties for Robin:

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Thank you for your attention