Bilinear forms for first-order systems

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Background: the text-book finite-element 'protocol'

(i) Boundary-value problem:

$$\begin{aligned} \mathscr{L}u &= f \quad \text{in } \Omega, \\ \mathscr{B}u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{PDE}$$

(ii) Reformulation:

$$\begin{aligned} &\text{Find } u \in V \text{ such that} \\ &a(v,u) = l(v) \qquad \forall v \in L \end{aligned} \tag{VP}$$

(iii) Discretization:

$$\begin{aligned} &\text{Find } u_h \in V_h \text{ such that} \\ &a(v_h,u_h) = l(v_h) \qquad \forall v_h \in L_h \end{aligned} \tag{FEM}$$

Typically, (VP) interpreted as the **meaning** of (PDE)

Well-posedness[1]

- l linear, continuous on L
- a bilinear and continuous on $L \times V$
- a defines a bounded linear operator $A : V \to L'$

(VP) well posed if and only if

(i) $\exists \alpha > 0$ such that, for each $u \in V$,

$$\sup_{\substack{v \in L \\ v \neq 0}} \frac{a(v,u)}{\|v\|_L} \geq \alpha \|u\|_V$$

$\Rightarrow$ The range of A is closed.

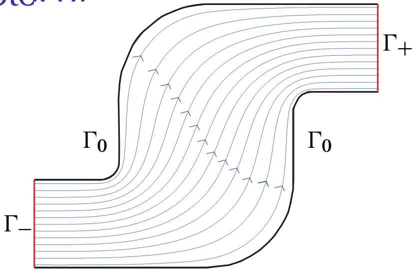
(ii) If $v \in L$ satifies

$$a(v,u) = 0 \qquad \forall u \in V$$

then $v = 0$.
$\Rightarrow$ A surjective.

[1] Nečas (1962)

Notable exception from the 'protocol'!



Discontinuous Galerkin (DG) methods for hyperbolic equations[2]

Standard model problem):

$$\begin{aligned} \boldsymbol{\beta} \cdot \nabla u + \rho u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_- \end{aligned}$$

Assuming $\rho(\boldsymbol{x}) \geq \rho_0 > 0$, $\nabla \cdot \boldsymbol{\beta} = 0$.

$$\Gamma_{-(+)} = \{ \boldsymbol{x} \in \partial\Omega \mid \boldsymbol{n} \cdot \boldsymbol{\beta} < (>) 0 \}$$
$$\Gamma_0 = \{ \boldsymbol{x} \in \partial\Omega \mid \boldsymbol{n} \cdot \boldsymbol{\beta} = 0 \}$$

- In DG literature, for well-posedness of hyperbolic problems, if at all discussed, reference to "theory for Friedrichs systems"
- Numerical method **not** based on discretization of a variational formulation!

[2]Reed & Hill (1973), Lesaint & Raviart (1974)

DG methods for hyperbolic equations

Variational form **only for the discrete problem**! Basic idea:

- Let V_h be a space of finite-dimensional, weakly differentiable (for now) functions
- Define $a_h(v_h, u_h) = l_h(v_h)$ for $u_h, v_h \in V_h$, where

$$a_h(v_h, u_h) = \int_\Omega v_h(\boldsymbol{\beta} \cdot \nabla u_h + \rho u_h) - \int_{\Gamma_-} \boldsymbol{n} \cdot \boldsymbol{\beta} v_h u_h,$$

$$l_h(v_h) = \int_\Omega v_h f - \int_{\Gamma_-} \boldsymbol{n} \cdot \boldsymbol{\beta} v_h g$$

Note:

- Variational expression **consistent** with BVP (i.e. $a_h(v_h, u) = l_h(v_h)$)
- Boundary condition weakly imposed

DG methods for hyperbolic equations

System matrix positive definite: for $u_h \neq 0$,

$$
\begin{aligned}
a_h(u_h, u_h) &= \int_{\Omega} u_h\big(\boldsymbol{\beta} \cdot \nabla u_h + \rho u_h\big) - \int_{\Gamma_-} \boldsymbol{n} \cdot \boldsymbol{\beta} (u_h)^2 \\
&= \frac{1}{2} \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{\beta} (u_h)^2 + \int_{\Omega} \rho (u_h)^2 - \int_{\Gamma_-} \boldsymbol{n} \cdot \boldsymbol{\beta} (u_h)^2 \\
&= \int_{\Omega} \rho (u_h)^2 + \frac{1}{2} \int_{\partial\Omega} |\boldsymbol{n} \cdot \boldsymbol{\beta}| (u_h)^2 > 0
\end{aligned}
$$

Thus, the linear system is solvable

DG methods for hyperbolic equations

- Stability for a_h as above too weak in practice
- Therefore:
  - Relax continuity; let V_h be **piecewise** polynomials
  - Impose interelement continuity **weakly**, analogous to BC
- Yields improved coercivity property

$$a_h(u_h, u_h) = \int_\Omega \rho (u_h)^2 + \frac{1}{2} \sum_{K \in \mathscr{T}_h} \int_{\partial K-} |\boldsymbol{n} \cdot \boldsymbol{\beta}| [\![u_h]\!]^2 + \frac{1}{2} \int_{\Gamma_+} |\boldsymbol{n} \cdot \boldsymbol{\beta}| (u_h)^2$$

$$[\![u_h]\!] = u_h^+ - u_h^-$$

- RHS discrete version of the norm

$$\|u\|^2 = \int_\Omega \rho u^2 + \int_\Omega (\boldsymbol{\beta} \cdot \nabla u)^2$$

A proper variational formulation?

Recall discrete expression

$$\int_{\Omega} v_h (\boldsymbol{\beta} \cdot \nabla u_h + \rho u_h) - \int_{\Gamma_-} \boldsymbol{n} \cdot \boldsymbol{\beta} v_h u_h = \int_{\Omega} v_h f - \int_{\Gamma_-} \boldsymbol{n} \cdot \boldsymbol{\beta} v_h g$$

- Data in $L^2(\Omega) \times L^2(\Gamma_-; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$
- Suggests test space $L = L^2(\Omega) \times L^2(\Gamma_-; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$
- Differential operator $Tu = \boldsymbol{\beta} \cdot \nabla u + \rho u$
- Suggests solutions bounded in $\|u\|_W^2 = \|u\|_{L^2(\Omega)}^2 + \|Tu\|_{L^2(\Omega)}^2$
- Boundary integral requires traces of u in $L^2(\Gamma_-; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$
- Our approach inspired by and developed from

  A. Ern, J.-L. Guermond, G. Caplain. An Intrinsic Criterion for the Bijectivity of Hilbert Operators Related to Friedrichs' Systems. *Comm. Partial Differential Equations* 32:317–341, 2007

The proposed variational formulation

- Domain Ω open, bounded, connected with Lipschitz boundary
- Function spaces

$$L = L^2(\Omega) \times L^2(\Gamma_-; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$$
$$W = \left\{ u \in L^2(\Omega) \mid Tu \in L^2(\Omega) \right\}$$

- Functions in W admits traces only in $H^{-1/2}(\partial\Omega)$ in general
- However, traces in $L^2(\partial\Omega; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$ when $\mathrm{dist}(\Gamma_-, \Gamma_+) > 0$ (Ern, Guermond, *SINUM*, 2006)
- Thus, assume $\mathrm{dist}(\Gamma_-, \Gamma_+) > 0$ and choose solution space $V = W$
- Denote by γ_- (γ_+) the trace operators $V \to L^2(\Gamma_-; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$ ($V \to L^2(\Gamma_+; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$)

The proposed variational formulation

For $v = (v_0, v_1) \in L$, define

$$a(v,u) = \int_\Omega v_0\big(\underbrace{\boldsymbol{\beta}\cdot\nabla u + \rho u}_{Tu}\big) - \int_{\Gamma_-} \boldsymbol{n}\cdot\boldsymbol{\beta} v_1 u \qquad u \in V$$

$$a^*(v,u) = \int_\Omega v_0\big(\underbrace{-\boldsymbol{\beta}\cdot\nabla u + \rho u}_{\tilde{T}u}\big) + \int_{\Gamma_+} \boldsymbol{n}\cdot\boldsymbol{\beta} v_1 u \qquad u \in V^* = V$$

$$l(v) = \int_\Omega v_0 f - \int_{\Gamma_-} \boldsymbol{n}\cdot\boldsymbol{\beta} v_1 g$$

where a^* satifies, $\forall u, v \in V$,

$$a^*\big((u, \gamma_+ u), v\big) = a\big((v, \gamma_- v), u\big)$$

Theorem

The variational problem

$$\begin{aligned} &\textit{Find } u \in V \textit{ such that} \\ &a(v,u) = l(v) \qquad \forall v \in L \end{aligned} \tag{VP}$$

is well posed

Proof strategy

1. Continuity of a, a^*, l (by construction)
2. V, V^* are closed
3. $C^1(\overline{\Omega})$ is dense in V
4. Boundary traces of functions in V are in $L^2(\partial\Omega; |\boldsymbol{n} \cdot \boldsymbol{\beta}|)$
5. Inf-sup in L: $\exists \alpha > 0$ s.t.

$$\sup_{v \in L \setminus \{0\}} \frac{a(v,u)}{\|v\|} \geq \alpha \left(\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma_-;|\boldsymbol{n}\cdot\boldsymbol{\beta}|)} \right)^{1/2} \quad \forall u \in V$$

$$\sup_{v \in L \setminus \{0\}} \frac{a^*(v,u)}{\|v\|} \geq \alpha \left(\|u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma_+;|\boldsymbol{n}\cdot\boldsymbol{\beta}|)} \right)^{1/2} \quad \forall u \in V^* \quad (*)$$

6. Inf-sup in V: $\exists \alpha > 0$ s.t.

$$\sup_{v \in L \setminus \{0\}} \frac{a(v,u)}{\|v\|} \geq \alpha \|u\|_V \qquad \forall u \in V$$

7. Surjectivity: if $v \in L$ such that

$$a(v,u) = 0 \qquad \forall u \in V,$$

then $v = 0$. Here, the "adjoint" inf-sup property $(*)$ in L is utilized.

Inf-sup

In L: for $u \in V$, let $\hat{u} = (u, \gamma\_u)$. Then

$$a(\hat{u}, u) = \int_{\Omega} u\big[(\boldsymbol{\beta} \cdot \nabla)u + \rho u\big] - \int_{\Gamma_-} \boldsymbol{n} \cdot \beta u^2$$

$$= \frac{1}{2} \int_{\partial\Omega} |\boldsymbol{n} \cdot \boldsymbol{\beta}| u^2 + \int_{\Omega} \rho u^2 \geq \frac{1}{2} \|\hat{u}\|_L^2,$$

from which the condition follows.

In V: let $\hat{v} = \big(\boldsymbol{\beta} \cdot \nabla u + \rho u, 0\big)$ $(u \neq 0)$

$$a(\hat{v}, u) = \left[\int_{\Omega} (\boldsymbol{\beta} \cdot \nabla u + \rho u)^2\right]^{1/2} = \|Tu\|,$$

which together with the condition in L yields the result.

Surjectivity

Recall

$$a(v,u) = \int_\Omega v_0\big(\boldsymbol{\beta}\cdot\nabla u + \rho u\big) - \int_{\Gamma_-} \boldsymbol{n}\cdot\boldsymbol{\beta}\, v_1 u$$

Let $v \in L$ such that $a(v,u) = 0$, $\forall u \in V$. Two properties of $v = (v_0, v_1)$ follow:

$$-(\boldsymbol{\beta}\cdot\nabla)v_0 + \rho v_0 = \tilde{T} v_0 = 0 \qquad \text{(by def. of weak derivative)}$$

$$\Rightarrow v \in W = V$$

$$v_0 = \begin{cases} 0 & \text{on } \Gamma_+, \\ v_1 & \text{on } \Gamma_- \end{cases} \qquad \text{(after integration by parts)}$$

$\tilde{T} v_0 = 0$ and $\gamma_+ v_0 = 0 \Rightarrow v_0 = 0$ by inf-sup condition in L of a^*.
Thus $v_1 = 0$ by above. □

The acoustic wave equation

$\Omega \in \mathbb{R}^d$, open, bounded, connected with a smooth boundary; $T < +\infty$

$$\partial_t \boldsymbol{u} + \nabla p = f \qquad \text{in } Q = \Omega \times (0,T) \tag{1a}$$

$$\partial_t p + \nabla \cdot \boldsymbol{u} = 0 \qquad \text{in } Q = \Omega \times (0,T) \tag{1b}$$

$$p - \boldsymbol{n} \cdot \boldsymbol{u} - \alpha(p + \boldsymbol{n} \cdot \boldsymbol{u}) = g \qquad \text{on } \Sigma = \partial\Omega \times (0,T), \tag{1c}$$

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{s}}, \qquad p = p_{\mathrm{s}} \qquad \text{on } \Sigma_{\mathbf{0}} = \Omega \times \{0\} \tag{1d}$$

Condition (1c) in alternative form:

$$(1-\alpha)p - (1+\alpha)\boldsymbol{n} \cdot \boldsymbol{u} = g$$

Restriction: $\alpha \in L^\infty(\partial\Omega)$, $|\alpha| \le \alpha_M < 1$

Here

$$\xi = \begin{pmatrix} \boldsymbol{\xi}_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^{d+1}, \quad T = \begin{pmatrix} \partial_t & \nabla \\ \nabla\cdot & \partial_t \end{pmatrix}, \quad T\xi = \begin{pmatrix} \partial_t \boldsymbol{\xi}_1 + \nabla \xi_2 \\ \partial_t \xi_2 + \nabla \cdot \boldsymbol{\xi}_1 \end{pmatrix},$$

$$\tilde{T} = -T = \begin{pmatrix} -\partial_t & -\nabla \\ -\nabla\cdot & -\partial_t \end{pmatrix}, \qquad \gamma_\Sigma^{\pm} \xi = (\xi_2 \pm \boldsymbol{n} \cdot \boldsymbol{\xi}_1)\big|_\Sigma$$

Acoustic wave equation: variational formulation

Let $\eta = (\hat{\eta}, \eta_\Sigma, \eta_|)$, $\eta \in L^2(Q)^{d+1} \times L^2(\Sigma) \times L^2(\Omega)^{d+1}$

$$a(\eta,\xi) = \int_Q \hat{\eta}^T T\xi + \int_\Sigma \eta_\Sigma \left[\gamma_\Sigma^- \xi - \alpha\gamma_\Sigma^+ \xi\right] + \int_{\Sigma_0} \eta_|^T \xi$$

$$l(\eta) = \int_Q \hat{\eta}^T f + \int_\Sigma \eta_\Sigma g + \int_{\Sigma_0} \eta_|^T \xi_0$$

$$a^*(\eta,\xi) = \int_Q \hat{\eta}^T \tilde{T}\xi + \int_\Sigma \eta_\Sigma \left[\gamma_\Sigma^+ \xi - \alpha\gamma_\Sigma^- \xi\right] - \int_{\Sigma_T} \eta_|^T \xi$$

Graph space:

$$W = \left\{ \xi \in L^2(Q)^{d+1} \mid T\xi \in L^2(Q)^{d+1} \right\}$$

Admits traces in $H^{-1/2}(\partial Q)$. Therefore following spaces well defined:

$$V = \left\{ \xi \in W \mid \gamma_\Sigma^- \xi \in L^2(\Sigma), \gamma_{\Sigma_0}\xi \in L^2(\Omega)^{d+1} \right\}$$

$$V^* = \left\{ \xi \in W \mid \gamma_\Sigma^+ \xi \in L^2(\Sigma), \gamma_{\Sigma_T}\xi \in L^2(\Omega)^{d+1} \right\}$$

Acoustic wave equation: steps of well-posedness proof

Lemma

V, V^* are closed.

Straightforward to prove.

Lemma

$C^1(\overline{Q})^{d+1}$ is dense in V and V^*

Not straightforward! The proof uses a density result by Rauch (1985).

Lemma

The trace operators $\gamma_\Sigma^+ \colon C^1(\overline{Q})^{d+1} \to L^2(\Sigma)$, $\gamma_{\Sigma_T} \colon C^1(\overline{Q})^{d+1} \to L^2(\Omega)^{d+1}$ defined by

$$\gamma_\Sigma^+ \xi = (\xi_2 + \boldsymbol{n} \cdot \boldsymbol{\xi}_1)|_\Sigma$$
$$\gamma_{\Sigma_T} = \xi\big|_{t=T}$$

extend uniquely to continuous operators on V.

Thus, integration by parts holds with $L^2(\partial Q)$ boundary integrals

Acoustic wave equation: inf-sup conditions

Lemma (inf-sup in L)

$\exists\alpha = C\min(1/T, 1-\alpha_M)$ *s.t.* $\forall\xi \in V$,

$$\sup_{\eta\in L\backslash\{0\}} \frac{a(\eta,\xi)}{\|\eta\|} \geq \alpha\left(\|\xi\|^2_{L^2(Q)^{d+1}} + \|\gamma^-_\Sigma\xi\|^2_{L^2(\Sigma)} + \|\gamma_{\Sigma_0}\xi\|^2_{L^2(\Omega)^{d+1}}\right)^{1/2}$$

and $\forall\xi \in V^*$,

$$\sup_{\eta\in L\backslash\{0\}} \frac{a^*(\eta,\xi)}{\|\eta\|} \geq \alpha\left(\|\xi\|^2_{L^2(Q)^{d+1}} + \|\gamma^+_\Sigma\xi\|^2_{L^2(\Sigma)} + \|\gamma_{\Sigma_T}\xi\|^2_{L^2(\Omega)^{d+1}}\right)^{1/2}$$

Proven by (for a) choosing $\eta = \left(e^{-t/T}\xi, \frac{1}{2}e^{-t/T}\gamma^-_\Sigma\xi, \gamma_{\Sigma_0}\xi\right)$, which gives the lower bound after integration by parts.

Acoustic wave equation: inf-sup and surjectivity

Lemma (inf-sup in V)

$\exists \alpha > 0$ *s.t.* $\forall \xi \in V$,

$$\sup_{\eta \in L \setminus \{0\}} \frac{a(\eta, \xi)}{\|\eta\|} \geq \alpha \|\xi\|_V \quad \forall \xi \in V$$

Choosing $\eta = (T\xi, 0, 0)^T$ gives $\|T\xi\|_{L^2(Q)^{d+1}}$ as lower bound, which together with the bound in L yields the result. □

Surjectivity proof same as before, but algebraically messier

Acoustic wave equation; well-posedness

$$\begin{aligned}
\partial_t \boldsymbol{u} + \nabla p &= f && \text{in } Q = \Omega \times (0,T) \\
\partial_t p + \nabla \cdot \boldsymbol{u} &= 0 && \text{in } Q = \Omega \times (0,T) \\
p - \boldsymbol{n} \cdot \boldsymbol{u} - \alpha(p + \boldsymbol{n} \cdot \boldsymbol{u}) &= g && \text{on } \Sigma = \partial\Omega \times (0,T) \\
\boldsymbol{u} = \boldsymbol{u}_\mathrm{s}, \qquad p &= p_\mathrm{s} && \text{on } \Sigma_0 = \Omega \times \{0\}
\end{aligned}$$

Theorem

The system admits a unique solution in V satisfying

$$\|(\boldsymbol{u}, p)\|_V \leq \frac{1}{\alpha} \|(f, g, \boldsymbol{u}_s, p_s)\|_L,$$

where $\alpha = C \min(1/T, 1 - \alpha_M)$

Final words

- Ongoing work, no manuscripts yet!
- Closes a “gap” in the typical protocol for FE analysis
- The variational form, although simple and natural, appears to be new
- Difference from Ern, Guermond, Caplain:
  - They **strongly** enforce **homogeneous** characteristic BC in definition of $V \subset W$
  - We **weakly** enforce **nonhomogeneous** BC through the bilinear form
- Next step:
  - Explore the method for additional problems (equations, boundary conditions)
  - Construct numerical methods based on the bilinear form