Bilinear forms for first-order systems

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Abstract

A finite-element approximation of a boundary-value problem is usually carried out according to a standard 'protocol:' (i) reformulation, and (ii) discretization. The *reformulation* collects the partial differential equation, the boundary conditions, and possibly other side conditions into a common expression. For linear boundary-value problem, the result is

$$\begin{aligned} &\text{find } u \in V \text{ such that} \\ &a(v,u) = l(v) \quad \forall v \in L, \end{aligned} \tag{1}$$

where l and a are continuous linear and bilinear forms on Hilbert spaces L and $L \times V$. Indeed, the solution theory for boundary-value problems is often based on such 'variational' formulations. The *discretization* step involves picking finite-dimensional spaces V_h and L_h approximating V and L as well as, sometimes, approximations of a and l.

A notable exception to this scheme is the treatment of first-order systems such as advection problems and the first-order formulation of the acoustic and Maxwell equations. A standard and successful finite element method for such problems is the discontinuous Galerkin method, where linear and bilinear forms are introduced *directly* on the finite-dimensional spaces, element by element. Regarding well-posedness of the problem before discretization, if discussed at all, authors typically refers to the theory of Friedrichs systems, which is not of the type (1).

Based on a reformulation of the theory of Friedrichs systems due to Ern, Guermond, and Caplain [1], we show a number of examples of how to incorporate the generalized Friedrichs systems in the standard variational framework (1). The critical step here concerns the fulfillment of characteristic boundary conditions, which Ern et al. [1] impose strongly in the definition of the solution space. We choose a weak enforcement in line with the practice in discontinuous Galerkin methods. With a proper choice of solution space V, test space L, and forms a and l, we show well-posedness of a few example first-order systems directly for the problem in the form (1). The variational formulations we present are not identical to the ones typical in discontinuous Galerkin methods and might constitute the first step towards a new class of numerical schemes for first-order systems.

References

[1] A. Ern, J.-L. Guermond, and G. Caplain. An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems. *Comm. Partial Differential Equations*, 32:317–341, 2007.