Localized pointwise error estimates of finite element approximations to the Stokes problem on convex polyhedra (and polygons)

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Outline

1. Applications and problem statement

2. Idea of the proof

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2. Idea of the proof

The situation for the elliptic problem

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\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Schatz/Wahlbin 1977 (interior error estimates)

For $\Omega_1 \Subset \Omega_2 \Subset \Omega$, $\operatorname{dist}(\Omega_1, \partial\Omega_2) \geq d$, $\operatorname{dist}(\Omega_2, \partial\Omega) \geq d$, then

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\|u - u_h\|_{L^\infty(\Omega_1)} \leq C_d \Big(h^l |\ln h|^r |u|_{W^{l,\infty}(\Omega_2)} + \|u - u_h\|_{L^2(\Omega_2)} \Big)
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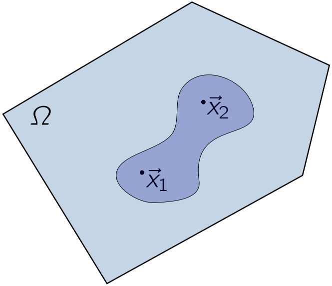

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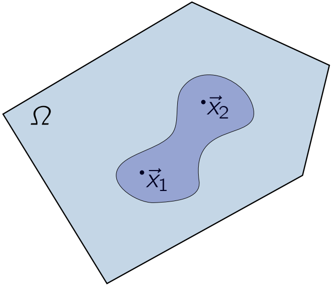
- L^∞ error estimates on graded meshes



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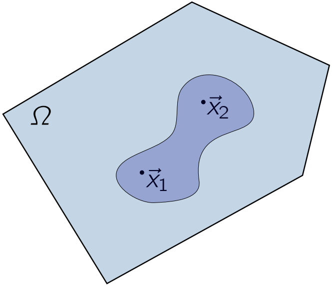
- L^∞ error estimates on graded meshes
- (Local) L^∞ error estimates for the parabolic problem (e.g. heat equation)



Problem statement

Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedral domain and $\vec{V}_h$ and M_h suitable finite element spaces for the Stokes problem (e.g. Taylor-Hood FEs).

Stokes equation and discretization

$(\vec{u}, p) \in H_0^1(\Omega)^3 \times L_0^2(\Omega)$ solve

$$\begin{aligned} -\Delta \vec{u} + \nabla p &= \vec{f} \quad \text{in } \Omega \\ \nabla \cdot \vec{u} &= 0 \quad \text{in } \Omega \\ \vec{u} &= \vec{0} \quad \text{on } \partial\Omega \end{aligned}$$

$(\vec{u}_h, p_h) \in \vec{V}_h \times M_h$ satisfy

$$\begin{aligned} (\nabla \vec{u}_h, \nabla \vec{v}_h) - (p_h, \nabla \cdot \vec{v}_h) + (\nabla \cdot \vec{u}_h, q_h) \\ = (\vec{f}, \vec{v}_h) \quad \forall (\vec{v}_h, q_h) \in \vec{V}_h \times M_h \end{aligned}$$

Discretization error estimate: global

$$\|\vec{u}_h - \vec{u}\|_{L^\infty(\Omega)} \leq \inf_{(\vec{v}_h, q_h) \in \vec{V}_h \times M_h} C|\ln h| \Big(|\ln h| \|\vec{u} - \vec{v}_h\|_{L^\infty(\Omega)} + h\|p - q_h\|_{L^\infty(\Omega)} \Big)$$

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Discretization error estimate: local

$\Omega_1 \subset \Omega_2 \subset \Omega$ with $dist(\bar{\Omega}_1, \partial\Omega_2) \geq d$

$$\begin{aligned} \|\vec{u}_h - \vec{u}\|_{L^\infty(\Omega_1)} \leq \inf_{(\vec{v}_h, q_h) \in \vec{V}_h \times M_h} \Big[& C|\ln h| \Big(|\ln h| \|\vec{u} - \vec{v}_h\|_{L^\infty(\Omega_2)} + h\|p - q_h\|_{L^\infty(\Omega_2)} \Big) + \\ & C_d |\ln h| \Big(h\|\vec{u} - \vec{v}_h\|_{H^1(\Omega)} + \|\vec{u} - \vec{v}_h\|_{L^2(\Omega)} + h\|p - q_h\|_{L^2(\Omega)} \Big) \Big] \end{aligned}$$

A short history

R. G. Durán, R. H. Nochetto, and J. P. Wang. Sharp maximum norm error estimates for finite element approximations of the Stokes problem in 2-D.

Math. Comp., 51(184):491–506, 1988

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Three dimensions, global, $W^{1,\infty}(\Omega)$, technique II.

Our results: Global results in three dimensions ($L^\infty(\Omega)$), local results in two and three dimensions ($L^\infty(\Omega)$,$W^{1,\infty}(\Omega)$)

Outline

Selected ingredients for the proof

For the Ritz projection $R_h\vec{z}$ of $\vec{z} \in H_0^1(\Omega)^3$ into the finite element space given by

$$(\nabla R_h\vec{z}, \nabla\vec{v}_h) = (\nabla\vec{z}, \nabla\vec{v}_h) \quad \forall\vec{v}_h \in \vec{V}_h$$

that for $\vec{z} \in H_0^1(\Omega)^3 \cap L^\infty(\Omega)^3$ the solution of the Laplace equation, it holds that

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A space fulfilling this assumption holds would be e.g. the space of Taylor-Hood finite elements of order greater or equal three.
To start off the proof, we define a regularized Green's function

$$\begin{aligned} -\Delta\vec{g} + \nabla\lambda &= \delta_h\vec{e}_i \quad &&\text{in } \Omega, \\ \nabla\cdot\vec{g} &= 0 &&\text{in } \Omega, \\ \vec{g} &= \vec{0} &&\text{on } \partial\Omega. \end{aligned}$$

Here $\delta_h \in C_0^1(T_{\vec{x}_0})$, $T_{\vec{x}_0}$ the cell where the maximum of $|\vec{u}_h|$ is attained, which satisfies for every $\vec{v}_h \in \vec{V}_h$:

$$\vec{v}_{h,i}(\vec{x}_0) = (\vec{v}_h, \delta_h\vec{e}_i)_{T_{\vec{x}_0}}.$$

$\mathbf{L}^\infty$ estimate

$$
\begin{aligned}
|\vec{u}_{h,i}(\vec{x}_0)| = |(\vec{u}_h, \delta_h \vec{e}_i)_{T_{\vec{x}_0}}| &= |(\nabla \vec{u}_h, \nabla \vec{g}_h) - (p_h, \nabla \cdot \vec{g}_h) + (\nabla \cdot \vec{u}_h, \lambda_h)| \\
&= |(\nabla \vec{u}, \nabla \vec{g}_h) - (p, \nabla \cdot \vec{g}_h)|
\end{aligned}
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&= |(\nabla \vec{u}, \nabla \vec{g}_h) - (p, \nabla \cdot \vec{g}_h)| \\
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&= |(\nabla R_h \vec{u}, \nabla \vec{g}_h) - (p, \nabla \cdot (\vec{g}_h - \vec{g}))|
\end{aligned}
$$

For the first term we get

$$
\begin{aligned}
|(\nabla R_h \vec{u}, \nabla \vec{g}_h)| &= |(\nabla R_h \vec{u}, \nabla(\vec{g}_h - \vec{g})) - (R_h \vec{u}, \Delta \vec{g})| \\
&= |(\nabla R_h \vec{u}, \nabla(\vec{g}_h - \vec{g})) - (R_h \vec{u}, -\delta_h \vec{e}_i + \nabla \lambda)| \\
&\leq h^{-1} \|R_h \vec{u}\|_{L^\infty(\Omega)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} + \|R_h \vec{u}\|_{L^\infty(\Omega)} \Big(C + \|\nabla \lambda\|_{L^1(\Omega)} \Big) \\
&\leq C h^{-1} |\ln h| \|\vec{u}\|_{L^\infty(\Omega)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} \\
&\quad + C |\ln h| \|\vec{u}\|_{L^\infty(\Omega)} \Big(1 + \|\nabla \lambda\|_{L^1(\Omega)} \Big)
\end{aligned}
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L^∞ estimate

$$\begin{aligned} |\vec{u}_{h,i}(\vec{x}_0)| = |(\vec{u}_h, \delta_h \vec{e}_i)_{T_{\vec{x}_0}}| &= |(\nabla \vec{u}_h, \nabla \vec{g}_h) - (p_h, \nabla \cdot \vec{g}_h) + (\nabla \cdot \vec{u}_h, \lambda_h)| \\ &= |(\nabla \vec{u}, \nabla \vec{g}_h) - (p, \nabla \cdot \vec{g}_h)| \\ &= |(\nabla R_h \vec{u}, \nabla \vec{g}_h) - (p, \nabla \cdot (\vec{g}_h - \vec{g}))| \end{aligned}$$

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and for the second term

$$|(p, \nabla \cdot (\vec{g}_h - \vec{g}))| \le C \|p\|_{L^\infty(\Omega)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)}$$

L^∞ estimate

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$$= |(\nabla \vec{u}, \nabla \vec{g}_h) - (p, \nabla \cdot \vec{g}_h)|$$
$$= |(\nabla R_h \vec{u}, \nabla \vec{g}_h) - (p, \nabla \cdot (\vec{g}_h - \vec{g}))|$$

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$$\leq h^{-1} \|R_h \vec{u}\|_{L^\infty(\Omega)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} + \|R_h \vec{u}\|_{L^\infty(\Omega)} \Big(C + \|\nabla \lambda\|_{L^1(\Omega)} \Big)$$
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and for the second term

$$|(p, \nabla \cdot (\vec{g}_h - \vec{g}))| \leq C \|p\|_{L^\infty(\Omega)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)}$$

Then, estimates

$\|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} \leq C h |\ln h|$ and $\|\nabla \lambda\|_{L^1(\Omega)} \leq C |\ln h|$ give the result.

Localization

Let x_0 be where $\|\vec{u}_h\|_{L^\infty(\Omega_1)}$ is maximal and consider the weight

$$\sigma = \sqrt{|\vec{x} - \vec{x}_0|^2 + \kappa^2 h^2}.$$

Note that $\sigma^{-1} \sim d^{-1}$ for $|\vec{x} - \vec{x}_0| \geq d$ and h small.

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$$|\vec{u}_{h,i}(\vec{x}_0)| = |(\nabla R_h \vec{u}, \nabla(\vec{g}_h - \vec{g})) - (R_h \vec{u}, -\delta_h \vec{e}_i + \nabla \lambda) - (p, \nabla \cdot (\vec{g}_h - \vec{g}))|.$$

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$$|\vec{u}_{h,i}(\vec{x}_0)| = |(\nabla R_h \vec{u}, \nabla(\vec{g}_h - \vec{g})) - (R_h \vec{u}, -\delta_h \vec{e}_i + \nabla \lambda) - (p, \nabla \cdot (\vec{g}_h - \vec{g}))|.$$

We consider only the first term (the others follow similarly)

$$\begin{aligned} |(\nabla R_h \vec{u}, \nabla(\vec{g}_h - \vec{g}))| &\leq \|\nabla R_h \vec{u}\|_{L^\infty(\Omega_2)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} \\ &\quad + \|\sigma^{-3/2} \nabla R_h \vec{u}\|_{L^2(\Omega \setminus \Omega_2)} \|\sigma^{3/2} \nabla(\vec{g}_h - \vec{g})\|_{L^2(\Omega)} \\ &\leq C h^{-1} |\ln h| \|\vec{u}\|_{L^\infty(\Omega_2)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} \\ &\quad + C_d h^{-1} |\ln h| \|\vec{u}\|_{L^2(\Omega \setminus \Omega_2)} \|\sigma^{3/2} \nabla(\vec{g}_h - \vec{g})\|_{L^2(\Omega)}. \end{aligned}$$

Localization

Let x_0 be where $\|\vec{u}_h\|_{L^\infty(\Omega_1)}$ is maximal and consider the weight

$$\sigma = \sqrt{|\vec{x} - \vec{x}_0|^2 + \kappa^2 h^2}.$$

Note that $\sigma^{-1} \sim d^{-1}$ for $|\vec{x} - \vec{x}_0| \geq d$ and h small. Then, as before

$$|\vec{u}_{h,i}(\vec{x}_0)| = |(\nabla R_h \vec{u}, \nabla(\vec{g}_h - \vec{g})) - (R_h \vec{u}, -\delta_h \vec{e}_i + \nabla \lambda) - (p, \nabla \cdot (\vec{g}_h - \vec{g}))|.$$

We consider only the first term (the others follow similarly)

$$\begin{aligned} |(\nabla R_h \vec{u}, \nabla(\vec{g}_h - \vec{g}))| &\leq \|\nabla R_h \vec{u}\|_{L^\infty(\Omega_2)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} \\ &\quad + \|\sigma^{-3/2} \nabla R_h \vec{u}\|_{L^2(\Omega \setminus \Omega_2)} \|\sigma^{3/2} \nabla(\vec{g}_h - \vec{g})\|_{L^2(\Omega)} \\ &\leq C h^{-1} |\ln h| \|\vec{u}\|_{L^\infty(\Omega_2)} \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} \\ &\quad + C_d h^{-1} |\ln h| \|\vec{u}\|_{L^2(\Omega \setminus \Omega_2)} \|\sigma^{3/2} \nabla(\vec{g}_h - \vec{g})\|_{L^2(\Omega)}. \end{aligned}$$

The result follows by the respective estimates for

$$\|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} \quad \text{and} \quad \|\sigma^{3/2} \nabla(\vec{g}_h - \vec{g})\|_{L^2(\Omega)}.$$

A word on the L^1 estimate of $\vec{g}_h - \vec{g}$

To prove the convergence result we use the well-known dyadic decomposition technique, where we split the domain in the following way. Put $d_j = 2^{-j}$ and consider the decomposition

$$\Omega = \Omega_* \cup \bigcup_{j=0}^{J} \Omega_j,$$

where

$$\Omega_* = \{\vec{x} \in \Omega : |\vec{x} - \vec{x}_0| \le Kh\},$$
$$\Omega_j = \{\vec{x} \in \Omega : d_{j+1} \le |\vec{x} - \vec{x}_0| \le d_j\},$$

K is a sufficiently large constant to be chosen later and J is an integer such that

$$2^{-(J+1)} \le Kh \le 2^{-J}.$$

The convergence of $\|\sigma^{3/2}\nabla(\vec{g}_h - \vec{g})\|_{L^2(\Omega)}$ follows then directly from the L^1 estimate.

Thank you very much for your attention and (potential) questions!

Preprint available on `http://www.igdk.eu/IGDK1754/Preprints`

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