On error estimates for solutions of the stationary thin obstacle problem

Darya Apushkinskaya

Saarland University, Saarbrücken

Based on joint work with Sergey Repin

Thin obstacle problem

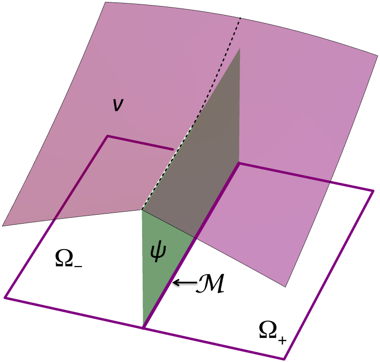


Figure: The thin obstacle problem

Thin obstacle problem

$$J(v) := \frac{1}{2} \int\limits_{\Omega} |\nabla v|^2 \to \min \qquad (\mathcal{P})$$

$$v \in \mathbb{K} := \{v \in H^1(\Omega), \quad v \geqslant \psi \text{ on } \mathcal{M}, \quad v = \varphi \text{ on } \partial\Omega\},$$

$$\Omega \subset \mathbb{R}^n, \quad \partial\Omega \in Lip,$$

$$\varphi : \partial\Omega \to \mathbb{R}, \quad \varphi \in H^{1/2}(\partial\Omega),$$

$\mathcal{M}$ is a smooth $(n-1)$ – dimensional manifold in $\mathbb{R}^n$,
which separates Ω into two Lipschitz subdomains $\Omega_{\pm}$,

$$\psi : \mathcal{M} \to \mathbb{R}, \quad \psi \; - \text{smooth},$$

Remark.

ψ is called "thin obstacle".

Applications:

- Elastic membranes
- Continuum mechanics
- Financial mathematics *(if the random variations of an underlying asset changes discontinuously)*

Historical Review:

- Thin obstacle problems have been actively studied from the early 1970s.
- Regularity of minimizers: (Frehse'75, Frehse'77, Richardson'78, Caffarelli'79, Uraltseva'85, Athanasopoulos & Caffarelli'04, Guillen'09).
- Properties of the free boundaries: (Lewy'72, Athanasopoulos, Caffarelli & Salsa'08, Caffarelli, Salsa & Silvestre'08, Garofalo & Petrosyan'09, Koch, Petrosyan & Shi'15, De Silva & Savin'16).

Known results:

1. $$\Delta u = 0 \quad \text{in} \quad \Omega_\pm,$$

2. $$u\text{, in general, is not a harmonic in } \Omega,$$

3. On $\mathcal{M}$, we have the so-called *complementarity conditions*
$$u - \psi \geqslant 0, \qquad \left[\frac{\partial u}{\partial \mathbf{n}}\right] \geqslant 0, \qquad (u-\psi)\left[\frac{\partial u}{\partial \mathbf{n}}\right] = 0,$$

4. $\mathcal{M}$, ψ - smooth $\quad \Rightarrow \quad u \in C^{1,\alpha}(\Omega_\pm), \ \alpha \in (0, 1/2]$

Our goal:

Let u be a minimizer of $(\mathcal{P})$.

For any $v \in \mathbb{K}$,

$$\|\nabla(u-v)\|_{2,\Omega} \leqslant \mathfrak{M}(v,\ \text{data of } (\mathcal{P}),\ \text{free functions}) \tag{1}$$

- $\mathfrak{M}$ is fully computable
- u and the coincidence set $\{u=\psi\}$ do not enter in $\mathfrak{M}$
- $\mathfrak{M}$ depends continuously on its arguments
- $\mathfrak{M}=0 \quad \Leftrightarrow \quad v=u$
- If $\{u=\psi\} \subset \{v=\psi\}$ then estimate (1) is sharp.

$$\frac{1}{2}\|\nabla(v-u)\|_{2,\Omega}^{2} \leqslant J(v) - J(u), \qquad \forall v \in \mathbb{K}$$

Perturbed problem:

$$J_\lambda(v) := J(v) - \int\limits_{\mathcal{M}} \lambda\,(v - \psi) \to \min \qquad (\mathcal{P}_\lambda)$$

$$v \in \varphi + H_0^1(\Omega) := \{w = \varphi + v : v \in H_0^1(\Omega)\},$$

$$\lambda \in \Lambda := \{\lambda \in L^2(\mathcal{M}) \; : \; \lambda(x) \geqslant 0 \text{ a.e. on } \mathcal{M}\}$$

Remark.

- $(\mathcal{P}_\lambda)$ is uniquely solvable for any $\lambda \in \Lambda$ since $\varphi + H_0^1(\Omega)$ is the affine subspace of $H^1(\Omega)$.

Let u_λ be the minimizer of $(\mathcal{P}_\lambda)$. Then

$$J(u) = \inf_{v\in\mathbb{K}} J(v) = \inf_{v\in\varphi+V_0(\Omega)} \sup_{\lambda\in\Lambda} J_\lambda(v) \geqslant \sup_{\lambda\in\Lambda} \inf_{v\in\varphi+V_0(\Omega)} J_\lambda(v)$$

$$\geqslant J_\lambda(u_\lambda) \quad \forall\lambda \in \Lambda.$$

Dual perturbed problem

$$J_\lambda^*(y^*) = \int_\Omega \left(y^* \cdot \nabla\varphi - \frac{1}{2}|y^*|^2 \right) dx - \int_{\mathcal{M}} \lambda\,(\varphi - \psi) \to \max \qquad (\mathcal{P}_\lambda^*)$$

$$y^* \in L^2(\Omega, \mathbb{R}^n)$$

$$\lambda \in \Lambda := \{\lambda \in L^2(\mathcal{M}) \ : \ \lambda(x) \geqslant 0 \text{ a.e. on } \mathcal{M}\}$$

Consider

$$Q_{\lambda,\mathcal{M}}^* := \left\{ y^* \in L^2(\Omega, \mathbb{R}^n) : \ \operatorname{div} y^* = 0 \quad \text{in} \quad \Omega_\pm, \qquad [y^* \cdot \mathbf{n}] = \lambda \right\}$$

Remark.

$$y^* \notin Q_{\lambda,\mathcal{M}}^* \quad \Rightarrow \quad J_\lambda^*(y^*) = -\infty$$

Dual perturbed problem

$$J^*_\lambda(y^*) = \int\limits_\Omega \left(y^* \cdot \nabla\varphi - \frac{1}{2}|y^*|^2 \right) dx - \int\limits_{\mathcal{M}} \lambda\,(\varphi - \psi) \to \max \qquad (\mathcal{P}^*_\lambda)$$

$$y^* \in Q^*_{\lambda,\mathcal{M}} := \{ y^* \in L^2(\Omega,\mathbb{R}^n) : \int\limits_\Omega y^* \cdot \nabla w dx = \int\limits_{\mathcal{M}} \lambda w d\mu \quad \forall w \in H^1_0(\Omega) \}$$

$$\lambda \in \Lambda := \{ \lambda \in L^2(\mathcal{M}) \ : \ \lambda(x) \geqslant 0 \text{ a.e. on } \mathcal{M} \}$$

Remark.

- $(\mathcal{P}^*_\lambda)$ is uniquely solvable for any $\lambda \in \Lambda$.
- $$J_\lambda(u_\lambda) = \inf_{v \in \varphi + V_0(\Omega)} J_\lambda(v) = \sup_{y^* \in Q^*_{\lambda,\mathcal{M}}} J^*_\lambda(y^*) = J^*_\lambda(y^*_\lambda)$$

Summary 1.

For any $\lambda \in \Lambda$ we have

$$J(u) \geqslant J_\lambda(u_\lambda) = J_\lambda^*(y_\lambda^*).$$

Therefore,

$$\begin{aligned} J(v) - J(u) \leqslant J(v) - J_\lambda^*(y_\lambda^*) &= J(v) - \sup_{y^* \in Q_{\lambda,\mathcal{M}}^*} J_\lambda^*(y^*) \\ &= J(v) + \inf_{y^* \in Q_{\lambda,\mathcal{M}}^*} (-J_\lambda^*(y^*)) = \inf_{y^* \in Q_{\lambda,\mathcal{M}}^*} [J(v) - J_\lambda^*(y^*)] \end{aligned}$$

and, consequently,

$$J(v) - J(u) \leqslant J(v) - J_\lambda^*(y^*) \qquad \forall v \in \mathbb{K},\ \lambda \in \Lambda,\ y^* \in Q_{\lambda,\mathcal{M}}^*.$$

Since $y^* \in Q^*_{\lambda,\mathcal{M}}$ and $v - \varphi \in H^1_0(\Omega)$ for any $v \in \mathbb{K}$, we find that

$$\int\limits_{\Omega} y^* \cdot \nabla\varphi dx = \int\limits_{\Omega} y^* \cdot \nabla v dx - \int\limits_{\Omega} y^* \cdot \nabla(v - \varphi) dx$$
$$= \int\limits_{\Omega} y^* \cdot \nabla v dx - \int\limits_{\mathcal{M}} \lambda(v - \varphi) d\mu.$$

Then

$$J(v) - J^*_\lambda(y^*) = \int\limits_{\Omega} \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{2}|y^*|^2 - y^* \cdot \nabla\varphi \right) dx + \int\limits_{\mathcal{M}} \lambda(\varphi - \psi) d\mu$$
$$= \frac{1}{2} \int\limits_{\Omega} |\nabla v - y^*|^2 dx + \int\limits_{\mathcal{M}} \lambda\,(v - \psi)\, d\mu.$$

First form of the majorant

For any $v \in \mathbb{K}$ we have

$$\|\nabla(v-u)\|^2_{2,\Omega} \leqslant \|\nabla v - y^*\|^2_{2,\Omega} + 2\int\limits_{\mathcal{M}} \lambda\,(v-\psi)\,d\mu,$$

where λ and y^* are arbitrary functions in Λ and $Q^*_{\lambda,\mathcal{M}}$, respectively.

$$\Lambda := \{\lambda \in L^2(\mathcal{M}) \,:\, \lambda(x) \geqslant 0 \text{ a.e. on } \mathcal{M}\}$$

$$Q^*_{\lambda,\mathcal{M}} := \left\{ y^* \in L^2\,(\Omega, \mathbb{R}^n) : \int\limits_{\Omega} y^* \cdot \nabla w dx = \int\limits_{\mathcal{M}} \lambda w d\mu \quad \forall w \in H^1_0\,(\Omega) \right\}$$

Remark:

Assume that

$$\{x \in \mathcal{M} : u(x) = \psi(x)\} \subset \{x \in \mathcal{M} : v(x) = \psi(x)\}.$$

In this case, the estimate

$$\|\nabla(v-u)\|_{2,\Omega}^2 \leqslant \|\nabla v - y^*\|_{2,\Omega}^2 + 2\int\limits_{\mathcal{M}} \lambda\,(v-\psi)\,d\mu \qquad (2)$$

is sharp in the sense that $\exists\ y^* = \nabla u$ and $\lambda = [\nabla u \cdot \mathbf{n}]$ such that the inequality holds in (2) as the equality.

Extension of the set of functions

$$H(\Omega_\pm, \mathrm{div}) := \{q^* \in L^2(\Omega, \mathbb{R}^n) : \mathrm{div}\,(q^*|_{\Omega_\pm}) \in L^2(\Omega_\pm), \quad [q^*\cdot\mathbf{n}] \in L^2(\mathcal{M})\}$$

Let $q^* \in H(\Omega_\pm, \mathrm{div})$ and $\lambda \in \Lambda$. For any $v \in \mathbb{K}$ and $y^* \in Q^*_{\lambda,\mathcal{M}}$ we have

$$\|\nabla v - y^*\|_{2,\Omega} \leqslant \|\nabla v - q^*\|_{2,\Omega} + \|q^* - y^*\|_{2,\Omega}$$

Advanced form of the majorant - 1

Let $q^* \in H(\Omega_\pm, \text{div})$ and $\lambda \in \Lambda$. For any $y^* \in Q^*_{\lambda,\mathcal{M}}$ we have

$$\|q^* - y^*\|_{2,\Omega} \leqslant C_{F_{\Omega_+}} \|\mathrm{div}\, q^*\|_{2,\Omega_+} + C_{F_{\Omega_-}} \|\mathrm{div}\, q^*\|_{2,\Omega_-} + C_{Tr_{\mathcal{M}}} \|\lambda - [q^* \cdot \mathbf{n}]\|_{2,\mathcal{M}}.$$

Advanced form of the majorant - 2

Let $q^* \in H(\Omega_\pm, \mathrm{div})$ and $\lambda \in \Lambda$ satisfy the additional restrictions

$$\int\limits_{\Omega_+} \operatorname{div} q^* dx = \int\limits_{\Omega_-} \operatorname{div} q^* dx = 0 \qquad \text{and} \qquad \int\limits_{\mathcal{M}} (\lambda - [q^* \cdot \mathbf{n}]) d\mu = 0.$$

Then, for any $\alpha \in [0, 1]$, we have

$$\|q^* - y^*\|^2_{2,\Omega} \leqslant (\mathfrak{D}_-(q^*) + \alpha \mathfrak{m}_-(q^*))^2 + (\mathfrak{D}_+(q^*) + (1-\alpha)\mathfrak{m}_+(q^*))^2,$$

where
$\mathfrak{D}_\pm(q^*) := C_{P_{\Omega_\pm}} \|\operatorname{div} q^*\|_{2,\Omega_\pm}$ and $\mathfrak{m}_\pm(q^*) = C_{P_{\mathcal{M}}}(\Omega_\pm)\|\lambda - [q^* \cdot \mathbf{n}]\|_{2,\mathcal{M}}$.

Remark

α can be defined in the optimal way.

Advanced form of the majorant - 3

Let $q^* \in H(\Omega_\pm, \mathrm{div})$ and $\lambda \in \Lambda$ satisfy only the restriction

$$\int\limits_{\mathcal{M}} (\lambda - [q^* \cdot \mathbf{n}]) d\mu = 0. \tag{3}$$

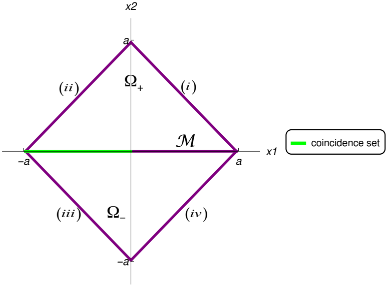
Then, for any $\alpha \in [0, 1]$, we have

$$\|q^* - y^*\|^2_{2,\Omega} \leqslant (\mathfrak{D}^F_-(q^*) + \alpha\mathfrak{m}_-(q^*))^2 + (\mathfrak{D}^F_+(q^*) + (1-\alpha)\mathfrak{m}_+(q^*))^2,$$

where
$\mathfrak{D}^F_\pm(q^*) := C_{F_{\Omega_\pm}} \|\mathrm{div}\, q^*\|_{2,\Omega_\pm}$ and $\mathfrak{m}_\pm(q^*) = C_{P_{\mathcal{M}}}(\Omega_\pm)\|\lambda - [q^* \cdot \mathbf{n}]\|_{2,\mathcal{M}}$.

Example: Ω_+ and Ω_-

Let $\Omega_\pm$ be the triangles in $\mathbb{R}^2$ defined as in figure below, let $\mathcal{M} := \{x_2 = 0\}$, and let $\psi \equiv 0$.



$$\frac{1}{C_{F_{\Omega_\pm}}} = \inf_{w \in V_0^\pm(\Omega_\pm)} \frac{\|\nabla w\|_{2,\Omega_\pm}}{\|w\|_{2,\Omega_\pm}} \quad \Longrightarrow \quad C_{F_{\Omega_+}} = C_{F_{\Omega_-}} = \frac{a}{\pi}.$$

Example: exact solution u

$$u(x_1, x_2) = \mathrm{Re}\left((x_1 + i|x_2|)^{3/2}\right) \quad \text{is the explicit solution in } \mathbb{R}^2$$

Setting the boundary condition φ on $\partial\Omega$ as the trace of $\mathrm{Re}\left((x_1 + i|x_2|)^{3/2}\right)$, we see that u is the exact solution in the bounded domain Ω as well.

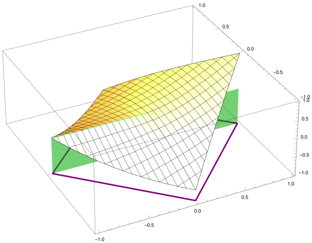


Figure: The exact solution u in Ω

Example: properties of u

1.
$$\Delta u = 0 \quad \text{in} \quad \Omega_\pm,$$

2.
$$u(x_1, 0) = \begin{cases} 0, & \text{if } x_1 \leqslant 0, \\ x_1^{3/2}, & \text{if } x_1 > 0 \end{cases}$$

3.
$$\left[\frac{\partial u}{\partial \mathbf{n}}\right] = \begin{cases} 3\sqrt{-x_1}, & \text{if } x_1 < 0, \\ 0, & \text{if } x_1 \geqslant 0. \end{cases}$$

Example.

$$v_1(x_1, x_2) := u(x_1, x_2) + \begin{cases} x_2^2(x_2 - x_1 - a)(x_2 + x_1 - a), & \text{if } x_2 \geqslant 0, \\ x_2^2(x_2 - x_1 + a)(x_2 + x_1 + a), & \text{if } x_2 < 0. \end{cases}$$

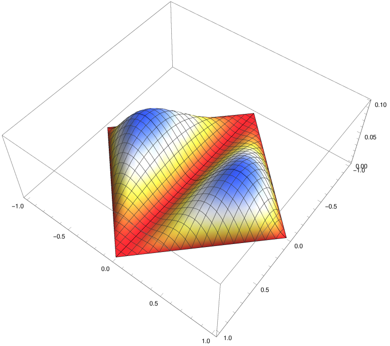


Figure: $v_1 - u$

Example.

$$\|\nabla(v_1 - u)\|_{2,\Omega} \leqslant \mathfrak{M}_1(v_1, q^*, \lambda, \psi),$$

$$\mathfrak{M}_1(v_1, q^*, \lambda, \psi) := \|\nabla v_1 - q^*\|_{2,\Omega} + \sqrt{2}\left(\int_{\mathcal{M}} \lambda(v_1 - \psi) d\mu\right)^{1/2}$$

$$+ C_{F_{\Omega_+}} \|\mathrm{div}\, q^*\|_{2,\Omega_+} + C_{F_{\Omega_-}} \|\mathrm{div}\, q^*\|_{2,\Omega_-} + C_{Tr_{\mathcal{M}}} \|\lambda - [q^* \cdot \mathbf{n}]\|_{2,\mathcal{M}},$$

Choose

$$q^* := \nabla v_1, \quad \lambda := \left[\frac{\partial v_1}{\partial \mathbf{n}}\right] \quad \Rightarrow \quad 1 \leqslant \frac{\mathfrak{M}_1\left(v_1, \nabla v_1, \left[\frac{\partial v_1}{\partial \mathbf{n}}\right], 0\right)}{\|\nabla(v_1 - u)\|_{2,\Omega}} \approx 2.382.$$

Example.

Observe also that for $q^* = \nabla v_1$ and $\lambda = \left[\frac{\partial v_1}{\partial \mathbf{n}}\right]$ the assumption (3) is satisfied. Thus, for any $\alpha \in [0,1]$ we can compute

$$\|\nabla(v_1 - u)\|_{2,\Omega} \leqslant \mathfrak{M}_3(v_1, \nabla v_1, \alpha, \left[\frac{\partial v_1}{\partial \mathbf{n}}\right], 0),$$

$$\mathfrak{M}_3(v_1, \nabla v_1, \alpha, \left[\frac{\partial v_1}{\partial \mathbf{n}}\right], 0) := \|\nabla v_1 - q^*\|_{2,\Omega} + \sqrt{2}\left(\int\limits_{\mathcal{M}} \lambda(v_1 - \psi)\,d\mu\right)^{1/2} + \left[C^2_{F_{\Omega_+}}\|\mathrm{div}\, q^*\|^2_{2,\Omega_+} + C^2_{F_{\Omega_-}}\|\mathrm{div}\, q^*\|^2_{2,\Omega_-}\right]^{1/2}.$$

$$1 \leqslant \frac{\mathfrak{M}_3\left(v_1, \nabla v_1, \alpha, \left[\frac{\partial v_1}{\partial \mathbf{n}}\right], 0\right)}{\|\nabla(v_1 - u)\|_{2,\Omega}} \approx 1.684.$$

Conclusions:

- In the above examples, rather simple functions q^* and λ provide quite realistic bounds of the error.
- In more complicated examples, so defined q^* and λ may be considered as a starting points for the iteration process of majorant minimization that generates a monotonically decreasing sequence of numbers, which are guaranteed upper bounds of the error.

References:

D.E. Apushkinskaya and S.I. Repin
Thin obstacle problem: estimate of the distance to the exact solution
Interfaces and Free Boundaries, vol. **20** (2018), No. 4, 483–595

The Last Slide

Thank You!