The Navier-Stokes equations – A never ending challenge?

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We consider the nonstationary nonlinear three-dimensional Navier-Stokes equations

$$\begin{aligned} v_t - \nu\Delta v + v\cdot\nabla v + \nabla p &= f, \qquad \nabla\cdot v = 0, \\ v|_{\partial\Omega} &= 0, \qquad v|_{t=0} = v_0. \end{aligned}$$

These equations describe the motion of a viscous incompressible fluid flow in $(0,T)\times\Omega$: The vector function $v = v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ denotes the velocity and the scalar function $p = p(t,x)$ the pressure of the fluid at time $t > 0$ in $x = (x_1, x_2, x_3) \in \Omega$. Here the constant $\nu > 0$ represents the kinematic viscosity, the vector function $f = (f_1(t,x), f_2(t,x), f_3(t,x))$ is the given external force density, and the steady vector function $v_0 = (v_{0_1}(x), v_{0_2}(x), v_{0_3}(x))$ denotes the prescribed initial velocity at time $t = 0$. In the following we consider the fluid flow always in a bounded domain $\Omega \subset \mathbf{R}^3$ with smooth boundary $\partial\Omega$ of class $C^{2,\mu}$ $(0 < \mu \leq 1)$.

The above system occupies a central position in the study of nonlinear partial differential equations, dynamical systems, scientific computation, and classical fluid dynamics. Because of the complexity and variety of fluid dynamical phenomena on the one hand, and the simplicity and exactitude of the equations' shape on the other hand, a strong depth and beauty is expected in the mathematical theory. It is a source of pleasure and fascination that many of the most important questions in the theory remain yet to be answered. So the famous American Clay Mathematics Institute created the Navier-Stokes Millennium Price Problem and offered one Million Dollar for its solution, stating: Although the Navier-Stokes equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory, which will unlock the secrets hidden in the Navier-Stokes equations.

The modern mathematical theory of the Navier-Stokes equations started with the pioneering work of Jean Leray in 1933 - 34. Leray was the first to use methods of functional analysis for the treatment of partial differential equations. He developed the concept of weak solutions for the Navier-Stokes Cauchy problem and proved their existence globally in time long before the theory of distributions was established by Schwartz and even before Sobolev systematically introduced the spaces which bear his name. Leray has laid the basis of the mathematical theory for the Navier-Stokes equations as we know it today, and he has introduced many tools and ideas used constantly since then.

The lecture introduces the Navier-Stokes equations from a historical and physical point of view, touches some fundamental mathematical problems of viscous incompressible fluid flow and ends up with recent regularity results on weak and strong solutions in Besov spaces.