A posteriori estimates for incompressible media problems

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Mathematical models of viscous fluids

$(x, t) \in Q := \Omega \times (0, T)$, $f \in L^2(\Omega)$. Find u, p, σ such that

Equation of motion $\quad \rho \frac{\partial u}{\partial t} + u_i \frac{\partial u}{\partial x_i} - \mathrm{div}\sigma = f$

Incompressibility $\quad \mathrm{div}\, u = 0$

Constitutive law $\quad \sigma = -p\mathbb{I} + \tau, \quad \tau \in \partial W(\varepsilon)$

Initial and boundary conditions

$$u(x,t) = 0 \qquad (x,t) \in \partial_1\Omega \times (0, +\infty),$$
$$\sigma\nu = F \qquad (x,t) \in \partial_2\Omega \times (0, +\infty),$$
$$u(x,0) = \varphi(x) \qquad x \in \Omega.$$

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- **How to get a fully computable a posteriori estimate for Stokes, Oseen, and Navier–Stokes problems?**
- **Variational (Raleigh-Ritz type) principles for computing the Inf-sup constant.**

Proofs and details can be found in:

- **S. R.** *Estimates of the distance to the set of divergence free fields*, **Zapiski. Nauchn. Semin. Steklov Inst. (POMI), 2014**
- **S. R.** *On variational representations of the constant in the Inf-Sup condition for the Stokes problem*, **J. Math. Sci., 2016**
- **S. R.** *Estimates of the distance to the set of solenoidal vector fields and applications to a posteriori error control*, **Comput. Math. Appl. Math., 2015**

Steady state problems. Generalized solution $u \in S_0(\Omega)$

$$\int_\Omega \pi(u) : \nabla w \, dx = \int_\Omega f \cdot w \, dx \qquad \forall w \in S_0(\Omega), \qquad (1)$$

where $\pi(u) := \sigma_D(u) - \varkappa(u)$, $\sigma_D(u) = \nu \nabla u$.
Stokes problem: $\varkappa(u) = 0$.
Oseen problem: $\varkappa(u) = \mathbf{a} \otimes u$.
Navier–Stokes problem: $\varkappa(u) = u \otimes u$.

Approximation $v \in S_0(\Omega)$, generates the functional

$$\mathcal{L}_v(w) := \int_\Omega \left(f \cdot w - \pi(v) : \nabla w\right) dx, \qquad w \in S_0(\Omega)$$

which contains all available (really computable) information.

Therefore, the quantity

$$|\mathcal{L}_v| := \sup_{w \in S_0} \frac{\mathcal{L}_v(w)}{\|w\|_{S_0}}$$

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The question we need to discuss now is **"what we could really obtain from the knowledge of $|\mathcal{L}_v|$?"**

It is easy to see that

$$|\mathcal{L}_v| \;=\; \mu_\pi(v, u),$$

$$\mu_\pi(v, u) := \sup_{w \in S_0} \frac{\int_\Omega (\pi(u) - \pi(v)) : \nabla w \; dx}{\|w\|_{S_0}}$$

It is easy to see that

$$|\mathcal{L}_{\boldsymbol{v}}| = \boldsymbol{\mu}_\pi(\boldsymbol{v}, \boldsymbol{u}),$$

$$\boldsymbol{\mu}_\pi(\boldsymbol{v}, \boldsymbol{u}) := \sup_{\boldsymbol{w} \in S_0} \frac{\int_\Omega (\pi(\boldsymbol{u}) - \pi(\boldsymbol{v})) : \nabla \boldsymbol{w} \, d\boldsymbol{x}}{\|\boldsymbol{w}\|_{S_0}}$$

It is easy to see that the $\boldsymbol{\mu}_\pi(v, u)$ is nonnegative and symmetric. Also, it satisfies the triangle inequality

$$\boldsymbol{\mu}_\pi(u, v) \leq \boldsymbol{\mu}_\pi(u, w) + \boldsymbol{\mu}_\pi(v, w).$$

$\boldsymbol{\mu}_\pi(v, u)$ is a certain measure (pseudometric) of the distance between v and u.

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For the Stokes problem,

$$\boldsymbol{\mu_\pi(u, v) = \nu \|\nabla(u - v)\|}$$

For the Oseen problem, we have

$$\int_\Omega (\mathbf{a} \otimes w) : \nabla w \, dx - \int_\Omega \mathrm{Div}(\mathbf{a} \otimes w) \cdot w \, dx = - \int_\Omega (\mathbf{a} \cdot \nabla w) \cdot w \, dx$$
$$= - \int_\Omega \mathbf{a} \cdot (\nabla w \cdot w) \, dx = -\frac{1}{2} \int_\Omega \mathbf{a} \cdot \nabla(|w|^2) \, dx = 0.$$

Therefore,

$$\sup_{w \in S_0^{1,2}(\Omega, \mathbb{R}^d)} \frac{\int_\Omega (\nu \nabla (u - v) : \nabla w - (\mathbf{a} \otimes (u - v)) : \nabla w) \, dx}{\|\nabla w\|} \geq \nu \|\nabla (u - v)\|$$

$$\boxed{\mu_\pi(u, v) \geq \nu \|\nabla(u - v)\|}$$

For the NS problem we can only prove that
$\boldsymbol{\mu}_\pi(v,u) \geq c\|\nabla(v-u)\|$, $c>0$
provided that ∇v is sufficiently small and the bound of this "smallness" depends on ν.

In general, $\boldsymbol{\mu}_\pi$ generated by NS equation is not a metric.

Comment: computational verification of non–uniqueness

"Whether or not there exists a unique flow u starting with the initial velocity u_0 and smoothly evolving in time from zero to 1?"

Millennium Problem

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Assume that we have

(A) a computable functional $M_+(v)$ such that

$$\boldsymbol{\mu}_\pi(u, v) \leq M_+(v),$$

and $M_+(u) = 0$ for any generalized solution.

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and $M_+(u) = 0$ for any generalized solution.

(B) we have found two very accurate approximations v_1 and v_2 satisfying

$$M_+(v_i) \leq \epsilon, \quad i = 1,2$$

In principle, this can always be achieved with the help of sufficiently powerful computers.

It seems that we have straightforward way:

We wish to verify that v_1 and v_2 approximate two different solutions u_1 and u_2 by showing that

$$\underbrace{\boldsymbol{\mu}_\pi(v_1, v_2)}_{\textbf{computable}} - \underbrace{\boldsymbol{\mu}_\pi(v_1, u_1)}_{\textit{error } v_1} - \underbrace{\boldsymbol{\mu}_\pi(v_2, u_2)}_{\textit{error } v_2} \geq \boldsymbol{\mu}_\pi(v_1, v_2) - 2\epsilon > 0.$$

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However, regardless of the computational efforts focused on computations of v_1 and v_2 and accuracy verification via M_+, the required (positive) result will never be achieved!

Indeed, if the problem is uniquely solvable, then a numerical method is unable to establish the opposite.

On the other hand, if the problem indeed possesses two different solutions, then

$$\int_\Omega (\pi(u_1) - \pi(u_2)) : \nabla w \; dx = 0 \qquad \forall w \in S_0.$$

Hence

$$\boldsymbol{\mu}_\pi(u_1, u_2) = 0$$

and the left hand side will be always nonpositive.

$$\boldsymbol{\mu}_\pi(v_1, v_2) \le \boldsymbol{\mu}_\pi(v_1, u_1) + \boldsymbol{\mu}_\pi(u_1, u_2) + \boldsymbol{\mu}_\pi(v_2, u_2)$$

A pessimistic conclusion: the question of uniqueness or non–uniqueness cannot be verified numerically if we are based only on analysis of numerical (e.g., Galerkin) approximations to generalized solutions and does not attract more sophisticated arguments.

Computable bounds of $\mathcal{L}_v$. Steady state equations

Key idea (earlier used in many other problems...):

The functional $\mathcal{L}_v$ should be split and transformed using suitable integration by parts relations.

Let $Y := \nabla S_0(\Omega)$ (i.e., Y contains tensor valued functions, which are gradients of all the functions in S_0) and $\mathcal{V}$ and $\mathcal{V}^*$ be another pair of mutually conjugate Banach spaces such that $V_0 \in \mathcal{V}$, $\mathcal{V}^* \subset S_0^*$

$$\|w\|_{\mathcal{V}} \leq C_F(\Omega)\|w\|_{S_0} \qquad \forall w \in S_0(\Omega), \tag{2}$$

(In our case norm of S_0 is $\|\nabla w\|$.)

$$\mathcal{L}_v(w) = \int_\Omega (f + \mathrm{Div}\tau) \cdot w\, dx + \int_\Omega (\tau - \pi(v) + q\mathbb{I}) : \nabla w\, dx. \quad (3)$$

Here

$$\tau \in H_{\mathrm{Div}}(\Omega) := \{\tau \in Y^*, \mid \mathrm{Div}\tau \in \mathcal{V}^*\}, \quad (4)$$

and q is a scalar function such that $q\mathbb{I} \in Y^*$.
Hence, we find that for $v \in S_0(\Omega)$, $\tau \in H_{\mathrm{Div}}$, and q:

$$|\mathcal{L}_v| \leq C_F(\Omega)\|f + \mathrm{Div}\tau\|_{\mathcal{V}^*} + \|\tau - \pi(v) + q\mathbb{I}\|_{Y^*}, \quad (5)$$

Extension to $v \in V_0$

$$\mu_\pi(v,u) \leq \mu_\pi(v_0,u) + \mu_\pi(v,v_0) \qquad \forall v_0 \in S_0(\Omega)$$

and

$$\mu_\pi(v,v_0) := \sup_{v_0 \in S_0} \frac{\int\limits_\Omega (\pi(v_0) - \pi(v)) : \nabla v_0 \, dx}{\|v_0\|_{S_0}} \leq \|\pi(v) - \pi(v_0)\|_{Y^*},$$

we find that

$$\mu_\pi(v,u) \leq C_F(\Omega)\|f + \mathrm{Div}\tau\|_{V^*} + \|\tau - \pi(v) + q\mathbb{I}\|_{Y^*} + 2 \inf_{v_0 \in S_0(\Omega)} \|\pi(v) - \pi(v_0)\|_{Y^*}. \quad (6)$$

For problems with Newtonian type potentials this problem can be reduced to

$$\inf_{v_0 \in S_0(\Omega)} \|\nabla(v - v_0)\|_{L^2(\Omega,\mathbb{M}^{d\times d})} =: \Pi_{S_0^{1,2}}(v)$$

Navier–Stokes problem: $\pi(v) = \nu \nabla v - v \otimes v$

Let $v_0 \in S_0^{1,2}(\Omega, \mathbb{R}^d)$.

$$\|\pi(v) - \pi(v_0)\| \leq \nu \|\nabla(v - v_0)\| + \|v \otimes v - v_0 \otimes v_0\|$$

$v \otimes v - v_0 \otimes v_0 = (v - v_0) \otimes v + v \otimes (v - v_0) - (v - v_0) \otimes (v - v_0)$.
Hence

$$\|\pi(v) - \pi(v_0)\| \leq \nu \|\nabla(v_0 - v)\| + 2\|v\|_{4,\Omega} \|v_0 - v\|_{4,\Omega} + \|v_0 - v\|^2_{4,\Omega}.$$

Due to embedding of $W^{1,2}$ to L^4, we have the estimate

$$\|v_0 - v\|_{4,\Omega} \leq \gamma(\Omega) \|\nabla(v_0 - v)\|$$

We have a majorant of the distance to the set of divergence free fields in terms of $\boldsymbol{\mu}_\pi$ generated by NS problem:

$$\inf_{v_0 \in S_0^{1,2}(\Omega,\mathbb{R}^d)} \|\pi(v) - \pi(v_0)\| \leq \Pi_{S_0^{1,2}}(v)\left(\nu + 2\gamma(\Omega)\|v\|^2_{4,\Omega} + \gamma^2(\Omega)\Pi_{S_0^{1,2}}(v)\right). \quad (7)$$

We arrive at the following result:

Theorem (Distance to a Hopf's solution $\boldsymbol{u}$)

For $\boldsymbol{v} \in \boldsymbol{V}_0$, we have the estimate

$$\mu_\pi(\boldsymbol{v}, \boldsymbol{u}) \leq C_{F\Omega} \|\boldsymbol{f} + \mathrm{Div}\boldsymbol{\tau}\| +$$

$$+ \|\boldsymbol{\tau} - \nu \nabla \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{v} + q\mathbb{I}\|$$

$$+ \Pi_{S_0^{1,2}}(\boldsymbol{v}) \left(\nu + 2\gamma(\Omega) \|\boldsymbol{v}\|^2_{4,\Omega} + \gamma^2(\Omega) \Pi_{S_0^{1,2}}(\boldsymbol{v}) \right).$$

The right hand side contains only known functions!

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The right hand side contains only known functions!

It remains to find $\Pi_{S_0^{1,2}}(v)$!

Stability Theorem/Lemma [Aziz-Babuska, Ladyzhenskaya–Solonnikov, Nečas]

Theorem

For any $f \in L^2(\Omega)$ *such that* $\{f\}_\Omega = 0$, *there exists a function* $w_f \in W_0^{1,2}(\Omega, \mathbb{R}^d)$ *such that*

$$\operatorname{div} w_f = f \quad \text{and} \quad \|\nabla w_f\| \leq \kappa_\Omega \|f\|, \tag{8}$$

where κ_Ω *is a positive constant depending on* Ω.

$$\inf_{\substack{p \in \widetilde{L}^2(\Omega) \\ \{p\}_\Omega = 0,\ p \neq 0}} \ \sup_{\substack{w \in V_0 \\ w \neq 0}} \frac{\int_\Omega p \operatorname{div} w \, dx}{\|p\| \, \|\nabla w\|} \geq c_\Omega. \tag{9}$$

Lemma (Distance to $S_0^{1,q}(\Omega, \mathbb{R}^d)$)

For any $v \in W_0^{1,q}(\Omega, \mathbb{R}^d)$,

$$\mathbf{d}(v, S_0^{1,q}(\Omega, \mathbb{R}^d)) \le \kappa_{\Omega,q} \|\mathrm{div}\, v\|_{q,\Omega}. \tag{10}$$

Lemma (Estimate based on decomposition)

Assume $v \in W^{1,q}(\Omega, \mathbb{R}^d)$ *satisfies*

$$\{\mathrm{div}\, v\}_{\Omega_i} = 0 \qquad i = 1, 2, ..., N,$$

and $\mathrm{div}\, v \in L^\delta(\Omega)$, *where* $\delta \ge q$. *Then, there exists* $v_0 \in W^{1,\gamma}(\Omega, \mathbb{R}^d)$ *such that* $\mathrm{div}\, v_0 = 0$, $v_0 = v$ *on* Γ, *and*

$$\|\nabla(v - v_0)\|_{\Omega,q} \le \sum_{i=1}^{N} \kappa_{\Omega_i,q} |\Omega_i|^{\frac{1}{q} - \frac{1}{\delta}} \|\mathrm{div}\, v\|_{\Omega_i,q}. \tag{11}$$

$\kappa_{\Omega_i,q}$ (or c_{Ω_i}) are required

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$$\mathbf{c_\Omega = \inf_{\phi \in \widetilde{L}^2(\Omega)} \frac{|\!|\!| \phi |\!|\!|}{\|\phi\|} = \inf_{\substack{\phi \in \widetilde{L}^2(\Omega) \\ \|\phi\|=1}} |\!|\!| \phi |\!|\!|,}$$

where

$$|\!|\!| \phi |\!|\!| := \sup_{w \in V_0} \frac{\int_\Omega \phi \operatorname{div} w \, dx}{\|\nabla w\|} = \|\nabla u_\phi\|,$$

where

$$\int_\Omega (\nabla u_\phi : \nabla w + \phi \operatorname{div} w) \, dx = 0 \qquad \forall w \in V_0.$$

A variational principle for c_Ω

Lemma

$$c_\Omega = \inf_{\substack{\phi \in \widetilde{L}^2(\Omega) \\ \|\phi\|=1}} \inf_{\tau_0 \in \mathbb{S}} \|\tau_0 - \phi \mathbb{I}\|, \tag{12}$$

where $\mathbb{S} := \left\{ \tau_0 \in U \mid \int_\Omega \tau_0 : \nabla w \, dx = 0 \quad \forall w \in V_0 \right\}.$

1. We apply duality arguments in order to estimate $\|\nabla u_\phi\|$, which is the minimizer of the functional

$$J_\phi(w) := \int_\Omega \left(\frac{1}{2}\|\nabla w\|^2 + \phi \operatorname{div} w\right) dx \quad \text{and} \quad J_\phi(u_\phi) = -\frac{1}{2}\|\nabla u_\phi\|^2.$$

Notice that $J_\phi(w) = \sup_{\tau \in U := L^2(\Omega, \mathbb{M}^{d\times d})} L_\phi(w,\tau)$,

$$L_\phi(w,\tau) := \int_\Omega (-\frac{1}{2}|\tau|^2 + \tau : \nabla w + \phi \mathrm{div} w)\, dx.$$

2. We have

$$-\frac{1}{2}\|\nabla u_\phi\|^2 = \inf_{v\in V_0} \sup_{\tau\in U} L_\phi(w,\tau) \geq \sup_{\tau \in U} \inf_{v\in V_0} L_\phi(w,\tau). \tag{13}$$

Since

$$\inf_{w\in V_0} L_\phi(w,\tau) = -\frac{1}{2}\|\tau\|^2 + \inf_{w\in V_0} \int_\Omega (\phi\mathbb{I} + \tau) : \nabla w\, dx,$$

we see that infimum is finite if and only if

$$\tau + \phi\mathbb{I} \in \mathbb{S} := \left\{ \tau_0 \in U \mid \int_\Omega \tau_0 : \nabla w\, dx = 0 \quad \forall w \in V_0 \right\},$$

i.e, $\mathrm{Div}\tau_0 = 0$ in a generalized form.

Hence τ must have the form $\tau = \tau_0 - \phi \mathbb{I}$, and

$$\inf_{w \in V_0} L_\phi(w, \tau) = -\frac{1}{2} \|\tau_0 - \phi \mathbb{I}\|^2.$$

Then

$$J(u_\phi) = \inf_{w \in V_0} J(w) = -\frac{1}{2} \|\nabla u_\phi\|^2 \geq \sup_{\tau_0 \in \mathbb{S}} -\frac{1}{2} \|\tau_0 - \phi \mathbb{I}\|^2$$

$$\|\nabla u_\phi\|^2 \leq -\sup_{\tau_0 \in \mathbb{S}} \{-\|\tau_0 - \phi \mathbb{I}\|^2\} = \inf_{\tau_0 \in \mathbb{S}} \|\tau_0 - \phi \mathbb{I}\|^2 \qquad (14)$$

3. To prove the opposite, we set $\tau_0 = \nabla u_\phi + \phi \mathbb{I}$. For any $w \in V_0$,

$$\int_\Omega \tau_0 : \nabla w \, dx = \int_\Omega (\nabla u_\phi : \nabla w + \phi \mathrm{div} w) \, dx = 0,$$

and, therefore, $\tau_0 \in \mathbb{S}$. We conclude that

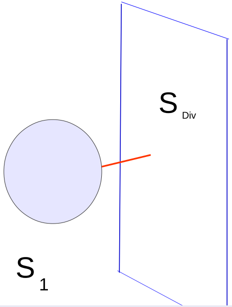
$$\inf_{\tau_0 \in \mathbb{S}} \|\tau_0 - \phi \mathbb{I}\| \leq \|\nabla u_\phi\|. \tag{15}$$

From (14) and (15), it follows that

$$|\!|\!| \phi |\!|\!| = \inf_{\tau_0 \in \mathbb{S}} \|\tau_0 - \phi \mathbb{I}\|$$

We see that

$$c_\Omega = \inf_{\substack{\phi \in \widetilde{L}^2(\Omega) \\ \|\phi\|=1}} \inf_{\tau_0 \in \mathbb{S}} \|\tau_0 - \phi \mathbb{I}\|. \tag{16}$$



c_Ω is the distance between two sets of tensor functions:
S_1 contains spheric tensors $q\mathbb{I}$, $\|q\| = 1$, $q \in \widetilde{L}^2$.
S_{Div} contains tensor functions τ such that $\mathrm{Div}\tau = 0$

Comment:

1. Notice that the intersection of these two sets is empty. Assume that there exists ϕ with zero mean such that $\|\phi\| = 1$ and

$$\int_{\Omega} \phi \mathbb{I} : \nabla w \, dx = 0 \quad \forall w \in V_0.$$

Then

$$\int_{\Omega} \phi \mathrm{div} w \, dx = 0.$$

For ϕ we can find $w_\phi \in V_0$ such that $\mathrm{div} w_\phi = \phi$ and therefore $\|\phi\| = 0$. We arrive at a contradiction.

2. It is easy to see that $c_\Omega \leq 1$. Set $\tau_{ij} = 0$ for $i \neq j$ $\tau_{11} = 0$, $\tau_{jj} = \phi$, $\phi = \phi(x_1)$.

Other forms of the variational principle

I. We can narrower the set $\mathbb{S}$

$$\boldsymbol{c_\Omega^2 = \inf_{\substack{\phi \in \widetilde{L}^2(\Omega) \\ \|\phi\|=1}} \inf_{\tau_0 \in \mathbb{S}^+ \cap \widetilde{\mathbb{S}}} \left\{ \|\tau_0^D\|^2 + d \| \frac{1}{d} \mathrm{Sp}\tau_0 - \phi \|^2 \right\}},$$

where $\widetilde{\mathbb{S}} := \{\tau_0 \in \mathbb{S}, \mid \{\mathrm{Sp}\tau_0\}_\Omega = 0\}$,

$$\mathbb{S}^+ := \left\{ \tau_0 \in \mathbb{S}, \mid \|\mathrm{Sp}\tau_0\| \leq d, \ \|\tau_0^D\| \leq \frac{\sqrt{d}}{2} \right\}.$$

II. We can exclude the function ϕ

$$c_\Omega^2 = \inf_{\tau_0 \in \mathbb{S}} \left\{ \|\tau_0^D\|^2 + \frac{1}{d}(\|\mathrm{Sp}\,\tau_0\| - d)^2 \right\}.$$

II. We can exclude the function ϕ

$$c_\Omega^2 = \inf_{\tau_0 \in \mathbb{S}} \left\{ \|\tau_0^D\|^2 + \frac{1}{d}(\|\mathrm{Sp}\,\tau_0\| - d)^2 \right\}.$$

or

$$c_\Omega^2 = \inf_{\tau_0 \in \mathbb{S}^+ \cap \widetilde{\mathbb{S}}} \left\{ \|\tau_0^D\|^2 + \frac{1}{d}(\|\mathrm{Sp}\,\tau_0\| - d)^2 \right\}.$$

III. $\tau_0 \in \mathbb{S}$ can be replaced by $\tau \in Q := H(\Omega, \mathrm{Div})$.

$$\inf_{\tau_0 \in \mathbb{S}} \|\tau_0 - \tau\| \leq C_F \|\mathrm{Div}\tau\|.$$

Therefore, we have

$$c_\Omega^2 = \inf_{\substack{\phi \in \widetilde{L}^2(\Omega) \\ \|\phi\|=1}} \inf_{\substack{\tau \in Q \\ \alpha > 0}} \left\{ \alpha \left(\|\tau^D\|^2 + d \left\| \frac{1}{d} \mathrm{Sp}\tau - \phi \right\|^2 \right) + \frac{\alpha - 1}{\alpha} C_F^2(\Omega) \|\mathrm{Div}\tau\|^2 \right\}$$

Thanks for attention