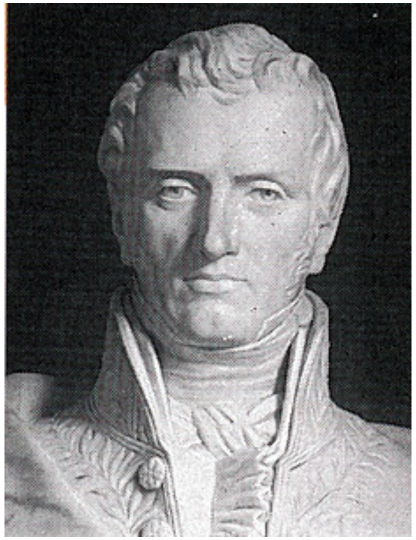
The Navier-Stokes equations –



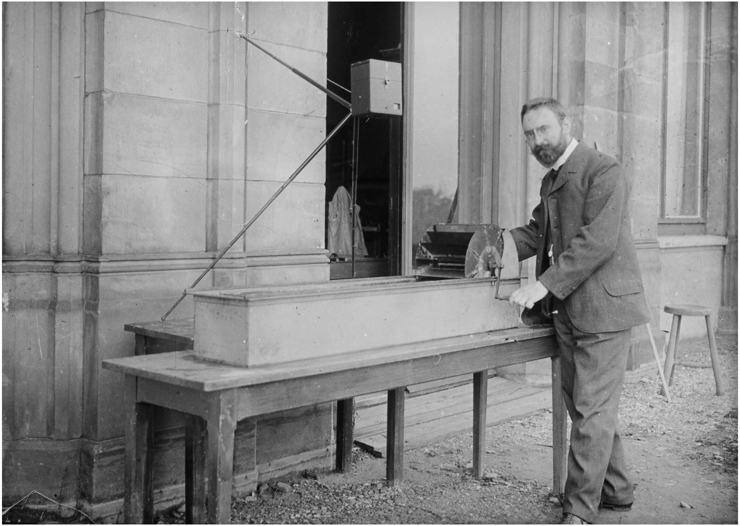
A never ending challenge?

Werner Varnhorn

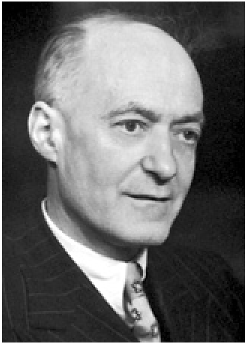
Institute of Mathematics – Kassel University – Germany

varnhorn@mathematik.uni-kassel.de

Workshop AANMPDE 10
Paleochora, Crete
October 5, 2017

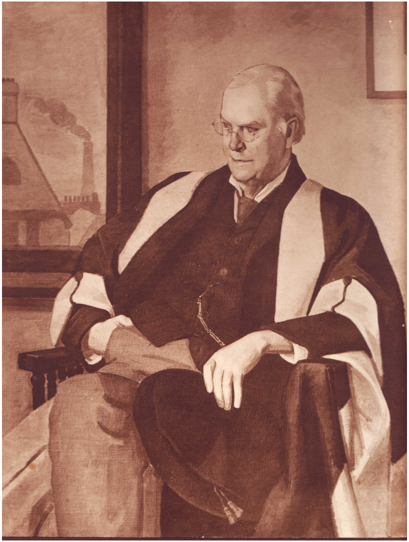


Ludwig Prandtl
(1875 – 1953)

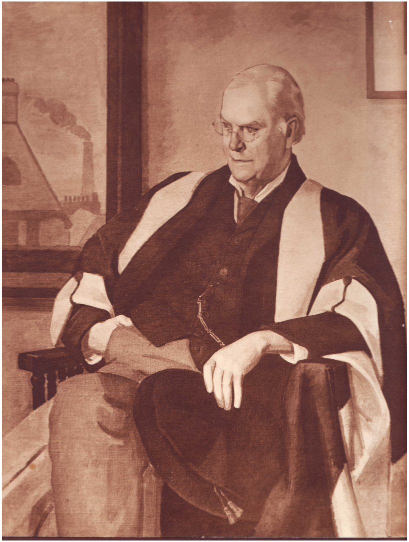


”‘Fluid dynamicists were divided into hydraulic engineers
who observed what could not be explained,
and mathematicians
who explained things that could not be observed.”’

Sir Cyril Hinshelwood
(1897 – 1967)



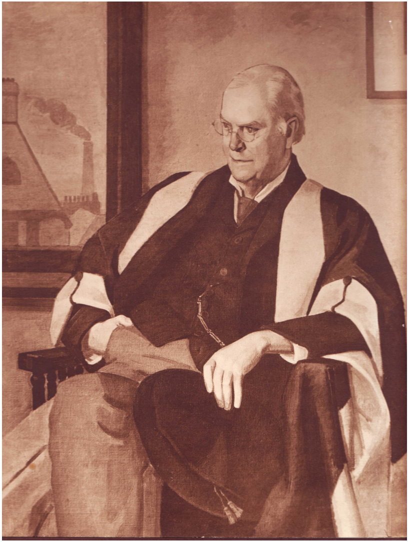
Sir Horace Lamb
(1849 – 1934)



Sir Horace Lamb
(1849 – 1934)

Sydney Goldstein
(1903 – 1989)

"'You can read all of Lamb without realizing that water is wet!"'



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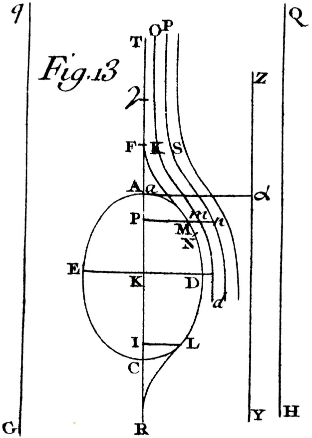
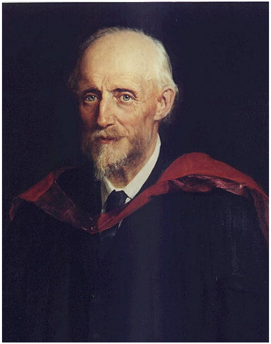
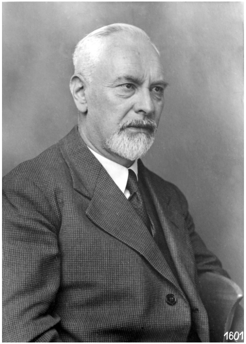
Jean D'Alembert
(1717 – 1783)



"'Opuscules mathématiques"'



Osborne Reynolds
(1842 – 1912)



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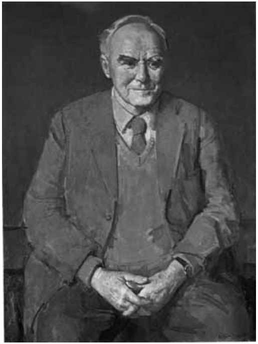
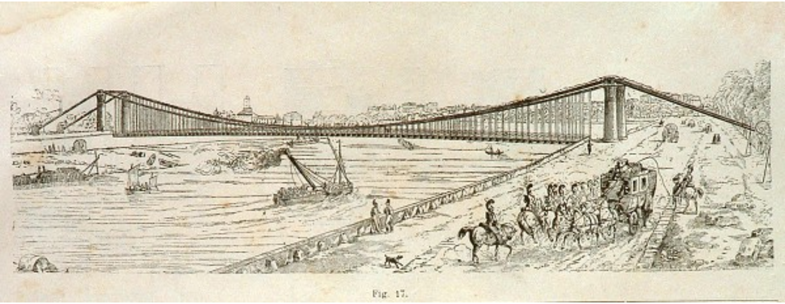
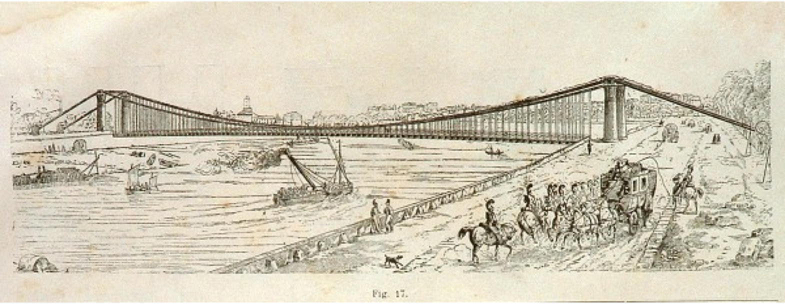


Sir Geoffrey Taylor
(1886 – 1975)

Zuse Z4 (1950) with Ferrit kernel storage
and multiple punch card reader



Pont des Invalides Paris, original drawing by Navier (1823)



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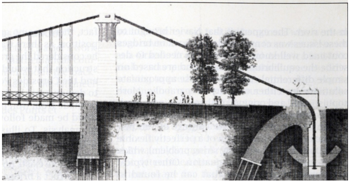
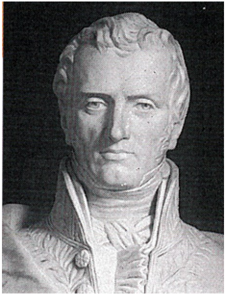


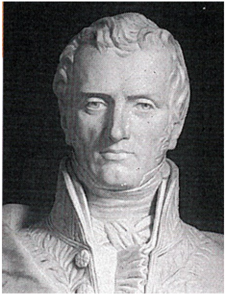
In 1826, September 6, the right pylon of Navier's bridge collapsed, caused by cracks in a water pipe!

Jacques Hadamard
(1865 – 1963)

"A mathematical problem is called well-posed, if there exists a solution which is uniquely determined and depends continuously on the data."

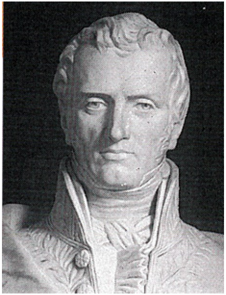


Henri Navier
(1785 – 1836)



Henri Navier
(1785 – 1836)

Saint-Venant
(1797 – 1886)



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(1785 – 1836)

Saint-Venant
(1797 – 1886)

Sir Gabriel Stokes
(1819 – 1903)

The equations of motion:

Henri Navier:
Mémoire sur les lois du mouvement des fluides (1822)

Saint-Venant:
Mémoire sur la dynamique des fluides (1834)

Gabriel Stokes:
On the Theories of the Internal Friction of Fluids in Motion (1845)

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Adhémar Jean Claude Barré de Saint-Venant:
Mémoire sur la dynamique des fluides (1834)

Sir George Gabriel Stokes:
On the Theories of the Internal Friction of Fluids in Motion (1845)

The Navier-Stokes equations:

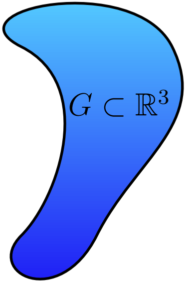
For $(t, \boldsymbol{x}) \in (0, T) \times G$:

$$\begin{aligned} \partial_t \boldsymbol{v} - \nu \Delta \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} + \boldsymbol{\nabla} p &= \boldsymbol{f} \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 \end{aligned}$$

On ∂G : $\quad \boldsymbol{v} = 0$

In $t = 0$: $\quad \boldsymbol{v} = \boldsymbol{v_0}$

∂G smooth



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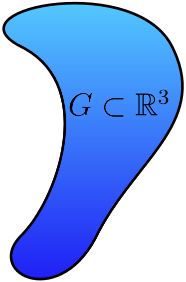

$\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$: velocity field

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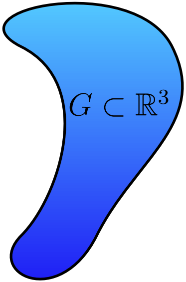

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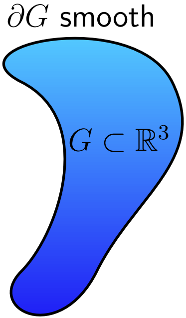

$\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$: velocity field
$p = p(t, \boldsymbol{x})$: kinematic pressure
$\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{x})$: external force density

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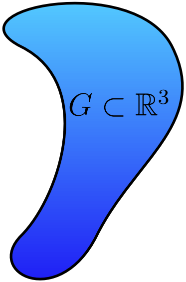

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$\nu > 0$: kinematic viscosity

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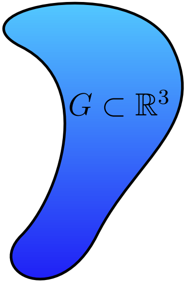

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$p = p(t, \boldsymbol{x})$:	kinematic pressure
$\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{x})$:	external force density
$\nu > 0$:	kinematic viscosity
$\boldsymbol{v_0} = \boldsymbol{v_0}(\boldsymbol{x})$:	initial velocity

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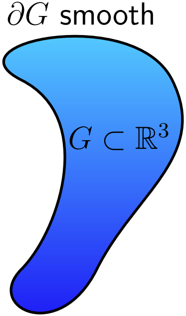
∂_t : partial time derivative



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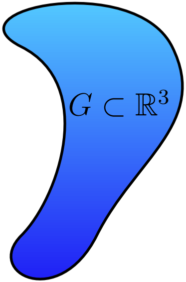

∂_t : partial time derivative

Δ : Laplace operator

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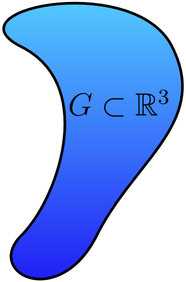

∂_t : partial time derivative
Δ : Laplace operator
$\boldsymbol{\nabla}$: gradient

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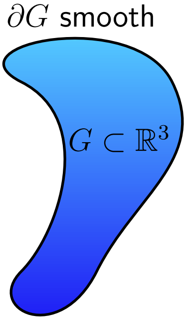

∂_t :	partial time derivative
Δ :	Laplace operator
$\boldsymbol{\nabla}$:	gradient
$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$:	nonlinear Term

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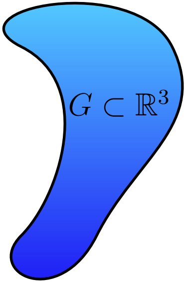

Balance of forces

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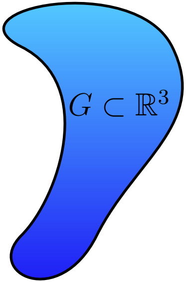

Balance of forces

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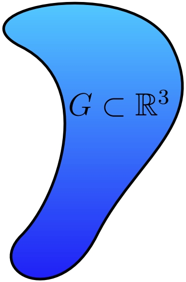

Balance of forces

Conservation of mass

No-slip condition on the boundary

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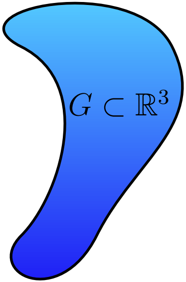
∂G smooth



Balance of forces
Conservation of mass
No-slip condition on the boundary
Initial condition

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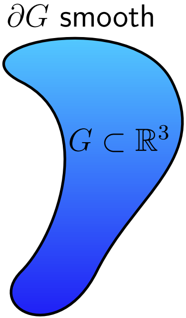

Given:

$\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{x})$, $\nu > 0$, $\boldsymbol{v_0} = \boldsymbol{v_0}(\boldsymbol{x})$

Construct:

$\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{x})$, $p = p(t, \boldsymbol{x})$.

Mathematical Difficulties:

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- Nonlinear system: Uniqueness? Global Existence?

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- The pressure acts non-locally
- Non-classical parabolic system

Consequences?

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  The Navier-Stokes equations mean an open door to hell!

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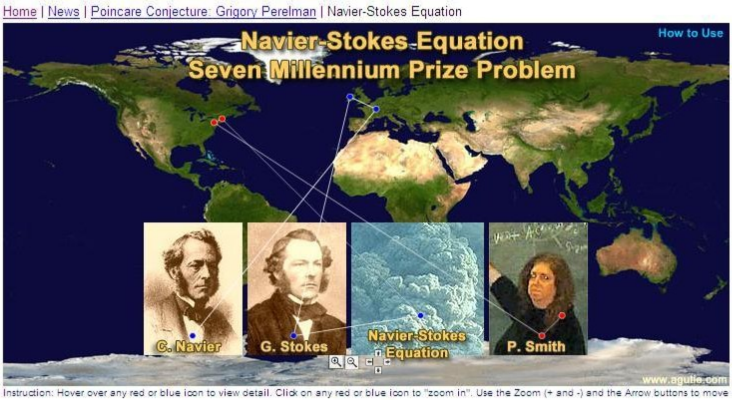
  The Navier-Stokes equations mean an open door to hell!

- Kyuya Masuda (Tohoku, Sendai, Japan):
  The Navier-Stokes equations are like a jungle with rare fruits hanging in high trees.
  But these fruits are delicious!

Clay Mathematics Institute:
Millennium Price Problem (1 Million $)

"'Although the Navier-Stokes equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory, which will unlock the secrets hidden in the Navier-Stokes equations."'

. . . only 5 problems left?



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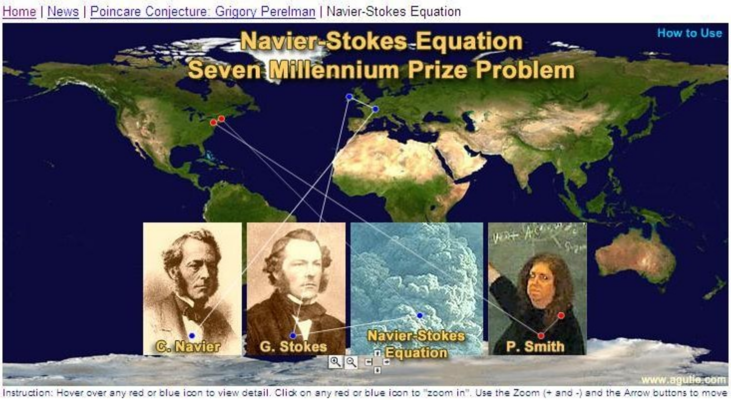


...only 5 problems left?

Penny Smith withdrew her paper "Immortal smooth solution of the 3 space dimensional Navier-Stokes system" in 2006, October 6, due to a "serious flaw"!

ASTANA. January 10, 2014, 16:33 (10:33 GMT). BNews.kz
Photo resource enu.kz

Academician from Astana Mukhtarbai Otelbayev has solved one of seven most difficult mathematical millennium problems, the press service of the Eurasian National University reports.

Mukhtarbay Otelbaev, Prof. Dr., Academician of the NAS of the RK, Director of the Eurasian Mathematical Institute of L.N. Gumilyov Eurasian National University, completed and published paper “Existence of a strong solution of the Navier-Stokes equations" in the press.□□The importance of the publication is that this problem is included in the 7 most complex mathematical problems, which are called “millennium problems". Note that for the solution of each of these problems Clay Mathematics Institute in early 2000 announced a prize of $ 1 million. Currently, only one of the seven Millennium problems (Poincaré conjecture) is solved. The Fields Prize for her decision was awarded to G.Perelman.□□Full Article of Muhtarbay Otelbaev was published in "Mathematical Journal" (2013, v.13 , № 4 (50)) http://www.math.kz/index.php/ru/513.

To my shame,
on page 56 the
inequality (6.34)
is incorrect,
therefore,
the proposition 6.3
(p. 54) isn’t proved!

Natural Science, 2015, 7, 88-99
Published Online February 2015 in SciRes. http://www.scirp.org/journal/ns
http://dx.doi.org/10.4236/ns.2015.72010

Scientific Research Publishing

Navier-Stokes Equations—Millennium Prize Problems

Asset A. Durmagambetov, Leyla S. Fazilova

System Research "Factor" Company, Astana, Kazakhstan
Email: asset.durmagambet@gmail.com

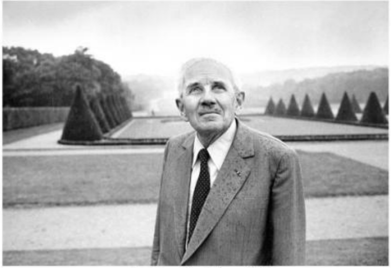
Received 6 February 2015; accepted 24 February 2015; published 27 February 2015

Abstract

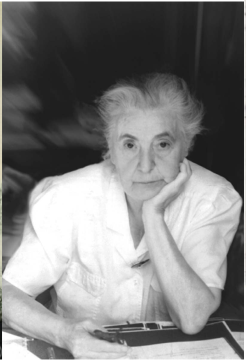
In this work, we present final solving Millennium Prize Problems formulated by Clay Math. Inst., Cambridge. A new uniform time estimation of the Cauchy problem solution for the Navier-Stokes equations is provided. We also describe the loss of smoothness of classical solutions for the Navier-Stokes equations.

Fundamental contributions since 1900:

- Jean Leray: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. 63 (1934)
- Eberhard Hopf: Über die Anfangsrandwertaufgabe für die hydrodynamischen Grundgleichungen. Math. Nachr. 4 (1951)
- A. A. Kiselev und O. A. Ladyzhenskaya: On existence and uniqueness of the solution of the nonstationary problem for a viscous incompressible fluid. Izv. Akad. Nauk. SSSR 21 (1957)
- James Serrin: The initial value problem for he Navier-Stokes equations. Univ. Wisconsin Press 9 (1963)



Jean Leray - Parc de Sceaux (1985)



Weak solutions

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Weyl:

$$L^2(G) = \mathcal{H}^0(G) \oplus \mathcal{G}(G)$$
$$u = v + \nabla p$$

with

$$\begin{aligned} \mathcal{H}^0(G) &= \{v | \nabla \cdot v = 0 \text{ in } G, v \cdot N = 0 \text{ on } \partial G\} \\ &= \overline{C^\infty_{0,\sigma}(G)}^{\|\cdot\|} \\ \mathcal{G}(G) &= \{v | v = \nabla p \text{ in } G\} \end{aligned}$$

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Hence: $$(v, \nabla p)_{L^2} = \int_G v(x) \cdot \nabla p(x) \,\mathrm{d}t = 0$$

Weak solutions

Let

$$P : L^2(G) \longrightarrow \mathcal{H}^0(G)$$

Then:

$$\begin{aligned} Pv &= v && \text{if} && v \in \mathcal{H}^0(G) \\ P\nabla p &= 0 && \text{if} && \nabla p \in \mathcal{G}(G). \end{aligned}$$

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Navier-Stokes equations

In $(0, T) \times G$:

$$\partial_t v - \nu P \Delta v + P(v \cdot \nabla v) = Pf$$

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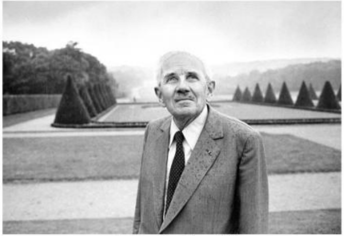
Theorem (Leray-Hopf): Let $T > 0$. For $v_0 \in \mathcal{H}^0(G)$, $\nu > 0$ and $f \in L^2(0, T; L^2(G))$ there is at least one weak solution v of the Navier-Stokes equations.

"There is at least ...



Jean Leray - Parc de Sceaux (1985)

Leray 1933, 1934

... one weak solution of (NS)!"

Hopf 1951

Weak solutions

Def: Let $v_0 \in \mathcal{H}^0(G)$, $f \in L^2(0,T;L^2(G))$. A function

$$v \in L^\infty(0,T;\mathcal{H}^0(G) \cap L^2(0,T;\mathcal{H}^1(G))$$

is called a weak solution of the NSE, if

$$\int_0^T \Big(\big(-v, \phi_t\big)_{L^2} + \nu \big(\nabla v, \nabla \phi\big)_{L^2} - \big(v \cdot \nabla \phi, v\big)_{L^2} \Big) \,\mathrm{d}t$$
$$= \big(v_0, \varphi(0)\big)_{L^2} + \int_0^T \big(f, v\big) \,\mathrm{d}t$$

for all $\phi \in C_0^\infty\big([0,t); C_{0,\sigma}^\infty(G)\big)$,
and if the energy inequality

$$\|v(t)\|^2 + 2\nu \int_0^T \|\nabla v(\tau)\|^2 \,\mathrm{d}\tau \leq \|v_0\|^2 + \int_0^t \big(f(\tau), v(\tau)\big) \,\mathrm{d}\tau$$

holds.

Strong solutions:

Strong solutions:

A weak solution u of (NS) is called a strong solution of (NS), if there are numbers s, q (the so-called Serrin exponents) with

$$2 < s < \infty, \quad 3 < q < \infty, \quad \frac{2}{s} + \frac{3}{q} = 1$$

such that additionally Serrin's condition

$$u \in L^s(0, T; L^q(\Omega))$$

is satisfied.

Serrin 1963

"Strong solutions are unique!"

Masuda 1984
Kozono & Sohr 1996

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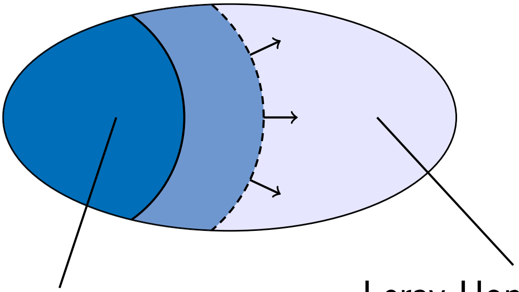
Serrin 1962, 1963
Heywood 1980, 1988
Galdi & Maremonti 1988
Beirao da Veiga 1995, 1997
Neustupa 1999

Serrin versus Leray-Hopf



Serrin class
of strong solutions:
Existence ?
Uniqueness !
Regularity !

Leray-Hopf class
of weak solutions:
Existence !
Uniqueness ?
Regularity ?

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Theorem (Prodi 1962, Heywood 1981): Let in addition $\nabla v_0 \in L^2(G)$ and $f \in L^2(0,T;H^1(G))$. Then there is a time $T^* = T^*(\|\nabla v_0\|, \nu, f) > 0$ such that there exists a unique strong solution v of the NSE in $(0,T^*) \times G$.

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Construct an a-priori estimate for

$$t \longrightarrow \|\nabla v(t,\cdot)\|^2, \quad 0 \le t \le T$$

Energy equation $(f = 0)$:

$$\Big(\partial_t v(t) - \nu\Delta v(t) + v(t)\cdot\nabla v(t) + \nabla p(t),\, v(t)\Big)_{L^2} = 0$$

L^2-orthogonality

$$\Big(v(t)\cdot\nabla v(t),\, v(t)\Big)_{L^2} = 0$$

implies

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 + \nu\|\nabla v(t)\|^2 = 0$$

Integration:

$$\|v(t)\|^2 + 2\nu\int_0^t \|\nabla v(s)\|^2\,\mathrm{d}s = \|v_0\|^2$$

implies

$$\int_0^T \|\nabla v(t)\|^2\,\mathrm{d}t < \infty$$

Gradient estimate $(f = 0)$

$$\Big(\partial_t v(t) - \nu\Delta v(t) + v(t)\cdot\nabla v(t) + \nabla p(t),\ -P\Delta v(t)\Big)_{L^2} = 0$$

Estimate

$$\big|\big(v(t)\cdot\nabla v(t), -P\Delta v(t)\big)\big| \le \tfrac{\nu}{2}\|P\Delta v(t)\|^2 + K_n\|\nabla v(t)\|^{2n}$$

implies

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla v(t)\|^2 + \nu\|P\Delta v(t)\|^2 \le \begin{cases} K_2\|\nabla v(t)\|^4 & (n=2) \\ K_3\|\nabla v(t)\|^6 & (n=3) \end{cases}$$

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$$\tfrac{\mathrm{d}}{\mathrm{d}t} \|\nabla v(t)\|^2 + \nu \|P\Delta v(t)\|^2 \leq \begin{cases} K_2 \|\nabla v(t)\|^4 & (n = 2) \\ K_3 \|\nabla v(t)\|^6 & (n = 3) \end{cases}$$

$$\psi'(t) \qquad \leq \qquad \begin{cases} K_2 \psi^2(t) & (n = 2) \\ K_3 \psi^3(t) & (n = 3) \end{cases}$$

Differential inequality $(n = 2)$

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Assume	$t \longrightarrow \psi(t) := \|\nabla v(t, \cdot)\|^2$	$(0 \leq t < T^*)$
solves DI	$\psi'(t) \leq K_2 \psi^2(t)$	
with IC	$\psi(0) = \|\nabla v_0\|^2$	

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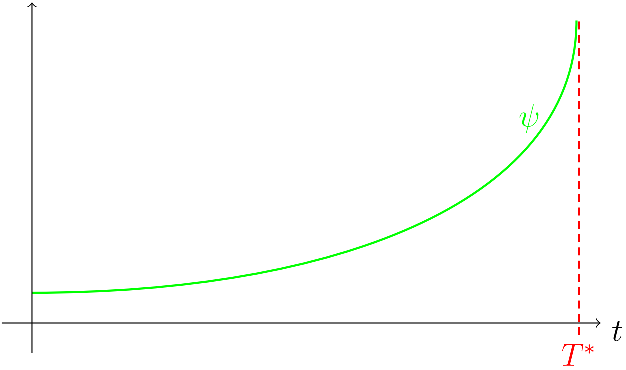

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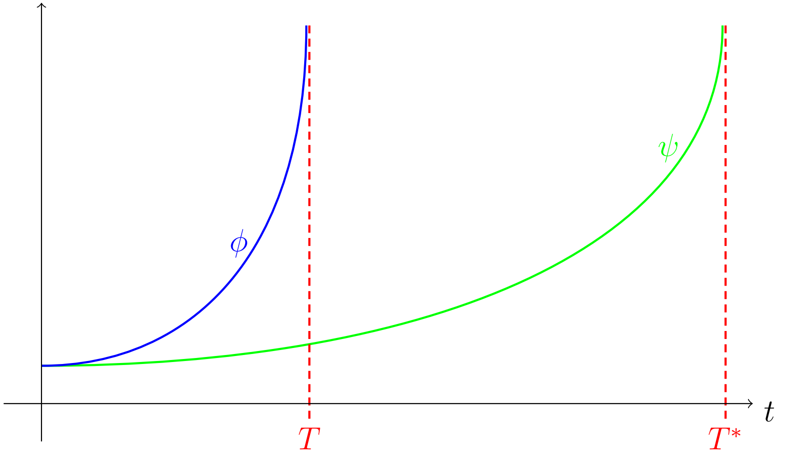
Note: $$\int_0^T \phi(t)\,\mathrm{d}t = \infty$$



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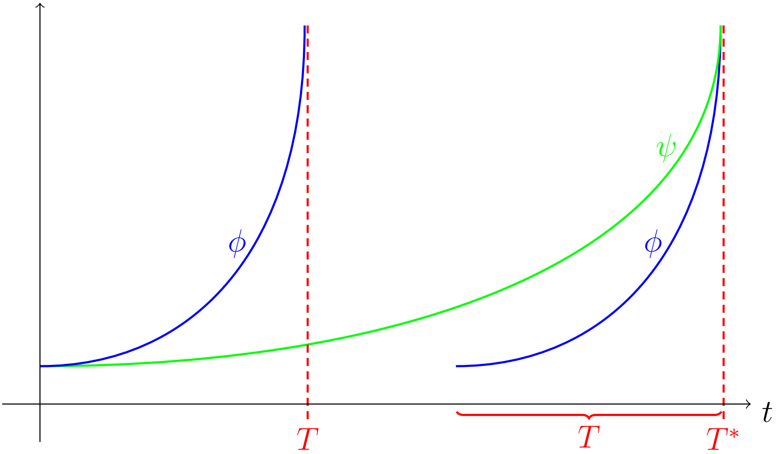

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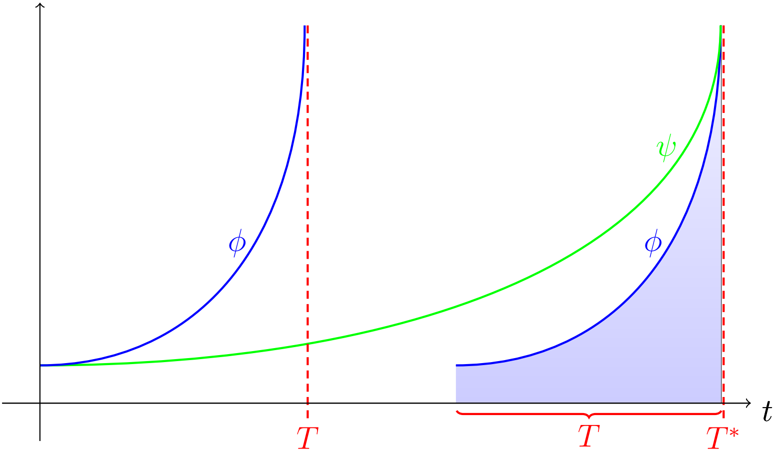

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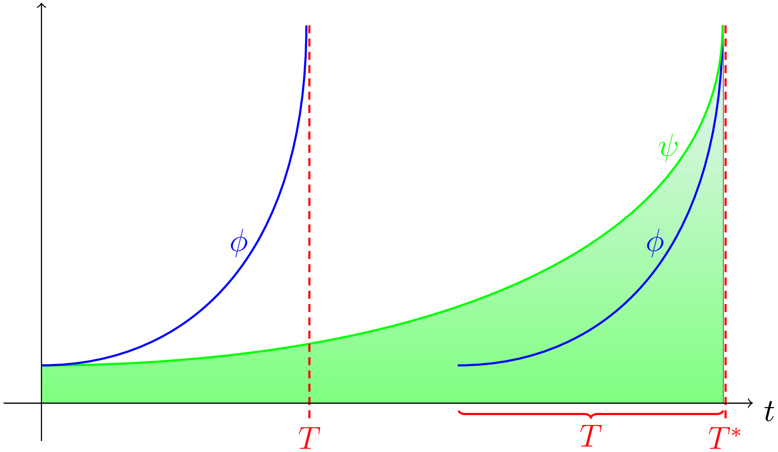

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solves DE $\qquad \phi'(t) = K_3 \phi^3(t), \quad \left(0 \le t < T = \frac{1}{2K_3 \cdot \|\nabla v_0\|^4}\right)$

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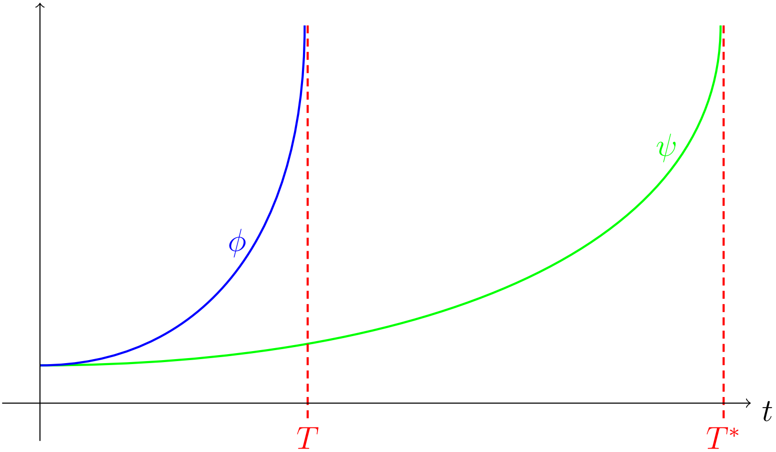
Catastrophy: $\int_0^T \phi(t)\, \mathrm{d}t = \frac{1}{K_3 \|v_0\|^2} < \infty$



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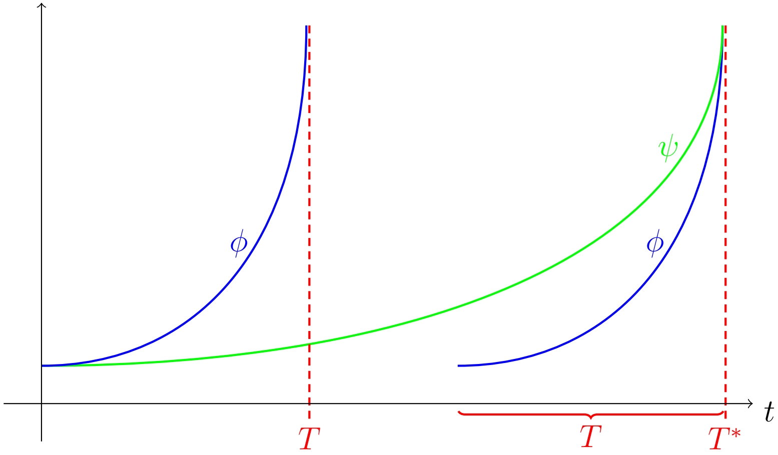

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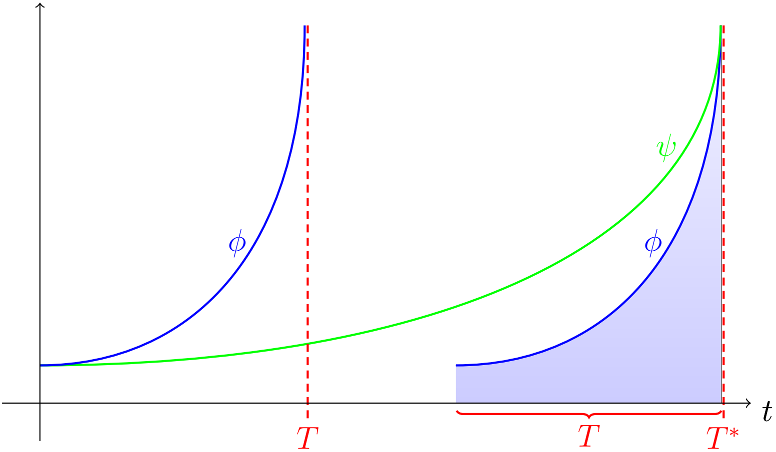

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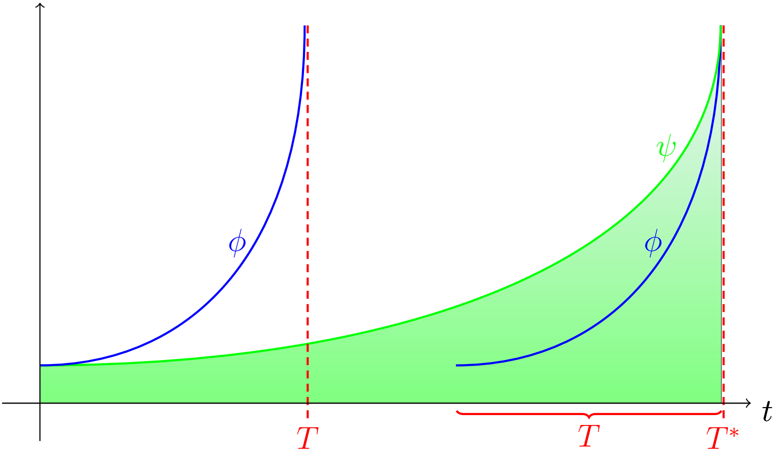

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Question:

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Let u be a weak solution of (NS) with data

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What additional regularity of the data is sufficient for u to be a strong solution in some intervall $(0,T)$, $0 < T \leq \infty$, i.e. what additional regularity of the data implies

$$u \in L^s(0,T; L^q(\Omega))$$

for some

$$2 < s < \infty, \quad 3 < q < \infty, \quad \frac{2}{s} + \frac{3}{q} = 1 \quad ?$$

Stokes Operator A_q:

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  - $\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$
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- Semigroup e^{-tA_q}, $t \geq 0$ generated by A_q in $L^q_\sigma(\Omega)$

Fractional Powers A_q^α:

Let $1 < q < \infty$, $-1 \leq \alpha \leq 1$:

- $A_q^\alpha : \mathcal{D}\big(A_q^\alpha\big) \to L^q_\sigma(\Omega)$ fractional power of A_q
  - $\mathcal{D}\big(A_q^\alpha\big) \subset L^q_\sigma(\Omega)$
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Moreover:

- $\big(A_q^\alpha\big)^{-1} = A_q^{-\alpha}$
- $(A_q)' = A_{q'} \quad$ with $\frac{1}{q} + \frac{1}{q'} = 1$

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- $u_0 \in L^{3,\infty}_\sigma(\Omega)$ & smallness, Kozono & Yamazaki 1995

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- $u_0 \in L^q_\sigma(\mathbb{R}^n)$, $q > n$, Fabes, Jones & Rivière 1972
- $u_0 \in L^q_\sigma(\Omega)$, $q > 3$, Miyakawa 1981
- $u_0 \in L^n_\sigma(\mathbb{R}^n)$, Kato 1984
- $u_0 \in L^3_\sigma(\Omega)$, Giga 1986
- $u_0 \in L^{3,\infty}_\sigma(\Omega)$ & smallness, Kozono & Yamazaki 1995
- $u_0 \in \mathbb{H}^{-2/s}_{q,0,\sigma}(\Omega)$, $2/s + 3/q = 1$, Amann 2000

$$
\begin{aligned}
\mathcal{D}(A) &\subset \mathcal{D}\left(A^{1/2}\right) \\
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What is the weakest possible condition on the data u_0 and f to get a (local) strong solution of (NS)?

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Can we formulate conditions on the data u_0 and f which are sufficient and necessary for u to be a (local) strong solution of (NS)?

Theorem 1:

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Let

- $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{2,1}$,
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Then:

1. The condition

$$\circledast \quad \int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, dt < \infty$$

   is sufficient and necessary for the existence of a unique strong solution $u \in L^s(0,T;L^q(\Omega))$ of (NS) with data u_0, f in some intervall $[0,T)$, $0 < T \leq \infty$.

2. Let u be a weak solution of (NS) in $[0, \infty) \times \Omega$ with data u_0, f, and let

$$\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, dt = \infty.$$

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$$\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s dt = \infty.$$

Then for every $0 < T \leq \infty$, Serrin's condition $u \in L^s(0, T; L^q(\Omega))$ does not hold, and in every intervall $[0, T)$, $0 < T \leq \infty$, the system (NS) does not have a strong solution with data u_0, f and Serrin exponents s, q.

Remark 1:

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- 1. means:

$$u \in L^s(0,T;L^q(\Omega)) \;\Leftrightarrow\; v \in L^s(0,\infty;L^q(\Omega))\,,$$

$v(t) := e^{-tA}u_0$ is the solution of a homogeneous linear Stokes system.

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- 2. means:

$$u_0 \in L^2_\sigma(\Omega)\,,\; \int_0^\infty \left\|e^{-tA}u_0\right\|_q^s\,dt = \infty$$

for all Serrin exponents s, q

$\Rightarrow$

For all q, s and all $0 < T \leq \infty$, (NS) has no strong solution $u \in L^s(0,T;L^q(\Omega))$ with data u_0, f.

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Use (A_q generates bounded analytic semigroup e^{-tA_q})

$$\left\| A_q^\alpha e^{-tA_q} v \right\|_q \le c\, t^{-\alpha} e^{-\delta t} \left\| v \right\|_q \;\text{ for } v \in L^q_\sigma(\Omega)$$

$$0 \le \alpha \le 1,\; t > 0,\; c = c(\Omega, q)\,,\; \delta = \delta(\Omega, q)$$

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to obtain

$$\begin{aligned} \left\|e^{-tA}u_0\right\|_q &= \left\|A^\alpha e^{-tA} A^{-\alpha} u_0\right\|_q = \left\|A_q^\alpha e^{-tA_q} A^{-\alpha} u_0\right\|_q \\ &\le c\,t^{-\alpha} e^{-\delta t} \left\|A^{-\alpha} u_0\right\|_q \le c\,t^{-\alpha} e^{-\delta t} \left\|u_0\right\|_2 \end{aligned}$$

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Use Sobolev embedding

$$\|v\|_q \leq c \left\| A_p^{\alpha} u_0 \right\|_p , \; v \in \mathcal{D}\left(A_p^{\alpha}\right) , \; c = c(\Omega, q)$$

$$1 < p \leq q, \quad 0 \leq \alpha = \tfrac{3}{2}\left(\tfrac{1}{p} - \tfrac{1}{q}\right) \leq 1$$

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to obtain

$$\left\|A^{-\alpha}u_0\right\|_q \leq c\left\|u_0\right\|_2$$

$$3 < q < \infty, \quad \tfrac{1}{4} < \alpha = \tfrac{3}{2}\left(\tfrac{1}{2} - \tfrac{1}{q}\right) < \tfrac{3}{4}$$

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shows for every $\varepsilon > 0$:

$$\int_0^\infty \left\|e^{-tA}u_0\right\|_q^s dt < \infty \quad \Leftrightarrow \quad \int_0^\varepsilon \left\|e^{-tA}u_0\right\|_q^s dt < \infty,$$

hence only integrability on $(0, \varepsilon)$ is important.

Theorem 2:

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Let

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Then there exists a constant $\varepsilon = \varepsilon(\Omega, q) > 0$ with:

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If

$$(\circledast\circledast) \quad \left(\int_0^T \left\|e^{-tA}u_0\right\|_q^s \, dt\right)^{1/s} + \left(\int_0^T \|F(t)\|_{q/2}^{s/2} \, dt\right)^{2/s} \leq \varepsilon,$$

then (NS) has a unique strong solution $u \in L^s(0,T;L^q(\Omega))$ with data u_0, f.

Question:

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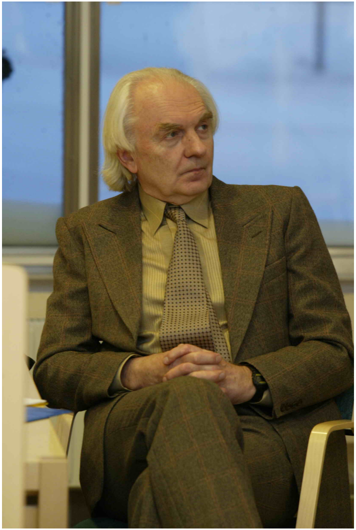

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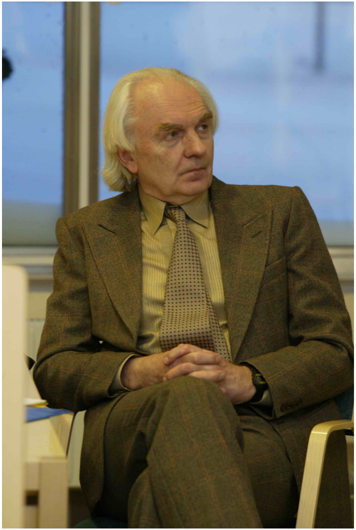

Who is this gentleman?

Oleg
Vladimirowitsch
Besov

Besov Spaces:

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$$\left(\int_0^\infty \left\| e^{-tA} u_0 \right\|_q^s \, dt \right)^{1/s} \approx \| u_0 \|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} ,$$

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where

$$\mathbb{B}_{q,s}^{-2/s}(\Omega) := \left(\mathbb{B}_{q',s'}^{2/s}(\Omega)\right)', \quad q' = \frac{q}{q-1}, \quad s' = \frac{s}{s-1}$$

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with (Amann 2002)

$$\begin{aligned}\mathbb{B}_{q',s'}^{2/s}(\Omega) : &= B_{q',s'}^{2/s}(\Omega) \cap L_\sigma^{q'}(\Omega) \\ &= \left(L_\sigma^{q'}, \mathcal{D}(A_{q'})\right)_{1/s,s'} \\ &= \left\{v \in B_{q',s'}^{2/s}(\Omega) \,|\nabla\cdot v = 0,\ v\cdot\nu|_{\partial\Omega} = 0\right\}\end{aligned}$$

$$\left(\int_0^\infty \left\|e^{-tA}u_0\right\|_q^s \, dt\right)^{1/s}$$

$$\left(\int_0^\infty \left\|e^{-tA}u_0\right\|_q^s \, dt\right)^{1/s} \approx \left\|A_q^{-1}u_0\right\|_{\left(L^q_\sigma, \mathcal{D}(A_q)\right)_{1-\frac{1}{s},s}}$$

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&\approx \left\|u_0\right\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)}
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$$

Theorem 1B:

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Let

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Then:

1. The condition
$$u_0 \in \mathbb{B}^{-2/s}_{q,s}(\Omega)$$
is sufficient and necessary for the existence of a unique strong solution $u \in L^s(0,T; L^q(\Omega))$ on (NS) with data u_0, f in some intervall $[0,T)$, $0 < T \leq \infty$.

2. Let u be a weak solution of (NS) in $[0, \infty) \times \Omega$ with data u_0, f, and let

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Then each of the following conditions are sufficient for the existence of a unique strong solution $u \in L^s(0,T; L^q(\Omega))$ with data u_0, f in some intervall $[0,T)$ with $0 < T \leq \infty$:

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Then each of the following conditions are sufficient for the existence of a unique strong solution $u \in L^s(0,T; L^q(\Omega))$ with data u_0, f in some intervall $[0,T)$ with $0 < T \leq \infty$:

1. $u_0 \in \mathbb{B}^{-2/s}_{q,s}(\Omega)$
2. $u_0 \in L^3_\sigma(\Omega)$, $q \leq s$
3. $u_0 \in \mathcal{D}\big(A^{1/4}\big)$

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2. Follows from $L^3_\sigma(\Omega) \subset \mathbb{B}_{q,s}^{-2/s}(\Omega)$ if $q \leq s$ (Amann 2002).

Remarks:

1. The condition $u_0 \in \mathbb{B}_{q,s}^{-2/s}(\Omega)$ is optimal.
2. Follows from $L^3_\sigma(\Omega) \subset \mathbb{B}_{q,s}^{-2/s}(\Omega)$ if $q \le s$ (Amann 2002).
3. Use $\|v\|_q \le c\,\|A^\alpha u_0\|_2$ for $v \in \mathcal{D}(A^\alpha)$, $c = c(\Omega, q)$ with $\alpha = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{1}{4} + \frac{1}{s}$ to obtain

$$
\begin{aligned}
\left\|e^{-tA}u_0\right\|_{q,s;\infty} &\le c\left\|A^\alpha e^{-tA}u_0\right\|_{2,s;\infty} \\
&= c\left\|A^{1/s}e^{-tA}A^{1/4}u_0\right\|_{2,s;\infty} \\
&\le c\left\|A^{1/4}u_0\right\|_2 .
\end{aligned}
$$

General domains $\Omega \subset \mathbb{R}^3$ ($s = 8$, $q = 4$):

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- Ω bounded or unbounded, $\partial\Omega$ bounded or unbounded
  $\partial\Omega$ smooth or non-smooth (corners, edges, cracks, ...)

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- However, Theorem 1 and Theorem 2 remain true also for general domains $\Omega \subset \mathbb{R}^3$ in the case $s = 8$, $q = 4$ $({}^2\!/\!_8 + {}^3\!/\!_4 = 1)$, since here only the L^2-approach for the Stokes Operator is used.

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- Moreover, in this case the constant $\varepsilon = \varepsilon(\Omega, q)$ from Theorem 2 does not depend on Ω and is therefore an absolute constant.

Theorem 4:

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Let

- $\Omega \subset \mathbb{R}^3$ be a general domain with boundary $\partial\Omega$,
- $u_0 \in L^2_\sigma(\Omega)$, $f = \nabla \cdot F$ with
  $F \in L^2(0, \infty; L^2(\Omega)) \cap L^4(0, \infty; L^2(\Omega))$.

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Then:

The condition

$$\int_0^\infty \left\| e^{-tA} u_0 \right\|_4^8 \, dt < \infty$$

is sufficient and necessary for the existence of a unique strong solution $u \in L^8(0,T; L^4(\Omega))$ of (NS) with data u_0, f in some intervall $[0,T)$, $0 < T \leq \infty$.

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Let

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  $F \in L^2(0,T;L^2(\Omega)) \cap L^4(0,T;L^2(\Omega))$, $0 < T \leq \infty$.

Then there exists an absolute constant $\varepsilon > 0$ with:

If

$$\left(\int_0^T \|e^{-tA}u_0\|_4^8 \, dt\right)^{1/8} + \left(\int_0^T \|F(t)\|_2^4 \, dt\right)^{1/4} \leq \varepsilon,$$

then (NS) has a unique strong solution $u \in L^8(0,T;L^4(\Omega))$ with data u_0, f.

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