AANMPDE 10

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Numerical integration of the Landau-Lifshitz-Gilbert equation

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joint work with

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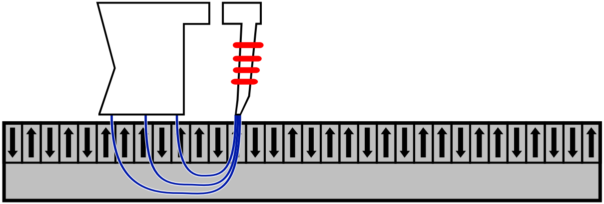
WIENER WISSENSCHAFTS-, FORSCHUNGS- UND TECHNOLOGIEFONDS

Model problem

Writing information on hard drives (HDD)



Source: https://en.wikipedia.org/wiki/Hard_disk_drive

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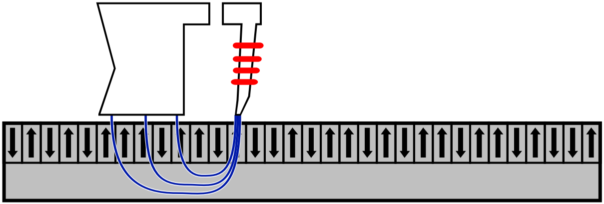

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- Micromagnetic phenomena (nanoseconds and nanometers)
- Evolution of magnetization $\boldsymbol{m} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ over time

Model problem

LLG equation

- $\boldsymbol{m}_t = -\boldsymbol{m} \times \boldsymbol{h}_{\text{eff}}(\boldsymbol{m}) + \alpha\, \boldsymbol{m} \times \boldsymbol{m}_t$ in $\Omega_T := (0,T) \times \Omega$
- $\partial_{\mathbf{n}} \boldsymbol{m} = \mathbf{0}$ on $(0,T) \times \partial\Omega$
- $\boldsymbol{m}(0) = \boldsymbol{m}^0$ on Ω

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- $\boldsymbol{h}_{\mathrm{eff}}(\boldsymbol{m})$

- Gilbert damping constant $\alpha \in (0,1]$
- Effective field $\boldsymbol{h}_{\mathrm{eff}}(\boldsymbol{m})$

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  - Exchange field $C_{\text{ex}} \Delta \boldsymbol{m}$, with $C_{\text{ex}} > 0$

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  - Stray field $\boldsymbol{\pi}$

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  - Stray field $\boldsymbol{\pi}$ $\rightsquigarrow$ FEM-BEM coupling

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  - External field $\boldsymbol{f}$ $\rightsquigarrow$ applied current

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- $\partial_{\mathbf{n}} \boldsymbol{m} = \mathbf{0}$ on $(0,T) \times \partial\Omega$
- $\boldsymbol{m}(0) = \boldsymbol{m}^0$ with $|\boldsymbol{m}^0| = 1$ on Ω on Ω
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- Effective field $\boldsymbol{h}_{\rm eff}(\boldsymbol{m})$
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  - Stray field $\boldsymbol{\pi}$ $\rightsquigarrow$ FEM-BEM coupling
  - External field $\boldsymbol{f}$ $\rightsquigarrow$ applied current
- $0 = \boldsymbol{m}_t \cdot \boldsymbol{m} = \frac{1}{2}\frac{\rm d}{\rm dt}|\boldsymbol{m}|^2 = 0$ in Ω_T $\implies$ $|\boldsymbol{m}| = 1$ in Ω_T

Challenges

Gilbert form

$$\boldsymbol{m}_t = -\boldsymbol{m} \times \boldsymbol{h}_{\mathrm{eff}}(\boldsymbol{m}) + \alpha\, \boldsymbol{m} \times \boldsymbol{m}_t$$

- Non-linear PDE
- PDE inherent constraint $|\boldsymbol{m}| = 1$

Alouges, Soyeur: *Nonlinear Anal.*, 18, 1992

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- Non-linear PDE
- PDE inherent constraint $|\boldsymbol{m}| = 1$
- Convergence rate (almost) always formal
- Ex. global weak solution, uniqueness may fail
  - Numerics: no surprises

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Numerical integrator

Overview

Gilbert form

$$\boldsymbol{m}_t = -\boldsymbol{m} \times \boldsymbol{h}_{\mathrm{eff}}(\boldsymbol{m}) + \alpha\, \boldsymbol{m} \times \boldsymbol{m}_t$$

- Finite element based schemes
  - Tangent plane scheme
  - Second-order tangent plane scheme
  - Midpoint scheme

Alouges: *Discrete Contin. Dyn. Syst. Ser. S.*, 1(2), 2008

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  - Alternative form liner in $\boldsymbol{v}$

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Idea (Alouges '08)

FEM for $\boldsymbol{v}$ in $\mathcal{K}(\boldsymbol{m}) := \{\boldsymbol{v} : \boldsymbol{v} \cdot \boldsymbol{m} = 0\}$

Alouges: *Discrete Contin. Dyn. Syst. Ser. S.*, 1(2), 2008

Discretization

- Time discretization
  - Number of uniform time-steps $\rightsquigarrow$ $M \in \mathbb{N}$
  - Time-step size $\rightsquigarrow$ $k := \frac{T}{M}$

Discretization

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  - Number of uniform time-steps $\rightsquigarrow$ $M \in \mathbb{N}$
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- Space discretization
  - Piecewise affine functions in every dimension $\rightsquigarrow$ $\boldsymbol{S}_h$
  - Number of nodes $\rightsquigarrow$ N
  - Tangent space: $\boldsymbol{\varphi}_h(\boldsymbol{z}) \cdot \boldsymbol{m}_h^i(\boldsymbol{z}) = 0$ for all nodes $\rightsquigarrow$ $\boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i) \subset \boldsymbol{S}_h$
  - $|\boldsymbol{\varphi}_h(\boldsymbol{z})| = 1$ for all nodes $\rightsquigarrow$ $\boldsymbol{\mathcal{M}}_h$

Numerical integrator

Tangent plane scheme (Alouges '08)

- **Input:** $\boldsymbol{m}_h^0 \in \boldsymbol{\mathcal{M}}_h$.
- **Loop:** For $0 < i \leq M - 1$:
  (a) Find $\boldsymbol{v}_h^i \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$, s. t. for all $\boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$

$$\alpha\langle \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i \,,\, \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i \,,\, \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) \,,\, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} \,,\, \boldsymbol{\varphi}_h\rangle$$

  (b) Compute $\boldsymbol{m}_h^{i+1}(\boldsymbol{z}) := \dfrac{\boldsymbol{m}_h^i(\boldsymbol{z}) + k\boldsymbol{v}_h^i(\boldsymbol{z})}{|\boldsymbol{m}_h^i(\boldsymbol{z}) + k\boldsymbol{v}_h^i(\boldsymbol{z})|}$ for all nodes $\boldsymbol{z}$
- **Output:** $\boldsymbol{\mathcal{M}}_h \ni (\boldsymbol{m}_h^i)_{i=0}^M \approx \boldsymbol{m}(t_i)$.

Alouges: *Discrete Contin. Dyn. Syst. Ser. S*, 1(2), 2008

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- **Loop:** For $0 < i \leq M-1$:
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$$\begin{aligned}\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle \\ = -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i)\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}_h\rangle\end{aligned}$$

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- Unconditionally convergent

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$$\alpha\langle \boldsymbol{v}_h^i\,,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}}k\langle \nabla \boldsymbol{v}_h^i\,,\nabla\boldsymbol{\varphi}_h\rangle$$
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- **Output:** $\boldsymbol{\mathcal{M}}_h \ni (\boldsymbol{m}_h^i)_{i=0}^M \approx \boldsymbol{m}(t_i)$.

- Unconditionally convergent
- Only one linear system per time-step in $\boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$

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Goals

- Linear algebra
  - Develop solution strategy for linear system in $\mathcal{K}_h(\boldsymbol{m}_h^i)$

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  - Numerically cheap
  - (Almost) independent of time-step
  - Robust in α

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Aim

Check $\boldsymbol{m}_h^i \rightsquigarrow$ choose preconditioner accordingly

Solve the linear system

Linear system in tangent space

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i , \nabla \boldsymbol{\varphi}_h\rangle$$
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Linear system in tangent space

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- $\boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i) \rightsquigarrow$ pointwise constraint $\boldsymbol{m}_h^i(\boldsymbol{z}) \cdot \boldsymbol{v}_h^i(\boldsymbol{z}) = 0$

Linear system in tangent space

Discrete variational formulation

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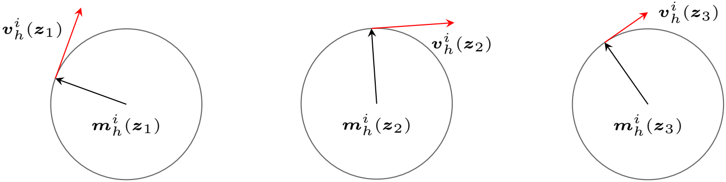
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i , \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} , \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$$

- $\boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i) \rightsquigarrow$ pointwise constraint $\boldsymbol{m}_h^i(\boldsymbol{z}) \cdot \boldsymbol{v}_h^i(\boldsymbol{z}) = 0$

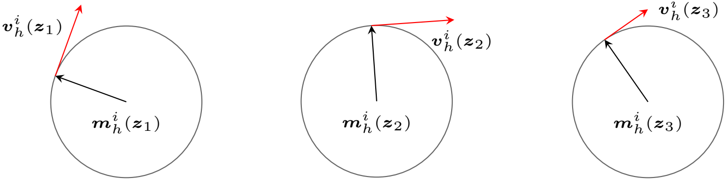


Linear system in tangent space

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i , \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i , \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} , \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$$

- $\boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i) \rightsquigarrow$ pointwise constraint $\boldsymbol{m}_h^i(\boldsymbol{z}) \cdot \boldsymbol{v}_h^i(\boldsymbol{z}) = 0$



- $\boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$ depends on time-step

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i , \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i , \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} , \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}}k\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i)\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \mathcal{K}_h(\boldsymbol{m}_h^i)$$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i \,,\, \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i \,,\, \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) \,,\, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} \,,\, \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i \, , \, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \, , \, \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i \, , \, \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i \, , \, \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) \, , \, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} \, , \, \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

- $\langle \boldsymbol{v}_h^i \, , \, \boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ mass matrix $\mathbf{M} \in \mathbb{R}^{3N \times 3N}$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}}k\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i)\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

- $\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ mass matrix $\mathbf{M} \in \mathbb{R}^{3N\times 3N}$
- $\langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ skew-symmetric matrix $\mathbf{S}(\boldsymbol{m}_h^i) \in \mathbb{R}^{3N\times 3N}$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}}k\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i)\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

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- $\langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ skew-symmetric matrix $\mathbf{S}(\boldsymbol{m}_h^i) \in \mathbb{R}^{3N\times 3N}$
- $\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ stiffness matrix $\mathbf{L} \in \mathbb{R}^{3N\times 3N}$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i \,,\, \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i \,,\, \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) \,,\, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} \,,\, \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

- $\langle \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle$ $\leadsto$ mass matrix $\mathbf{M} \in \mathbb{R}^{3N\times 3N}$
- $\langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \,,\, \boldsymbol{\varphi}_h\rangle$ $\leadsto$ skew-symmetric matrix $\mathbf{S}(\boldsymbol{m}_h^i) \in \mathbb{R}^{3N\times 3N}$
- $\langle \nabla \boldsymbol{v}_h^i \,,\, \nabla \boldsymbol{\varphi}_h\rangle$ $\leadsto$ stiffness matrix $\mathbf{L} \in \mathbb{R}^{3N\times 3N}$
- Right-hand side $\mathbf{r} \in \mathbb{R}^{3N}$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i \,, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \,, \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i \,, \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i \,, \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) \,, \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} \,, \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

- $\langle \boldsymbol{v}_h^i \,, \boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ mass matrix $\mathbf{M} \in \mathbb{R}^{3N\times 3N}$
- $\langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \,, \boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ skew-symmetric matrix $\mathbf{S}(\boldsymbol{m}_h^i) \in \mathbb{R}^{3N\times 3N}$
- $\langle \nabla \boldsymbol{v}_h^i \,, \nabla \boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ stiffness matrix $\mathbf{L} \in \mathbb{R}^{3N\times 3N}$
- Right-hand side $\mathbf{r} \in \mathbb{R}^{3N}$

Full-space system

$$\big(\alpha\mathbf{M} + C_{\mathrm{ex}} k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\,\mathbf{v} := \mathbf{A}(\boldsymbol{m}_h^i)\,\mathbf{v} = \mathbf{r}$$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i \, , \, \boldsymbol{\varphi}_h \rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \, , \, \boldsymbol{\varphi}_h \rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i \, , \, \nabla \boldsymbol{\varphi}_h \rangle$$
$$= -C_{\mathrm{ex}} \langle \nabla \boldsymbol{m}_h^i \, , \, \nabla \boldsymbol{\varphi}_h \rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) \, , \, \boldsymbol{\varphi}_h \rangle + \langle \boldsymbol{f} \, , \, \boldsymbol{\varphi}_h \rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

- $\langle \boldsymbol{v}_h^i \, , \, \boldsymbol{\varphi}_h \rangle$ $\rightsquigarrow$ mass matrix $\mathbf{M} \in \mathbb{R}^{3N \times 3N}$
- $\langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i \, , \, \boldsymbol{\varphi}_h \rangle$ $\rightsquigarrow$ skew-symmetric matrix $\mathbf{S}(\boldsymbol{m}_h^i) \in \mathbb{R}^{3N \times 3N}$
- $\langle \nabla \boldsymbol{v}_h^i \, , \, \nabla \boldsymbol{\varphi}_h \rangle$ $\rightsquigarrow$ stiffness matrix $\mathbf{L} \in \mathbb{R}^{3N \times 3N}$
- Right-hand side $\mathbf{r} \in \mathbb{R}^{3N}$

Full-space system

$$\big(\alpha \mathbf{M} + C_{\mathrm{ex}} k \mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\, \mathbf{v} := \mathbf{A}(\boldsymbol{m}_h^i)\, \mathbf{v} = \mathbf{r} \quad \rightsquigarrow \quad \boldsymbol{v}_h^i = \sum_{j=1}^{3N} \mathbf{v}_j \boldsymbol{\phi}_j$$

Full-space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\rm ex}k\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
$$= -C_{\rm ex}\langle \nabla \boldsymbol{m}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i)\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_h$$

- $\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ mass matrix $\mathbf{M} \in \mathbb{R}^{3N\times 3N}$
- $\langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ skew-symmetric matrix $\mathbf{S}(\boldsymbol{m}_h^i) \in \mathbb{R}^{3N\times 3N}$
- $\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$ $\rightsquigarrow$ stiffness matrix $\mathbf{L} \in \mathbb{R}^{3N\times 3N}$
- Right-hand side $\mathbf{r} \in \mathbb{R}^{3N}$

Full-space system

$$\big(\alpha\mathbf{M} + C_{\rm ex}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\,\mathbf{v} := \mathbf{A}(\boldsymbol{m}_h^i)\,\mathbf{v} = \mathbf{r} \quad \rightsquigarrow \quad \boldsymbol{v}_h^i = \sum_{j=1}^{3N} \mathbf{v}_j \boldsymbol{\phi}_j$$

- Performs well $\rightsquigarrow$ convergence mathematically open

Tangent space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}}k\langle\nabla\boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle\nabla\boldsymbol{m}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle + \langle\boldsymbol{\pi}(\boldsymbol{m}_h^i)\,,\,\boldsymbol{\varphi}_h\rangle + \langle\boldsymbol{f}\,,\,\boldsymbol{\varphi}_h\rangle \quad \forall\boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$$

Tangent space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k\langle \nabla \boldsymbol{v}_h^i , \nabla\boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i , \nabla\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} , \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$$

- $\mathbf{H}(\boldsymbol{m}_h^i(z)) \in \mathbb{R}^{3\times 2}$, s.t. $\left[\,\mathbf{H}(\boldsymbol{m}_h^i(z)) \,\middle|\, \boldsymbol{m}_h^i(z)\,\right] \in \mathbb{R}^{3\times 3}$ orthonormal

Tangent space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
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- $\mathbf{H}(\boldsymbol{m}_h^i(z)) \in \mathbb{R}^{3\times 2}$, s.t. $\left[\,\mathbf{H}(\boldsymbol{m}_h^i(z))\,\middle|\,\boldsymbol{m}_h^i(z)\,\right] \in \mathbb{R}^{3\times 3}$ orthonormal
- $\mathbf{Q}(\boldsymbol{m}_h^i) := \begin{pmatrix} \mathbf{H}(\boldsymbol{m}_h^i(z_1)) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{H}(\boldsymbol{m}_h^i(z_N)) \end{pmatrix} \in \mathbb{R}^{3N\times 2N}$
- Project to $\mathbb{R}^{2N} \rightsquigarrow \mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i) := \mathbf{Q}(\boldsymbol{m}_h^i)^T \mathbf{A}(\boldsymbol{m}_h^i)\mathbf{Q}(\boldsymbol{m}_h^i) \in \mathbb{R}^{2N\times 2N}$

Tangent space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}}k\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
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- $\mathbf{H}(\boldsymbol{m}_h^i(z)) \in \mathbb{R}^{3\times 2}$, s.t. $\left[\,\mathbf{H}(\boldsymbol{m}_h^i(z))\,\middle|\,\boldsymbol{m}_h^i(z)\,\right] \in \mathbb{R}^{3\times 3}$ orthonormal
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Tangent space system

Find $\mathbf{w} \in \mathbb{R}^{2N}$ s.t $\mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\,\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)^T\mathbf{r}$

Tangent space system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i , \boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}} k \langle \nabla \boldsymbol{v}_h^i , \nabla \boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i , \nabla \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i) , \boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f} , \boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$$

- $\mathbf{H}(\boldsymbol{m}_h^i(z)) \in \mathbb{R}^{3\times 2}$, s.t. $\left[\mathbf{H}(\boldsymbol{m}_h^i(z)) \,\middle|\, \boldsymbol{m}_h^i(z)\right] \in \mathbb{R}^{3\times 3}$ orthonormal
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Tangent space system

Find $\mathbf{w} \in \mathbb{R}^{2N}$ s.t $\mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\,\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)^T \mathbf{r} \quad \rightsquigarrow \boldsymbol{v}_h^i = \sum_{j=1}^{3N} (\mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{w})_j \boldsymbol{\phi}_j$

Preconditioning

Symmetric preconditioner

Unconstrained system

$$\big(\alpha\mathbf{M} + C_{\mathrm{ex}} k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\,\mathbf{v} =: \mathbf{A}(\boldsymbol{m}_h^i)\mathbf{v} = \mathbf{r}$$

Symmetric preconditioner

Unconstrained system

$$\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\,\mathbf{v} =: \mathbf{A}(\boldsymbol{m}_h^i)\mathbf{v} = \mathbf{r}$$

- Symmetric preconditioner $\rightsquigarrow$ $\mathbf{P} := \big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\big)^{-1}$

Symmetric preconditioner

Unconstrained system

$$\left(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\right)\mathbf{v} =: \mathbf{A}(\boldsymbol{m}_h^i)\mathbf{v} = \mathbf{r}$$

- Symmetric preconditioner $\rightsquigarrow$ $\mathbf{P} := \left(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\right)^{-1}$

Theorem (Kraus, Pfeiler, Praetorius, Ruggeri, S. '17+)

$$\mathrm{cond}_2(\mathbf{PA}(\boldsymbol{m}_h^i)) := \|\mathbf{PA}(\boldsymbol{m}_h^i)\|\,\|(\mathbf{PA}(\boldsymbol{m}_h^i))^{-1}\| \lesssim \left(1 + \frac{1}{\alpha}\right)^2$$

Symmetric preconditioner

Unconstrained system

$$\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\,\mathbf{v} =: \mathbf{A}(\boldsymbol{m}_h^i)\mathbf{v} = \mathbf{r}$$

- Symmetric preconditioner $\rightsquigarrow$ $\mathbf{P} := \big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\big)^{-1}$

Theorem (Kraus, Pfeiler, Praetorius, Ruggeri, S. '17+)

$$\mathrm{cond}_2(\mathbf{PA}(\boldsymbol{m}_h^i)) := \|\mathbf{PA}(\boldsymbol{m}_h^i)\|\,\|(\mathbf{PA}(\boldsymbol{m}_h^i))^{-1}\| \lesssim \left(1 + \frac{1}{\alpha}\right)^2$$

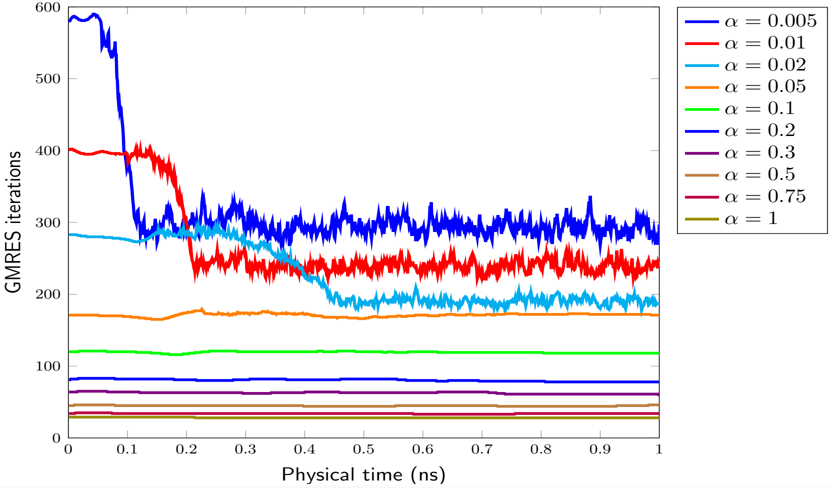
- Preconditioner independent of the time-step

GMRES @ full space system

$$\mathrm{cond}_2(\mathbf{PA}) := \|\mathbf{PA}\|\,\|(\mathbf{PA})^{-1}\| \lesssim \left(1 + \frac{1}{\alpha}\right)^2$$

GMRES @ full space system

$$\mathrm{cond}_2(\mathbf{PA}) := \|\mathbf{PA}\|\,\|(\mathbf{PA})^{-1}\| \lesssim \left(1 + \frac{1}{\alpha}\right)^2$$



Implicit preconditioner

Tangent space system

$$\mathbf{Q}(\boldsymbol{m}_h^i)\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{w} := \mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{r}$$

Implicit preconditioner

Tangent space system

$$\mathbf{Q}(\boldsymbol{m}_h^i)\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{w} := \mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{r}$$

- Symmetric $\mathbf{P}(\boldsymbol{m}_h^i) := \Big(\mathbf{Q}(\boldsymbol{m}_h^i)^T\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\big)\mathbf{Q}(\boldsymbol{m}_h^i)\Big)^{-1}$

Implicit preconditioner

Tangent space system

$$\mathbf{Q}(\boldsymbol{m}_h^i)\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{w} := \mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{r}$$

- Symmetric $\mathbf{P}(\boldsymbol{m}_h^i) := \big(\mathbf{Q}(\boldsymbol{m}_h^i)^T\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\big)\mathbf{Q}(\boldsymbol{m}_h^i)\big)^{-1}$ ✓

Implicit preconditioner

Tangent space system

$$\mathbf{Q}(\boldsymbol{m}_h^i)\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{w} := \mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{r}$$

- Symmetric $\mathbf{P}(\boldsymbol{m}_h^i) := \big(\mathbf{Q}(\boldsymbol{m}_h^i)^T\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\big)\mathbf{Q}(\boldsymbol{m}_h^i)\big)^{-1}$ ✓
  - Depends on $\boldsymbol{m}_h^i \rightsquigarrow$ rebuild every time-step

Implicit preconditioner

Tangent space system

$$\mathbf{Q}(\boldsymbol{m}_h^i)\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{w} := \mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{r}$$

- Symmetric $\mathbf{P}(\boldsymbol{m}_h^i) := \big(\mathbf{Q}(\boldsymbol{m}_h^i)^T\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\big)\mathbf{Q}(\boldsymbol{m}_h^i)\big)^{-1}$ ✓
  - Depends on $\boldsymbol{m}_h^i \rightsquigarrow$ rebuild every time-step
  - Idea: Reuse same preconditioner from old time-step $\boldsymbol{m}_h^{i-p}$

Implicit preconditioner

Tangent space system

$$\mathbf{Q}(\boldsymbol{m}_h^i)\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L} + \mathbf{S}(\boldsymbol{m}_h^i)\big)\mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{w} := \mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)\mathbf{w} = \mathbf{Q}(\boldsymbol{m}_h^i)\mathbf{r}$$

- Symmetric $\mathbf{P}(\boldsymbol{m}_h^i) := \Big(\mathbf{Q}(\boldsymbol{m}_h^i)^T\big(\alpha\mathbf{M} + C_{\mathrm{ex}}k\mathbf{L}\big)\mathbf{Q}(\boldsymbol{m}_h^i)\Big)^{-1}$ ✓
  - Depends on $\boldsymbol{m}_h^i \rightsquigarrow$ rebuild every time-step
  - Idea: Reuse same preconditioner from old time-step $\boldsymbol{m}_h^{i-p}$

Theorem (Kraus, Pfeiler, Praetorius, Ruggeri, S. '17+)

$$\mathrm{cond}_2(\mathbf{P}(\boldsymbol{m}_h^{i-p})\mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)) \lesssim \left(1 + \frac{1}{\alpha} + \frac{C_{ex}k}{\alpha h^2}\max_z \|\mathbf{H}(\boldsymbol{m}_h^i(z)) - \mathbf{H}(\boldsymbol{m}_h^{i-p}(z))\|\right)^2$$

Implicit preconditioner

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- More updates for smaller h

Explicit preconditioning

Tangent space system

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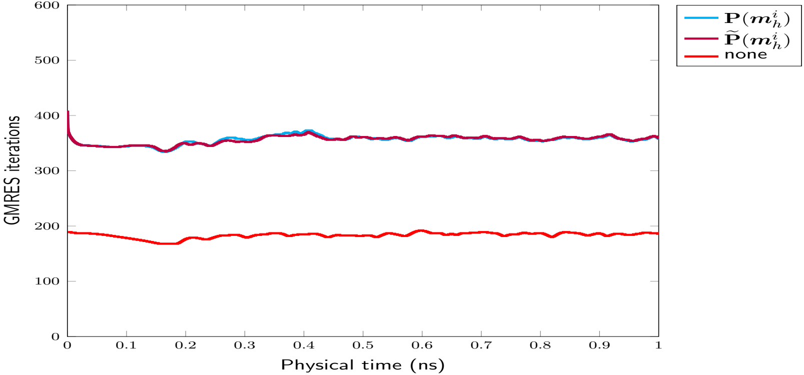
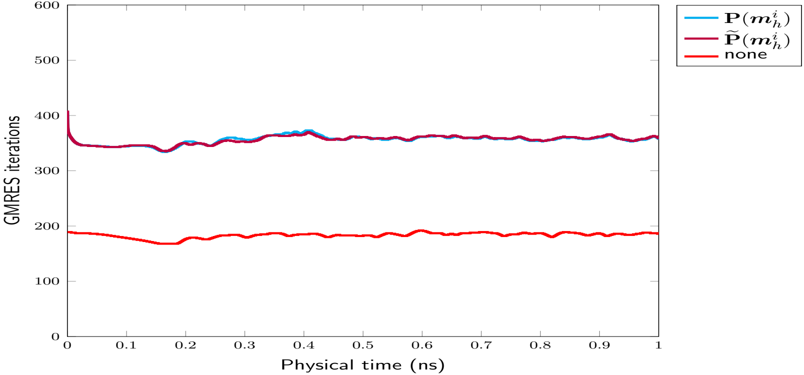
- *Householder* + $1 + (\boldsymbol{m}_h^i(z))_3 \geq \gamma > 0$ *for all nodes* z*:*

$$\mathrm{cond}_2(\widetilde{\mathbf{P}}(\boldsymbol{m}_h^i)\mathbf{A}_{\mathbf{Q}}(\boldsymbol{m}_h^i)) \lesssim \left(1 + \frac{1}{\alpha}\right)^2 \left(1 + \frac{C_{ex}k}{\alpha\gamma^4}\|\nabla\boldsymbol{m}_h^i\|^2_{\boldsymbol{L}^\infty(\Omega)}\right)$$

GMRES @ Tangent space system



GMRES @ Tangent space system



- $\alpha = 0.02 \rightsquigarrow$ condition number grows with $1/\alpha$
- $\Omega = (0, 500) \times (0, 250) \times (0, 3) \rightsquigarrow$ condition number depends on Ω

Conlusion & Outlook

- Problem in $\mathcal{K}_h(\boldsymbol{m}_h^i) \cong \mathbb{R}^{2N}$
  - Projected system with $\operatorname{span} \mathbf{H}(\boldsymbol{m}_h^i(z)) \perp \boldsymbol{m}_h^i(z)$
  - Solve only in $\mathbb{R}^{2N}$

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- Outlook: Different approaches for constrained system
  - What is the best approach?
  - Preconditioning?

Thanks for listening!

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Additional slides

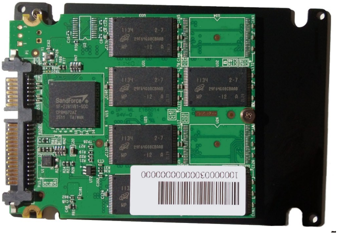
Data storage technologies

Hard disk drive (HDD)

Source: https://en.wikipedia.org/wiki/Hard_disk_drive

- Since 1954
- Magnetic storage
- Recording head

Solid state drive (SSD)



Source: https://de.wikipedia.org/wiki/Solid-State-Drive

- Since 1979
- Semi-conductor technology
- Faster, but more expensive

Data storage technologies

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- 1 Terrabyte $\sim$ 40 €

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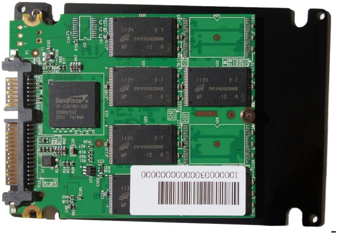

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- Since 1979
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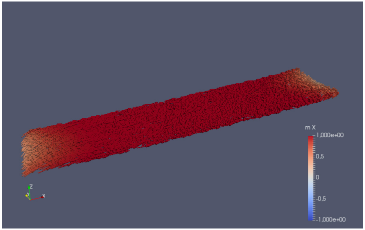
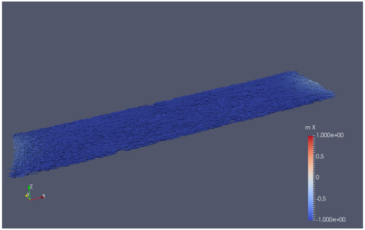
Reversal of magnetization



- Schöberl: NGSolve Finite Element Library
- Śmigaj, Betcke et al.: *ACM Trans. Math. Software*, 41, 2015
- Haberl et al.: `SolveLLG`

Reversal of magnetization



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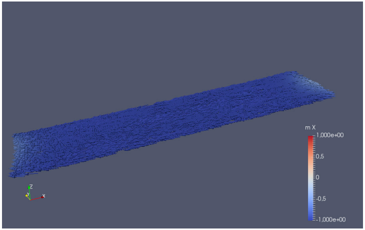

- FEM-part ⇝ `NGS/Py`
- BEM-part ⇝ `BEM++`
- Simulation tool for micromagnetics ⇝ `SolveLLG`

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Weak solution of LLG

Weak solution

- $\boldsymbol{m} \in \boldsymbol{H}1\Omega_T$, $|\boldsymbol{m}| = 1$ a.e. in Ω_T
- $\boldsymbol{m}l0r = \boldsymbol{m}^0$
- For all $\boldsymbol{\varphi} \in \boldsymbol{H}1\Omega_T$ it holds

$$\int_0^T \prod \partial_t \boldsymbol{m} \boldsymbol{\varphi} \,\mathrm{d}t = C_{\mathrm{ex}} \int_0^T \prod \boldsymbol{m} \times \nabla \boldsymbol{m} \nabla \boldsymbol{\varphi} \,\mathrm{d}t$$
$$- \int_0^T \prod \boldsymbol{m} \times \boldsymbol{\pi}(\boldsymbol{m}) \boldsymbol{\varphi} \,\mathrm{d}t + \alpha \int_0^T \prod \boldsymbol{m} \times \partial_t \boldsymbol{m} \boldsymbol{\varphi} \,\mathrm{d}t$$

- For all $\tau \in (0, T]$ it holds: $\mathcal{E}\left(\boldsymbol{m}(\tau)\right) + \alpha \int_0^\tau \|\partial_t \boldsymbol{m}\|^2_{\boldsymbol{L}2\Omega} \,\mathrm{d}t \leq \mathcal{E}\left(\boldsymbol{m}^0\right)$

Saddle-point system

Discrete variational formulation

$$\alpha\langle \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{m}_h^i \times \boldsymbol{v}_h^i\,,\,\boldsymbol{\varphi}_h\rangle + C_{\mathrm{ex}}k\langle \nabla \boldsymbol{v}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle$$
$$= -C_{\mathrm{ex}}\langle \nabla \boldsymbol{m}_h^i\,,\,\nabla\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{\pi}(\boldsymbol{m}_h^i)\,,\,\boldsymbol{\varphi}_h\rangle + \langle \boldsymbol{f}\,,\,\boldsymbol{\varphi}_h\rangle \quad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i)$$

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- Nodewise $\boldsymbol{m}_h^i(\boldsymbol{z}) \cdot \boldsymbol{v}_h^i(\boldsymbol{z}) = 0 \quad \rightsquigarrow$ Lagrange multiplier $\boldsymbol{\lambda} \in \mathbb{R}^N$

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Find $(\mathbf{v}, \boldsymbol{\lambda})$, s.t $\begin{pmatrix} \mathbf{A}(\boldsymbol{m}_h^i) & \mathbf{B}(\boldsymbol{m}_h^i)^T \\ \mathbf{B}(\boldsymbol{m}_h^i) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ \mathbf{0} \end{pmatrix}$

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- Solve system in $\mathbb{R}^{4N} \quad \rightsquigarrow \quad \boldsymbol{\mathcal{K}}_h(\boldsymbol{m}_h^i) \cong \mathbb{R}^{2N}$