On mathematical morphology, non-linear filters, and length scale control in topology optimization

Eddie Wadbro and Linus Hägg

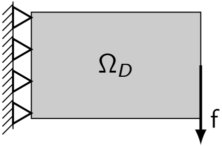
Department of Computing Science
Umeå University

AANMPDE 10: October 2-6, 2017 Paleochora, Crete, Greece

This work is financially supported by the Swedish Foundation for Strategic Research (No. AM13-0029) and the Swedish Research Council (No. 621-3706)

Material distribution (Topology optimization)

- Placement of material arbitrarily in region Ω_D
- Material distribution function ρ constant in each element



- $\rho = 0$ if void and 1 if solid
- Want to solve:

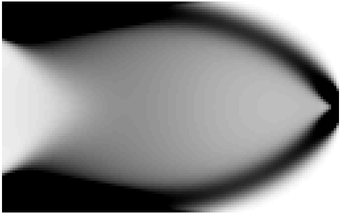
$$\min_{\rho} \quad J(\rho) \quad \text{(compliance)}$$

$$\text{s.t.} \quad \rho(1-\rho) = 0 \text{ a.e.}$$

$$\int \rho \leq V$$

governing PDE

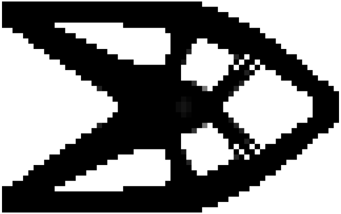
Typical approach—Relaxation



$$\min_{\rho} \; J(\rho)$$
$$\text{s.t. } 0 < \varepsilon \leq \rho \leq 1$$
$$\int \rho \leq V$$
governing PDE

Theorem For the continuous case, there exists a solution to the relaxed minimal compliance problem

Typical approach—Penalization

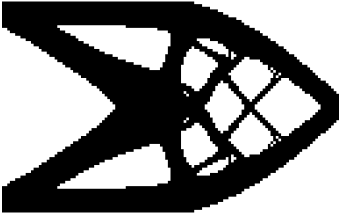


$$\min_{\rho} \; J(\rho^p)$$
$$\text{s.t. } 0 < \varepsilon \le \rho \le 1$$
$$\int \rho \le V$$
governing PDE

Theorem The continuous problem is ill-posed (it lacks solutions within the set of feasible designs)

Can also add explicit penalty term to the objective function

Typical approach—Penalization

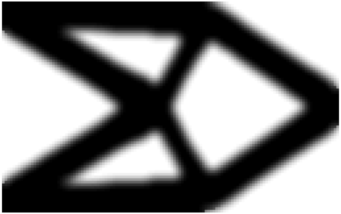


$$\min_{\rho} \; J(\rho^p)$$
$$\text{s.t. } 0 < \varepsilon \le \rho \le 1$$
$$\int \rho \le V$$
governing PDE

Theorem The continuous problem is ill-posed (it lacks solutions within the set of feasible designs)

Can also add explicit penalty term to the objective function

Typical approach—Filtering



$$\min_{\rho} \; J\big(F(\rho)^p\big)$$
$$\text{s.t. } 0 < \varepsilon \leq \rho \leq 1$$
$$\int F(\rho) \leq V$$
governing PDE

Here F is some averaging operator

Theorem For the continuous case: if $F(\rho)$ is a convolution product of a filter kernel ϕ and the density ρ, then there exists a solution to the relaxed minimal compliance problem (Bourdin 2001)

Quasi-arithmetic means (f-means)

- Arithmetic mean

  $$M_x(\boldsymbol{x};\boldsymbol{w}) = \boldsymbol{w}^T\boldsymbol{x} \equiv \sum_{i=1}^{m} w_i x_i$$

  $$\boldsymbol{w}^T\mathbf{1}_{\boldsymbol{m}} = 1$$

  $$w_i > 0$$

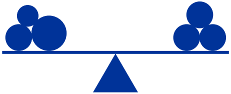
- Harmonic mean

  $$M_{x^{-1}}(\boldsymbol{x};\boldsymbol{w}) = \left(\boldsymbol{w}^T\boldsymbol{x}^{-\mathbf{1}}\right)^{-1}$$

- Geometric mean

  $$M_{\ln x}(\boldsymbol{x};\boldsymbol{w}) = \prod_{i=1}^{m} x_i^{w_i} \equiv \exp\left(\boldsymbol{w}^T\mathbf{ln}\,\boldsymbol{x}\right)$$

- ...



Quasi-arithmetic mean (f-mean)

$$M_f(\boldsymbol{x};\boldsymbol{w}) = f^{-1}\left(\boldsymbol{w}^T\boldsymbol{f}(\boldsymbol{x})\right) \iff f(M_f) = \boldsymbol{w}^T\boldsymbol{f}(\boldsymbol{x})$$

Properties of f-means

$M_f(\boldsymbol{x}) = M_f(\boldsymbol{x}; \frac{1}{m}\mathbf{1}_{\boldsymbol{m}})$
$\boldsymbol{x} \in [0,1]^m$

- P1 $M_f(\boldsymbol{x})$ is continuous and strictly increasing in each variable
- P2 $M_f(\boldsymbol{x})$ is *symmetric*, that is, $M_f(\boldsymbol{Px}) = M_f(\boldsymbol{x})$ for all permutation matrices $\boldsymbol{P} \in \mathbb{R}^{m \times m}$
- P3 $M_f(\boldsymbol{x})$ is *reflexive*, that is, for $c \in [0,1]$, we have $M_f(c\mathbf{1}_{\boldsymbol{m}}) = c$
- P4 $M_f(\boldsymbol{x})$ is *associative*, that is, for $k \in \{1, \ldots, m-1\}$, we have $M_f(x_1, \ldots, x_m) = M_f(c\mathbf{1}_k, x_{k+1}, \ldots, x_m)$, where $c = M_f(x_1, \ldots, x_k)$

Theorem (Kolmogorov 1930, Nagumo 1930)

Any sequence of functions satisfying P1–P4 is of the form $M_f(\boldsymbol{x}; \frac{1}{m}\mathbf{1}_{\boldsymbol{m}}) = f^{-1}\left(\frac{1}{m}\mathbf{1}_{\boldsymbol{m}}{}^T \boldsymbol{f}(\boldsymbol{x})\right)$ for some continuous function f

fW-mean filters

Replace the value of the design variable in one element with the f-mean of the values of its neighboring elements

fW-mean filter

- $\boldsymbol{F}(\rho) = \boldsymbol{f}^{-1}(\boldsymbol{W}\boldsymbol{f}(\rho))$
  $\boldsymbol{W} = [w_{ij}] \in \mathbb{R}^{n\times n}$
  $w_{ij} \geq 0$ and $\boldsymbol{W}\mathbf{1_n} = \mathbf{1_n}$
- $w_{ij} > 0$ iff $j \in \mathcal{N}_i \subset \{1, \dots, n\}$
- Replace f^{-1} with g, then a vast majority of available filters
  - Heaviside filter (Guest et al. 2004)
  - Morphology-based filters (Sigmund 2007)
  - Pythagorean mean based filters (Svanberg and Svärd 2014)

  can be handled in a similar manner
- Filters can be applied in a cascade: $\boldsymbol{F}^{(N)} \circ \boldsymbol{F}^{(N-1)} \circ \dots \circ \boldsymbol{F}^{(1)}$, where $\boldsymbol{F}^{(K)}(\rho) = \boldsymbol{f}_K^{-1}\left(\boldsymbol{W}^{(K)}\boldsymbol{f}_K(\rho)\right)$, $K \in \{1, \dots, N\}$

Existence of solutions: continuous min compliance

- Let $\Omega \subset \mathbb{R}^d$ be bounded and connected
  - Lipschitz boundary $\partial\Omega$
  - Structure is fixed at $\Gamma_D \subset \partial\Omega$
- Admissible displacements $\mathcal{U} = \left\{ u \in H^1(\Omega)^d \mid u|_{\Gamma_D} \equiv 0 \right\}$
- Design variable ρ
- Physical design $\tilde{\rho}(\rho) = \underline{\rho} + (1 - \underline{\rho}) P(F(\rho))$ in which
  - $\underline{\rho} > 0$
  - P is an invertible penalty function
  - $F(\rho)$ is a continuous version of the filtering
- Equilibrium displacement $u \in \mathcal{U}$ solves $a(\rho; u, v) = \ell(v) \ \forall v \in \mathcal{U}$
  - $\ell(v) = \int\limits_{\Omega} b \cdot v + \int\limits_{\Gamma_L} t \cdot v$
  - $a(\rho; u, v) = \int_\Omega \tilde{\rho}(\rho) E \epsilon(u) : \epsilon(v)$,

    where E is a constant elasticity tensor

Existence of solutions: continuous min compliance

- Admissible designs
  $$\mathcal{A} = \left\{ \rho \mid 0 \le \rho \le 1 \text{ a.e. on } \Omega,\ \int_\Omega F(\rho) \le V \right\} \subset L^\infty(\Omega)$$
- Continuous filtering: $\big(F(\rho)\big)(x) = f^{-1}\left(\frac{1}{|\mathcal{N}_x|} \int_{\mathcal{N}_x} (f \circ \rho)(y)\,\mathrm{d}y \right)$
  - $\mathcal{N}_x$ neighborhood of x with measure (area or volume) $|\mathcal{N}_x| > 0$
  - f smooth and invertible function $f : [0,1] \to [f_{\min}, f_{\max}] \subset \mathbb{R}$
- A standard problem formulation

  Find $\rho^* \in \mathcal{A}$ and $u^* \in \mathcal{U}$ such that
  $\ell(u^*) \le \ell(u)\ \forall u \in \mathcal{U}^*$ and $a(\rho^*; u^*, v) = \ell(v)\ \forall v \in \mathcal{U}$
- Alternative equivalent problem formulation

$$\text{Find } u^* \in \mathcal{U}^* \text{ such that } \ell(u^*) = \inf_{u \in \mathcal{U}^*} \ell(u), \tag{1}$$

$$\text{where}\quad \mathcal{U}^* = \left\{ u \in \mathcal{U} \;\middle|\; \begin{array}{l} \exists \rho \in \mathcal{A} \text{ such that} \\ a(\rho; u, v) = \ell(v)\ \forall v \in \mathcal{U} \end{array} \right\}$$

Existence of solutions: continuous min compliance

Theorem If $|\mathcal{N}_x| > 0$ for all $x \in \Omega$, then there exists a solution to problem (1)

Proof (melody) For details, see Hägg & Wadbro (2017)

- Pick minimizing sequence (u_m), $u_m \in \mathcal{U}^*$
  - let (ρ_m) sequence so that $a(\rho_m; u_m, v) = \ell(v) \ \forall v \in \mathcal{U}$
  - bilinear form a coercive so subsequence (u_m) converges weakly to u^* in $H^1(\Omega)^d$

Existence of solutions: continuous min compliance

Theorem If $|\mathcal{N}_x| > 0$ for all $x \in \Omega$, then there exists a solution to problem (1)

Proof (melody) For details, see Hägg & Wadbro (2017)

- Pick minimizing sequence (u_m), $u_m \in \mathcal{U}^*$
  - let (ρ_m) sequence so that $a(\rho_m; u_m, v) = \ell(v) \;\; \forall v \in \mathcal{U}$
  - bilinear form a coercive so subsequence (u_m) converges weakly to u^* in $H^1(\Omega)^d$
- Define $\tau_m = f \circ \rho_m \in L^\infty(\Omega)$
  - Banach–Agaoglu: subsequence (τ_m) converges weak* to τ^*
  - ...define $\rho^* = f^{-1} \circ \tau^*$.
  - Banach–Agaoglu: $F(\rho_m) \to F(\rho^*)$ pointwise
  - Lebesques dominated convergece theorem: $\rho^* \in \mathcal{A}$

Existence of solutions: continuous min compliance

Theorem If $|\mathcal{N}_x| > 0$ for all $x \in \Omega$, then there exists a solution to problem (1)

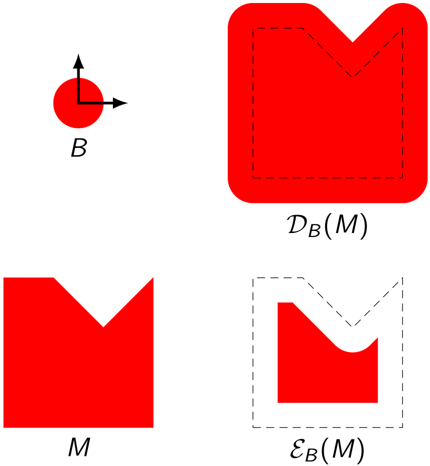
Proof (melody) For details, see Hägg & Wadbro (2017)

- Pick minimizing sequence (u_m), $u_m \in \mathcal{U}^*$
  - let (ρ_m) sequence so that $a(\rho_m; u_m, v) = \ell(v)\ \forall v \in \mathcal{U}$
  - bilinear form a coercive so subsequence (u_m) converges weakly to u^* in $H^1(\Omega)^d$
- Define $\tau_m = f \circ \rho_m \in L^\infty(\Omega)$
  - Banach–Agaoglu: subsequence (τ_m) converges weak* to τ^*
  - ...define $\rho^* = f^{-1} \circ \tau^*$.
  - Banach–Agaoglu: $F(\rho_m) \to F(\rho^*)$ pointwise
  - Lebesques dominated convergece theorem: $\rho^* \in \mathcal{A}$
- ...some further arguments to show that
  - $a(\rho_m; u_m, v) - a(\rho^*; u^*, v) \to 0$ for any $v \in \mathcal{U}$
- Thus $u^* \in \mathcal{U}$ and since ℓ is linear & bounded $\ell(u_m) \to \ell(u^*)$

□

Morphological operators (review: Heijmans 1995)

Gain information about a set M by probing it by a convex set B

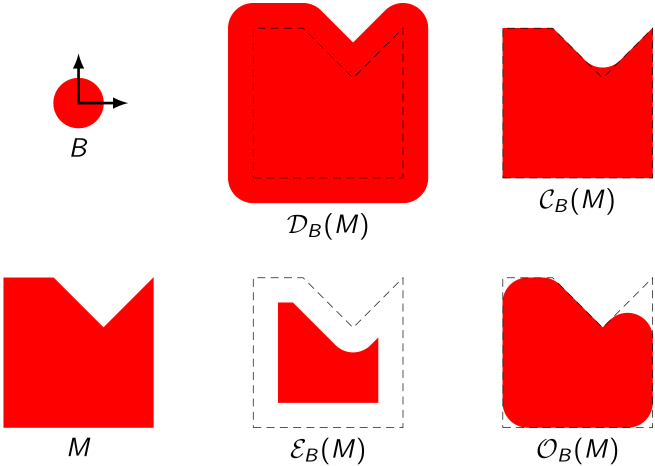


Dilation: $\mathcal{D}_B(M) = \{m + b \mid m \in M, b \in B\} = \bigcup_{b \in B}(b + M)$

Erosion: $\mathcal{E}_B(M) = \left(\mathcal{D}_{-B}(M^{\complement})\right)^{\complement}$

Morphological operators (review: Heijmans 1995)

Gain information about a set M by probing it by a convex set B



Closing: $\mathcal{C}_B(M) = \mathcal{E}_B(\mathcal{D}_B(M))$

Opening: $\mathcal{O}_B(M) = \mathcal{D}_B(\mathcal{E}_B(M))$

$$\mathcal{O}_B(M) \subset M \subset \mathcal{C}_B(M)$$

Definition of the minimum length scale of M

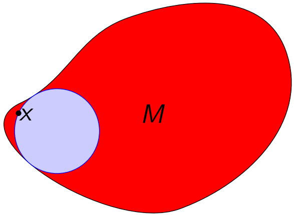
- M open
- B is the open unit ball for some metric on $\mathbb{R}^d$

Local length scale

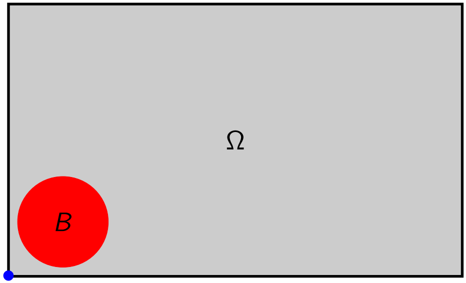
- $R_B(M; x) = \sup\{r > 0 \mid \exists y \in M \text{ s.t. } x \in y + rB \subset M\}$
  - Radius of "largest" ball in M containing x
  - $R_B(M; x) > 0$

Minimum length scale

- $R_B(M) = \inf_{x \in M} R_B(M; x)$
  - "Smallest" local length scale

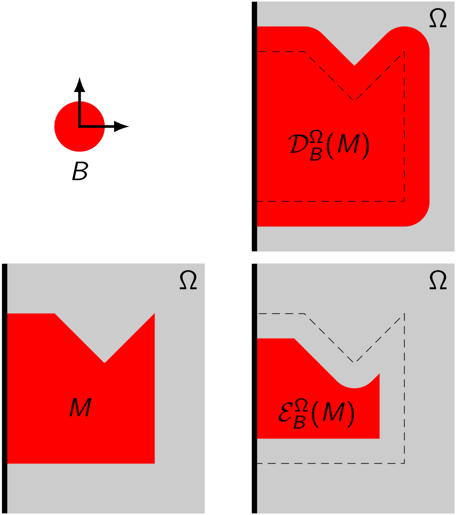


Problem with length scale $R_B(M)$



- The design domain Ω is typically a hyperrectangle
- B is often an open Euclidian ball
- $R_B(\Omega, x) \to 0$ as $x \to \bullet \implies R_B(\Omega) = 0$
- So $M \subset \Omega$ and $V = \Omega \setminus \overline{M}$ cannot both possess minimum length scale (w.r.t. R_B)

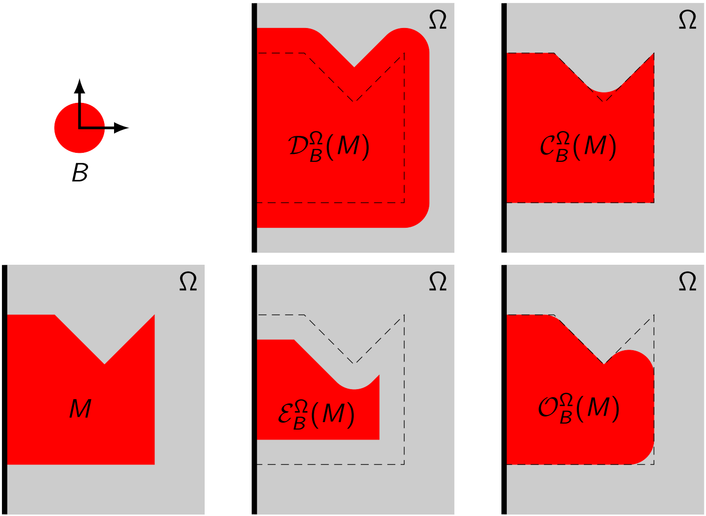
Morphological operators—bounded domain



Dilation: $\mathcal{D}_B^\Omega(M) = \mathcal{D}_B(M) \cap \Omega$

Erosion: $\mathcal{E}_B^\Omega(M) = \mathcal{E}_B(\Omega^\complement \cup M) \cap \Omega$

Morphological operators—bounded domain



Closing: $\mathcal{C}_B^\Omega(M) = \mathcal{E}_B^\Omega(\mathcal{D}_B^\Omega(M))$

Opening: $\mathcal{O}_B^\Omega(M) = \mathcal{D}_B^\Omega(\mathcal{E}_B^\Omega(M))$

$$\mathcal{O}_B^\Omega(M) \subset M \subset \mathcal{C}_B^\Omega(M)$$

Minimum length scale of M relative Ω

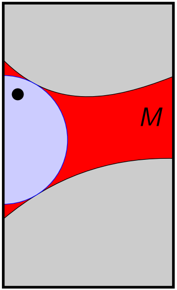
- M and Ω open
- B is the open unit ball for some metric on $\mathbb{R}^d$

Local length scale

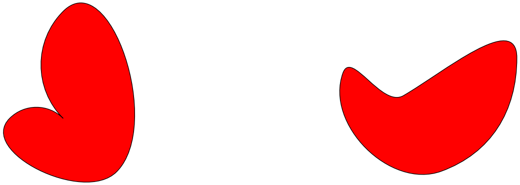
- $R_B^\Omega(M;x) = \sup\{r > 0 \mid E_{rB}^\Omega(M;x) \neq \emptyset\}$
  - $E_{rB}^\Omega(M;x) = \{y \in M \mid x \in (y + rB) \cap \Omega \subset M\}$.

Minimum length scale

- $R_B^\Omega(M) = \inf_{x \in M} R_B^\Omega(M;x)$



B-open and B-regular sets



- M is B-open (relative Ω) iff $M = \mathcal{O}_B^{\Omega}(M)$
- M is B-regular (rel. Ω) iff M and $V = \Omega \setminus \overline{M}$ both are B-open

This extends the work on r-regular sets: B Euclidian ball with radius $r > 0$ and $\Omega = \mathbb{R}^d$ (Serra 1982, Pavlidis 1982)

Math. morphology ∼ minimum length scale

Theorem 1

If $M \neq \emptyset$ *is* rB*-open relative* Ω *for some* $r > 0$*, then* $R_B^\Omega(M) \geq r$

Theorem 2

If $M \neq \emptyset$ *and* $R_B^\Omega(M) > 0$*, then* M *is* rB*-open for any* r *satisfying* $0 < r < R_B^\Omega(M)$

Theorem 3 (Alternative definition of $R_B^\Omega(M)$)

If $M \neq \emptyset$*, then* $R_B^\Omega(M) = sup\{r > 0 \mid M = \mathcal{O}_{rB}^\Omega(M)\}$

- *Convention:* $\sup \emptyset = 0$

Math. morphology ~ minimum length scale

Theorems 1–3 interrelates B-open and B-regular sets to sets whose interior and/or exterior exhibit positive minimum length scales

Natural generalization: $M = \mathcal{O}_B^\Omega(M)$ and $V = \mathcal{O}_{\hat{B}}^\Omega(V)$ for $B \neq \hat{B}$

- Duality $\implies$ $V = \mathcal{O}_{\hat{B}}^\Omega(V)$ iff $\overline{M} \cap \Omega = \mathcal{C}_{\hat{B}}^\Omega(\overline{M} \cap \Omega)$

Minimum length scale constraints

$$M = \mathcal{O}_{rB}^\Omega(M) \implies R_B^\Omega(M) \geq r$$

$$\overline{M} \cap \Omega = \mathcal{C}_{\hat{r}\hat{B}}^\Omega(\overline{M} \cap \Omega) \implies R_{\hat{B}}^\Omega(V) \geq \hat{r}$$

Mathematical morphology for density based topology optimization

- Ω discretized using a regular grid
  - x_i centroid of element i
- $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)^T \in [0,1]^n$
  - ρ_i determines the material state of element i
- $\mathcal{D}_i(\boldsymbol{\rho}) = \max_{j \in \mathcal{N}_i} \rho_j$ and $\mathcal{E}_i(\boldsymbol{\rho}) = \min_{j \in \mathcal{N}_i} \rho_j$
  - Neighborhoods $\mathcal{N}_i = \{j \mid x_j - x_i \in rB\}$, $r > 0$

Minimum length scale constraints

$$\boldsymbol{\mathcal{O}}_{rB}(\boldsymbol{\rho}) = \boldsymbol{\mathcal{C}}_{\hat{r}\hat{B}}(\boldsymbol{\rho})$$

$$\boldsymbol{\rho}^T(\mathbf{1} - \boldsymbol{\rho}) = 0$$

Note that by definition $\boldsymbol{\mathcal{O}}_{rB}(\boldsymbol{\rho}) \leq \boldsymbol{\rho} \leq \boldsymbol{\mathcal{C}}_{\hat{r}\hat{B}}(\boldsymbol{\rho})$

Heuristic method for compliance problems

- Physical density $\mathcal{P}(\boldsymbol{\rho}) = \underline{\rho} + (1 - \underline{\rho})\boldsymbol{\mathcal{O}}_h(\boldsymbol{\rho})^p$
  - $\underline{\rho}$ small positive parameter
  - p SIMP penalty parameter
  - $\boldsymbol{\mathcal{O}}_h$ is an approximation of $\mathcal{O}_B^\Omega$
- Admissible designs
  $\mathcal{A} = \left\{\boldsymbol{\rho} \in \mathbb{R}^n \mid \mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1} \text{ and } \boldsymbol{v}^T\boldsymbol{\mathcal{C}}_h(\boldsymbol{\rho}) \leq V^*\right\}$
  - $\boldsymbol{\mathcal{C}}_h$ is an approximation of $\mathcal{C}_{\hat{B}}^\Omega$.
  - $\boldsymbol{v} \in \mathbb{R}^n$ holds the fractional volume $(|E|/|\Omega|)$ of the elements
  - V^* is the maximum volume fraction

Quality measures

Measure of non-discreteness (suggested by Sigmund 2007):

$$M_{\text{ND}} = \frac{4}{n}\widetilde{\mathcal{P}}(\boldsymbol{\rho})^T\left(\mathbf{1} - \widetilde{\mathcal{P}}(\boldsymbol{\rho})\right)$$

where $\widetilde{\mathcal{P}}(\boldsymbol{\rho})$ is the physical design

Two new quality measures

Measure of difference between open and close:

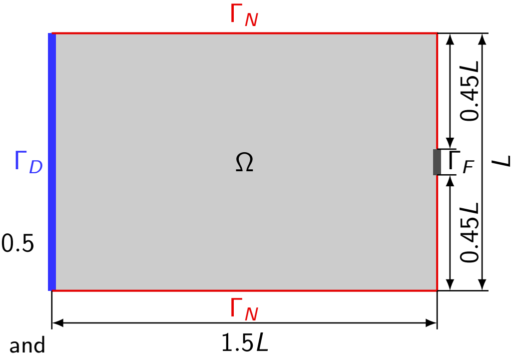
$$M_{\text{DOC}} = \frac{1}{n}\left\|\mathcal{C}(\boldsymbol{\rho}) - \mathcal{O}(\boldsymbol{\rho})\right\|_1$$

A related quality measure is

$$F_{\text{DOC}} = \frac{1}{n}\mathrm{card}\left\{i \mid \left(\mathcal{C}(\boldsymbol{\rho}) - \mathcal{O}(\boldsymbol{\rho})\right)_i > 0.5\right\}$$

If $M_{\text{ND}} = M_{\text{DOC}} = 0$, then we have a binary design with minimum size control of both materials

Cantilever beam

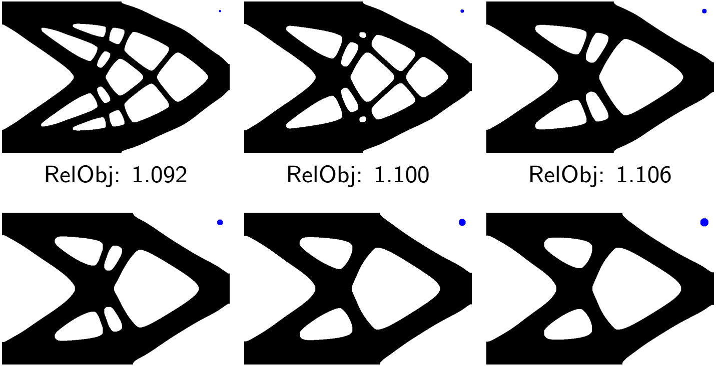


- Fixed at Γ_D
- Downward force uniformly over Γ_F
- OC damping $\eta = 1/2$
- Volume fraction $V^* = 0.5$
- Harmonic *fW*-mean filters with
  - $f_{\mathcal{E}^H_\alpha}(x) = (x + \alpha)^{-1}$ and
  - $f_{\mathcal{D}^H_\alpha}(x) = f_{\mathcal{E}^H_\alpha}(1 - x)$

  approximate the erode and dilate operator, respectively (Svanberg & Svärd 2013)
- Continuation for SIMP penalty p and filter parameter α
- Solved using a standard desktop computer
- Modified version of multigrid-CG code by Amir et al. (2014)

Cantilever beam

768×512 elements
Continuation approach first $\alpha = 10$ and $p = 1, 1.5, \ldots, 3$;
then $p = 3$ and $\alpha = 10^{1-m/2}$ for $m = 1, 2, \ldots, 18$



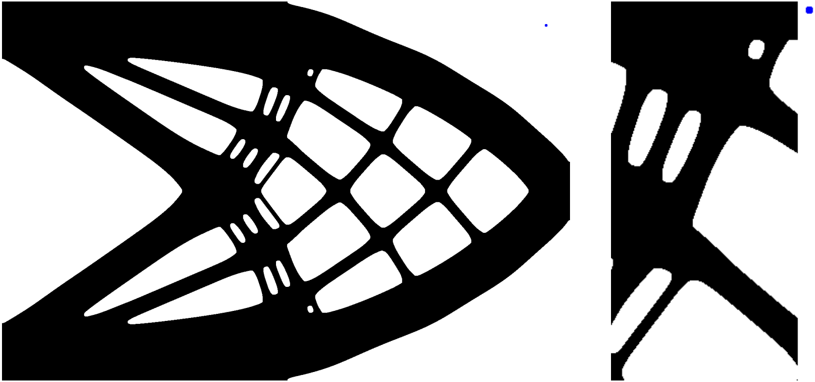
RelObj: 1.092 | RelObj: 1.100 | RelObj: 1.106

RelObj: 1.116 | RelObj: 1.119 | RelObj: 1.125

$M_{ND} < 1.2 \cdot 10^{-5}$ %, $M_{DOC} < 5.0 \cdot 10^{-6}$ %, and $F_{DOC} = 0$

Cantilever beam—mesh convergence

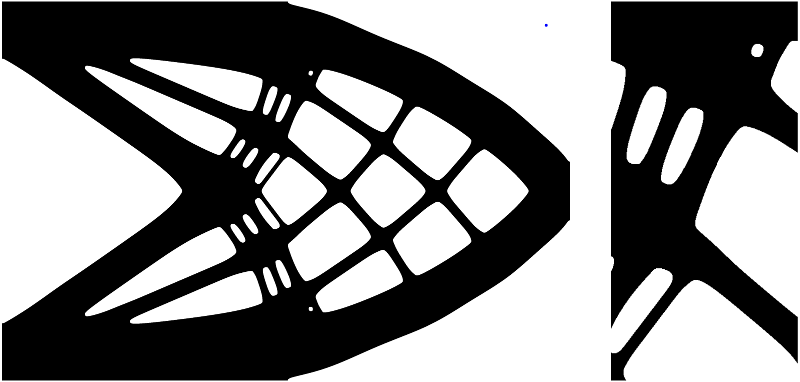
1536×1024 elements
Continuation approach first $\alpha = 10$ and $p = 1, 1.5, \ldots, 3$;
then $p = 3$ and $\alpha = 10^{1-m/2}$ for $m = 1, 2, \ldots, 18$



$M_{ND} < 1.3 \cdot 10^{-6}$ %, $M_{DOC} < 5.0 \cdot 10^{-7}$ %, and $F_{DOC} = 0$

Cantilever beam—mesh convergence

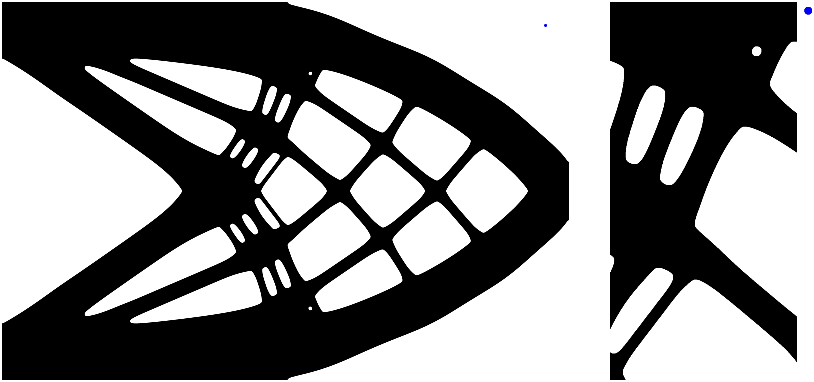
3072×2048 elements
Continuation approach first $\alpha = 10$ and $p = 1, 1.5, \ldots, 3$;
then $p = 3$ and $\alpha = 10^{1-m/2}$ for $m = 1, 2, \ldots, 18$



$M_{ND} < 2.3 \cdot 10^{-6}\,\%$, $M_{DOC} < 1.0 \cdot 10^{-6}\,\%$, and $F_{DOC} = 0$

Cantilever beam—mesh convergence

6144×4096 elements
Continuation approach first $\alpha = 10$ and $p = 1, 1.5, \ldots, 3$;
then $p = 3$ and $\alpha = 10^{1-m/2}$ for $m = 1, 2, \ldots, 18$

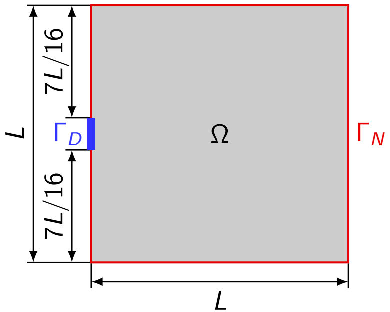


$M_{ND} < 3.9 \cdot 10^{-6}\,\%$, $M_{DOC} < 1.8 \cdot 10^{-6}\,\%$, and $F_{DOC} = 0$

Minimum heat compliance

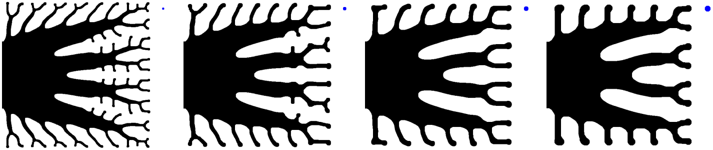
- Fixed at Γ_D
- Uniform force distrubuted over Ω
- OC damping $\eta = 0.5$
- Volume fraction $V^* = 0.5$
- Harmonic fW-mean filters with
  - $f_{\mathcal{E}^H_\alpha}(x) = (x+\alpha)^{-1}$ and
  - $f_{\mathcal{D}^H_\alpha}(x) = f_{\mathcal{E}^H_\alpha}(1-x)$

  approximate the erode and dilate operator, respectively (Svanberg & Svärd 2013)
- Continuation for SIMP penalty p and filter parameter α
- Solved using a standard desktop computer

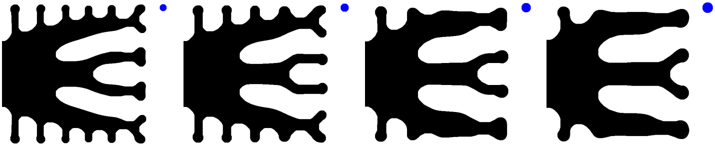


Minimum heat compliance

512×512 elements
Continuation approach first $\alpha = 10$ and $p = 1, 1.5, \ldots, 3$;
then $p = 3$ and $\alpha = 10^{1-m/2}$ for $m = 1, 2, \ldots, 18$



RelObj: 1.225 RelObj: 1.363 RelObj: 1.475 RelObj: 1.623



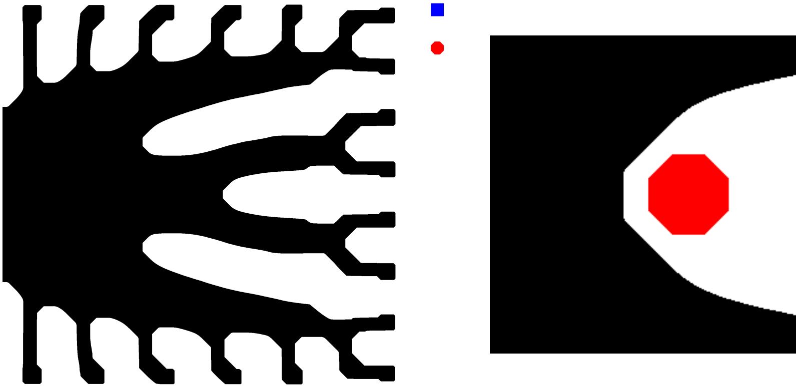
RelObj: 1.802 RelObj: 1.977 RelObj: 2.259 RelObj: 2.433

$M_{ND} < 0.14\,\%$, $M_{DOC} < 0.018\,\%$, and $F_{DOC} \leq 4/512^2$

Minimum heat compliance—different neighborhoods

2028×2048 elements

Continuation approach first $\alpha = 10$ and $p = 1, 1.5, \ldots, 3$;
then $p = 3$ and $\alpha = 10^{1-m/2}$ for $m = 1, 2, \ldots, 18$



$M_{ND} < 0.012\,\%$, $M_{DOC} < 0.0030\,\%$, and $F_{DOC} = 24/2048^2$