Uniqueness of inverse boundary value problem for dynamical anisotropic elasticty systems

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Collaborators of this study are

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Outline of my talk

- Anisotropic elasticity and ND map, localized DN map
- Global uniqueness for identifying piecewise homogeneous/analytic density and elasticity tensor with known interfaces
- Some important ingredients of the proof
- Global uniqueness for identifying piecewise homogeneous/analytic density and elastic tensor with unknown interfaces
- Key tool for the proof: theory of subanalytic sets

Anisotropic elasticity and its equation

- $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$ bounded domain with smooth boundary $\partial\Omega$

- *elasticity tensor*: $C = (C_{ijkl}(x)) \in L^\infty(\Omega)$;
  for any a.e. $x \in \overline{\Omega}$ and indices $i, j, k, l \in \{1, 2, 3\}$, it satisfies

$$\textit{symmetry} \begin{cases} C_{ijkl}(x) = C_{ijlk}(x) \ (\textit{minor symmetry}), \\ C_{ijkl}(x) = C_{klij}(x) \ (\textit{major symmetry}) \end{cases}$$

- *strong convexity*:

  There exists $\delta > 0$ s.t. for any symmetric matrix $\epsilon = (\epsilon_{ij})$,
  $\epsilon : (C :: \epsilon) = \sum_{i,j,k,l=1}^{3} C_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq \delta(\epsilon : \epsilon)$ (a.e. $x \in \Omega$)

- e.g. *isotropic* case: $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$

Continued

- the *displacement* (column) vector $u = (u_1, u_2, u_3)$: satisfies the *elasticity equations of system*

$$\begin{aligned}(\rho\partial_t^2 u - L_C u)_i &= \left(\rho\partial_t^2 u - \operatorname{div}(C :: \nabla u)\right)_i \\ &:= \rho\partial_t^2 u_i - \sum_{j,k,l=1}^{3} \partial_j \left(C_{ijkl}(x)\partial_l u_k\right) = 0 \text{ in } \Omega_T = \Omega \times (0,T),\end{aligned}$$

  where $\partial_j = \partial/(\partial x_j)$, $0 < \rho_0 \leq \rho \in L^\infty(\Omega)$ is the density, where ρ_0 is a constant.

- $$\partial\Omega = \overline{\Gamma^D} \cup \overline{\Gamma^N},$$

  where Γ^D, $\Gamma^N \subset \partial\Omega$: open, connected, nonempty, disjoint sets with smooth boundaries.

Forward problem

Consider

$$
\text{(MP)} \begin{cases} (\rho\partial_t^2 u - L)u = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma_T^D, \\ \partial_L u := (C :: \nabla u)\nu = f \in H_0^2((0,T); H^{-1/2}(\Gamma^N)) & \text{on } \Gamma_T^N, \\ u\big|_{t=0} = 0,\ \partial_t u\big|_{t=0} = 0, & \text{in } \Omega, \end{cases} \tag{1}
$$

where $\Gamma_T^D := \Gamma^D \times (0,T)$, $\Gamma_T^N := \Gamma^N \times (0,T)$ and ν is the outer unit normal of $\partial\Omega$.

Well-posedness of (MP) :
$^{\exists!}u \in C^0([0,T]; H^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$: solution to (MP) such that

$$
\|u(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot,t)\| \lesssim \|f\|_{H^2((0,T);H^{-1/2}(\Gamma^N))},\ t \in [0,T].
$$

Neumann-to-Dirichlet map=ND map

- **ND map** Λ^T

  $$\Lambda^T : H_0^2((0,T); H^{-1/2}(\Gamma^N) \ni f \mapsto u^f\big|_{\Gamma_T^N} \in C^0([0,T]; H_0^{1/2}(\overline{\Gamma^N}))$$

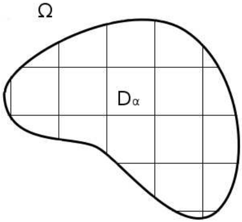
  with the solution $u = u^f$ of (MP). Also, define $\Lambda := \Lambda^T$ with $T = \infty$.

- **influence domain** $\tilde{E}^T$ is the maximal subdomain of Ω in which all the solutions in $C^0([0,2T]; H^1(\Omega)) \cap C^1([0,2T]; L^2(\Omega))$ of the equation of (MP) become zero at $t = T$ if their Cauchy data on Γ_{2T}^N are zero.

- **filling time** $T^* := \inf\{T > 0 : \tilde{E}^T = \Omega\}$.

- **extension of ND map**: If the equation of (MP) has the unique continuation property=UCP of solutions, then $T^* < \infty$ exists and Λ^{2T} with $T > T^*$ can be extended to Λ.

Further assumptions on the density and elastic tensor



- finitely many subdomains $D_\alpha \subset \Omega$, $\alpha \in A$, s.t.

$$\bar{\Omega} = \cup_{\alpha \in A} \overline{D_\alpha}, \quad D_\alpha \cap D_\beta = \emptyset \text{ if } \alpha \neq \beta$$

  $\{D_\alpha\}_{\alpha \in A} / \{\overline{D_\alpha}\}_{\alpha \in A}$ and each ∂D_α are called *cover* of Ω and *interface*, respectively. Each interface is piecewise analytic.

- ρ, C are analytic up to the boundary in each D_α.

  Refer these as saying (ρ, C) is piecewise analytic.

Inverse problem and its reduction

Inverse problem: Show the uniqueness of identifying the density ρ and elasticity tensor C by knowing Λ^T.

This is a typical question asked for the vibroseis exploration technique in reflection seismology.

By the piecewise analyticity of (ρ, C), we have UCP by the Holmgren uniqueness theorem. Hence there is a filling time $T^* < \infty$ and Λ^{2T} with $T > T^*$ can be extended to Λ.

Take $f = t^2 g(x)$ with $g \in H^{-1/2}(\Gamma^N)$. Then due to $(\mathcal{L}t^2)(\tau) := \int_0^\infty e^{-\tau t} t^2\, dt = 2\tau^{-3}$ for $\tau > 0$, we can reduce the inverse problem as follows by taking the Laplace transform of the equation and boundary condition of (MP) with respect to t.

Continued

Intermediate reduced inverse problem: Show the uniqueness of identifing the density ρ and elasticity tensor C by knowing the ND maps Λ_τ for $\tau > 0$ defined as

$$\Lambda_\tau : H^{-1/2}(\Gamma^N) \ni g \mapsto v_\tau^g \in H_0^{1/2}(\overline{\Gamma^N}),$$

where $v = v_\tau^g \in H^1(\Omega)$ is the solution to the boundary value problem (BP):

$$\text{(BP)} \begin{cases} M_\tau v := (\rho\tau^2 - L)v = 0 \text{ in } \Omega, \\ v = 0 \text{ on } \Gamma^D,\ \partial_L v = g \text{ on } \Gamma^N. \end{cases}$$

Remark Let Φ_τ be the localized Dirichlet to Neumann (=DN) map on Γ^N, then Λ_τ and Φ_τ are inverse to each other. Here $\Phi_\tau h = \partial_L w_\tau^h$ on Γ^N for $h \in H_0^{1/2}(\overline{\Gamma^N})$ with $w = w_\tau^h$ solving

$$M_\tau w = 0 \text{ in } \Omega,\ \ w = h \text{ on } \partial\Omega.$$

Reduced inverse problem

Finally we have reduced the inverse problem to

Reduced inverse problem: Show the uniqueness of identifying the density ρ and elasticity tensor C by knowing the localized DN maps Φ_τ for $\tau > 0$.

Consider two cases for the interfaces.
(i) Interfaces are known.
(ii) Interfaces are unknown.

Consider three cases for piecewise analytic pair (ρ, C).
(i) (ρ, C) is piecewise homogeneous. i.e. (ρ, C) is homogeneous in each subdomain.
(ii) C is transversally isotropic.
(iii) C is orthorhombic.

Transversally isotropic/orthorhombic elasticity tensor

(i) If x_3 axis coincide with the axis of symmetry, the 5 non-zero components of transversally isotropic elastic tensor C are

C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{2233}, C_{2323}, C_{1313}, C_{1212}

with relations

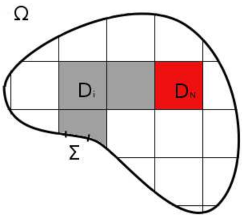
$$C_{1111} = C_{2222}, \quad C_{1133} = C_{2233},$$

$$C_{2323} = C_{1313}, \quad C_{1212} = \frac{1}{2}(C_{1111} - C_{1122}).$$

(ii) If the coordinates (x_1, x_2, x_3) are aligned within the symmetry planes, the 9 non-zero components of orthorhombic elastic tensor C are

C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{2233}, C_{2323}, C_{1313}, C_{1212}.

Main result assuming interfaces are known



- for any $\alpha \in A$, there exists a *chain* $D_i := D_{\alpha_i}$ $(i = 1, \ldots, N)$ with $\alpha_N = \alpha$ and nonempty surfaces $\Gamma_i \subset \partial D_i$ s.t. $\Gamma_1 = \Sigma = \Gamma^N$, and $\bar{D}_i \cap \bar{D}_{i+1} \supset \Gamma_{i+1}$ $(i = 1, \ldots, N-1)$

Let $(\rho_j, C^{(j)})$, $j = 1, 2$ be the two pairs of density and elastic tensor with the same known interfaces.
Denote the localized DN maps for them by $\Phi_{\tau,j}$, $j = 1, 2$. Then we have the following theorem.

Continued and some previous works

Theorem
If $\Phi_{\tau,1} = \Phi_{\tau,2}$ *for* $\tau > 0$*, then this implies* $(\rho_1, C^{(1)}) = (\rho_2, C^{(2)})$ *in* D_N*, i.e. the* global uniqueness *for the following cases:*

(i) (ρ, C) *is piecewise homogeneous and each* Γ_i *has nonzero curvature in its some open subset* (curvature condition)*.*

(ii) (ρ, C) *is piecewise analytic and* C *is either transversally isotropic with known symmetric axis or orthorhombic with known symmetry planes.*

Remark The regularity assumption on the interfaces can be relaxed to piecewise smooth when we know the interfaces.

Continued

Some previous works for the case $\tau = 0$

(i) [Kohn, Vogelius, 1984, 1985] uniqueness for scalar conductivity even with unknown interfaces for 2D.

(ii) [Alessandrini, de Hoop, Gaburro, 2016] uniqueness for anisotropic conductivity

(iii) [Carstea, Honda, Nakamura, 2017] uniqueness for piecewise homogeneous anisotropic elasticity even with unknown interfaces.

We will adapt the argument developed in (iii).

Structure of the argument

- Determination at the boundary: Φ_τ for $\tau > 0$ determine (ρ, C) and if it is not piecewise homogeneous, also its all the derivatives at Σ.

- Interior determination: iteratively show that

$$(\rho_1,\, C^{(1)})\big|_{D_i} = (\rho_2,\, C^{(2)})\big|_{D_i}$$
$$\Longrightarrow (\rho_1,\, C^{(1)})\big|_{D_{i+1}} = (\rho_2,\, C^{(2)})\big|_{D_{i+1}}.$$

  - usually this has been done by using Green functions or singular solutions similar to these
  - for elliptic systems, Green functions are not always known to exist with necessary properties
  - we use a more abstract method adapted from [Ikehata; 2002]

Determination at the boundary

Theorem

$(\rho, C)|_{D_1}$ can be recovered from $\Phi_\tau,\ \tau > 0$ for the following cases.

(i) *(ρ, C) is piecewise homogeneous and Σ satisfies the curvature condition.*

(ii) *(ρ, C) is piecewise analytic and C is either transversally isotropic with known symmetric axis or orthorhombic with known symmetry planes.*

Some important ingredients for the proof

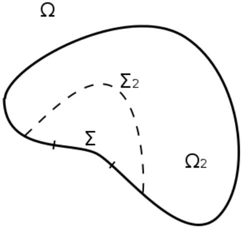
for (ii) Φ_τ is a pseudodifferential operator with large parameter τ. Its principal symbol $Z = Z(x', \xi', \tau)$ with $(x', \xi') \in T^*(\Sigma)$ is called the surface impedance tensor. The normal derivatives of $Z(x', \xi', \tau)$ can be recovered from the full symbol of Φ_τ.

Continued

Z only depends on $\xi' = m \times n$ with $m \perp n$ and $n =$ unit normal of Σ (referred as (*)). Using these and by an explicit computations we have (ii).

for (i) Let $\Gamma(x, \tau)$ be the fundamental solution of M_τ. Then at each fix point of Σ, $\Gamma(x, \tau)$ for $x \perp n$ can be recovered from Z (cf. (*)). Since Σ satisfies the curvature condition, we can recover $\Gamma(x, \tau)$ for x in an open subset of $\mathbb{R}^3$ by doing this recovery along Σ. Then due to the analyticity of $\Gamma(x, \tau)$ with respect to $x \in \mathbb{R}^3$, we can recover $\Gamma(x, \tau)$ and hence also (ρ, C).

Interior uniqueness: adapting Ikehata's argument



- $\Omega_2 = \Omega \setminus \bar{D}_1$, $\Sigma_2 = \partial\Omega_2 \setminus \overline{\partial\Omega}$; we write $\rho = \rho_0 + \chi_{\Omega_2}\rho'$, $C = C^0 + \chi_{\Omega_2}C'$
- it is enough to show that $\Phi_\tau = \Phi_{\tau,\Sigma}$ determines Φ_{τ,Σ_2}
- Green operator $G : H^{-1}(\Omega) \to H_0^1(\Omega)$: the inverse of $-L_C$ with Dirichlet boundary conditions
- if $f \in H^{-1/2}(\Sigma_2)$ define $T_f \in H^{-1}(\Omega)$ by

$$T_f(\phi) = \langle f, \phi|_{\Sigma_2} \rangle \text{ for any } \phi \in H_0^1(\Omega)$$

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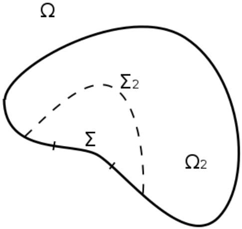

- *single layer operator* $S^{\Sigma_2} : H^{-1/2}(\Sigma_2) \to H_0^{1/2}(\overline{\Sigma_2})$,

$$S^{\Sigma_2}(f) = (GT_f)|_{\Sigma_2}$$

- by *Runge approximation* ($\Longleftarrow$ *UCP*), (ρ_0, C^0) and $\Phi_{\tau,\Sigma}$ determine $H(GF)$ for any $F, H \in H^{-1}(\Omega)$, $\text{supp } F, \text{supp } H \subset \Omega \setminus \bar{\Omega}_2$

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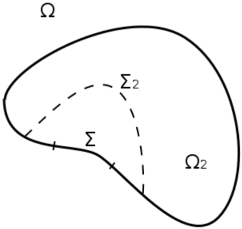

- limiting procedure $\longrightarrow C^0$ and $\Phi_{\tau,\Sigma}$ determine S^{Σ_2}
- Φ^+_{τ,Σ_2} the loc. DN map in the domain $\Omega \setminus \bar{\Omega}_2$
- $\Phi_{\tau,\Sigma_2} + \Phi^+_{\tau,\Sigma_2}$ is injective (i.e. one to one)
- $(\Phi_{\tau,\Sigma_2} + \Phi^+_{\tau,\Sigma_2})S^{\Sigma_2} f = f$ for any $f \in H^{-1/2}(\Sigma_2)$
- $\Rightarrow \Phi_{\tau,\Sigma_2} + \Phi^+_{\tau,\Sigma_2} = (S^{\Sigma_2})^{-1}$

Main result without assuming interfaces are known

Definition

Let $\{D_\alpha\}_{\alpha\in A}$ be a cover with piecewise analytic interfaces.

(i) (ρ, C) and the associated cover are called piecewise homogeneous if (ρ, C) is constant in each D_α.

(ii) If the analytic part of each interfaces have non-zero curvature, we say that the strong curvature condition is satisfied.

Theorem

The global uniqueness holds for the following cases.

(i) (ρ, C) *is piecewise homogeneous and unknown surfaces satisfy the strong curvature condition.*

(ii) (ρ, C) *is piecewise analytic and* C *is either transversally isotropic with known symmetric axis or orthorhombic with known symmetric planes.*

Key lemma

Key is to generalize Lemma 1 of [Kohn-Vogelius ; 1985] given for the two dimensional (isotropic conductivity) case to the three dimesional case by using the *theory of subanalytic sets*.

Lemma

$(\rho_1, C_1), (\rho_2, C_2)$: *picewise analytic* $\Longrightarrow$
there exists a piecewise analytic cover (refered as common cover) such that $(\rho_1, C_1), (\rho_2, C_2)$ *are piecewise analytic with respect to this cover.*
If the covers of $(\rho_1, C_1), (\rho_2, C_2)$ *satisfy the strong curvature condition, then this cover satisfy the strong curvature condition.*

Once having this, the proof of the second global uniqueness theorem can be done by just repeating the previous proof of the first global uniqueness theorem for the common cover.

A tool to prove the key lemma

Let X be a real analytic manifold countable at infinity.

- $Z \subset X$ is said to be semi-analytic if, for any point $x \in \overline{Z}$, there exists an open neighborhood V of x satisfying

$$Z \cap V = \bigcup_i \bigcap_j \{x \in V \ : \ f_{ij}(x) \ \ *_{ij} \ \ 0\}$$

  for a finite number of analytic functions f_{ij} on V. Here the binary relation $*_{ij}$ is either $>$ or $=$ for each i,j.

- Each subdomain D_α of our cover is a semi-analytic.

- A semi-analytic set is subanalytic.

Structure of subanalytic set

Any closed subanaytic set Z admits a so called subanalytic stratification $\{Z_\alpha\}_{\alpha \in A}$. Namely it is a family of locally closed subsets with the following conditions.

1. Z is a locally finite disjoint union of Z_α's. Each Z_α is called a stratum.
2. For every point $p \in Z_\alpha$, Z_α is an analytic submanifold of X in some open neighborhood of p.
3. If $Z_\alpha \cap \overline{Z}_\beta \neq \emptyset$ for $\alpha, \beta \in \Lambda$, then $Z_\alpha \subset \overline{Z}_\beta$ holds. In particular, we have $Z_\alpha \subset \partial Z_\beta$ and $\dim_{\mathbb{R}} Z_\alpha < \dim_{\mathbb{R}} Z_\beta$.

Example of subanalytic stratification

Let $X = \mathbb{R}^2$ and let Z be a closed triangle abc with vertexes a, b, c. Then Z has a subanalytic stratification consisting of 7-strata, the interior of the triangle, open segments ab, bc, ca and points a, b, c.

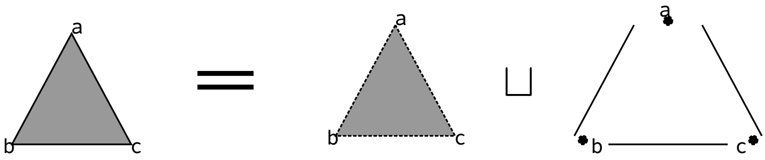

Figure: A stratification of a closed triangle Z.

Properties of subanalytic sets and construction of a common cover

Let X be a real analytic manifold countable at infinity. Then here are some properties of subanalytic sets.

1. Let Z be a subanalytic subset in X. Then its closure, its interior and its complement in X are again subanalytic in X.
2. A finite union and a finite intersection of subanalytic subsets in X are subanalytic in X.

Construction of common cover

Let $\{D'_\beta\}_{\beta \in B}$, $\{D''_\gamma\}_{\gamma \in C}$ be covers. Consider all the non-empty connected components of the closure of strata with dimension 3 for each $\overline{D'_\beta} \cap \overline{D''_\gamma}$. Then all of these non-empty connected components give a desire common cover $\{D_\alpha\}_{\alpha \in A}$.

Thank you for your attention !

Z is said to be subanalytic if for any $x \in \overline{Z}$ there exist an open neighborhood U of x, real analytic compact manifolds $Y_{i,j}$, $i = 1, 2,\ 1 \le j \le N$ and real analytic maps $\Phi_{i,j} : Y_{i,j} \to X$ such that

$$Z \cap U = \bigcup_{j=1}^{N} (\Phi_{1,j}(Y_{1,j}) \setminus \Phi_{2,j}(Y_{2,j})) \bigcap U.$$