Time-harmonic electro-magnetic scattering in exterior weak Lipschitz domains with mixed boundary conditions

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UNIVERSITÄT DUISBURG ESSEN

Open-Minded

Interested in weak solutions for the time-harmonic Maxwell equations/ scattering problem/ radiation problem:

$$\begin{aligned} -\operatorname{rot} H + i\omega\varepsilon E &= F \quad \text{on} \quad \Omega, & n \times E &= 0 \quad \text{on} \quad \Gamma_1, \\ \operatorname{rot} E + i\omega\mu H &= G \quad \text{on} \quad \Omega, & n \times H &= 0 \quad \text{on} \quad \Gamma_2. \end{aligned}$$

where:

- $\Omega \subset \mathbb{R}^3$ an exterior weak Lipschitz domain,
- Γ_1, Γ_2 relatively open subsets of $\Gamma := \partial\Omega$ with $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.
- $\varepsilon, \mu \in \mathsf{L}^\infty(\Omega, \mathbb{R}^{3\times 3})$ are symmetric and uniformly positive definite,
- F, G given data, $\omega \in \mathbb{C}$.

Rewrite:

$$\begin{aligned} -\operatorname{rot} H + i\omega\varepsilon E &= F \\ \operatorname{rot} E + i\omega\mu H &= G \end{aligned} \quad \Longleftrightarrow \quad (\mathrm{M} - \omega)\, u = f,$$

where

$$\mathrm{M} = i\Lambda^{-1} \begin{pmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}, \quad u = \begin{pmatrix} E \\ H \end{pmatrix}, \quad f = i\Lambda^{-1} \begin{pmatrix} F \\ G \end{pmatrix}.$$

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Polynomial weighted Sobolev Spaces

With $\rho = \rho(x) := \left(1 + |x|^2\right)^{1/2}$, $x \in \mathbb{R}^3$ we define for $s \in \mathbb{R}$:

$$\mathsf{L}^2_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \mid \rho^s u \in \mathsf{L}^2(\Omega) \right\},$$

$$\mathbf{H}^m_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \partial^\alpha u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \ \forall\, |\alpha| \leq m \right\},$$
$$\mathsf{H}^m_s(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \partial^\alpha u \in \mathsf{L}^2_{\mathrm{s}+|\alpha|}(\Omega) \ \forall\, |\alpha| \leq m \right\},$$

$$\mathbf{R}_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \right\}, \quad \mathsf{R}_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{s}+1}(\Omega) \right\},$$
$$\mathbf{D}_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{div} u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \right\}, \quad \mathsf{D}_s(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{div} u \in \mathsf{L}^2_{\mathrm{s}+1}(\Omega) \right\},$$

and to formulate boundary conditions: $\left(\mathrm{V} \in \left\{\mathsf{H}, \mathbf{H}, \mathsf{R}, \mathbf{R}, \mathsf{D}, \mathbf{D}\right\}\right)$

$$\mathrm{V}_{s,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathrm{V}_s(\Omega)}}, \qquad i = 1, 2$$

where

$$\mathsf{C}^\infty_{\Gamma_i}(\Omega) := \left\{ u_{|\Omega} \mid u \in \mathsf{C}^\infty(\mathbb{R}^3), \operatorname{supp} u \text{ cpt.}, \operatorname{dist}\left(\operatorname{supp} u, \Gamma_i\right) > 0 \right\}.$$

Moreover, kernels are denoted by 0 in the lower left corner, e. g.

$$ {}_0\mathbf{R}_{\mathrm{s}}(\Omega) := \big\{ u \in \mathbf{R}_{\mathrm{s}}(\Omega) \; \big| \; \operatorname{rot} u = 0 \big\}, $$
$$ {}_0\mathsf{D}_{s,\Gamma_1}(\Omega) := \big\{ u \in \mathsf{D}_{s,\Gamma_1}(\Omega) \; \big| \; \operatorname{div} u = 0 \big\}, \; \ldots $$

and if V_t is any of the spaces above, we write:

$$ \mathrm{V}_{<s} := \bigcap_{t<s} \mathrm{V}_t \qquad \text{and} \qquad \mathrm{V}_{>s} := \bigcup_{t>s} \mathrm{V}_t $$

Further notations:

- we call γ *κ-admissible*, iff

$$ \gamma = \gamma_0 \cdot \mathbb{I} + \hat{\gamma}, \; \gamma_0 \in \mathbb{R}_+ \text{ and } \hat{\gamma} = \mathcal{O}(r^{-\kappa}) \text{ for } r \longrightarrow \infty. $$

- $\mathsf{L}^2_\Lambda(\Omega) := \Big(\mathsf{L}^2(\Omega) \times \mathsf{L}^2(\Omega), < \,.\,,\,.\, >_\Lambda \Big)$, where

$$ < \,.\,,\,.\, >_\Lambda = < \,.\,, \varepsilon\,.\, >_{\mathsf{L}^2(\Omega)} + < \,.\,, \mu\,.\, >_{\mathsf{L}^2(\Omega)} . $$

- $\mathbb{C}^+ := \{ \omega \in \mathbb{C} \mid \operatorname{Im}(\omega) > 0 \}$.

Weck's local selection theorem

Definition

Let $\gamma \in \{\varepsilon, \mu\}$ be κ-admissible with $\kappa \geq 0$. A domain $\Omega \subset \mathbb{R}^3$ satisfies

* „*Weck's selection theorem*“ (WST), iff the imbedding

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Remarks:

(1) (cf. Bauer, Pauly, Schomburg 2016) :

$$\Omega \text{ bounded, weak Lipschitz} \implies \Omega \text{ satisfies (WST)}\,.$$

direct consequence:

$$\Omega \text{ ext. domain, weak Lipschitz} \implies \Omega \text{ satisfies (WLST)}\,.$$

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(2) An ext. domain Ω satisfies (WLST), iff for all $s, t \in \mathbb{R}$ with $t < s$ the imbedding

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$$u \longmapsto \mathrm{M}u = i \begin{pmatrix} 0 & -\varepsilon^{-1}\,\mathrm{rot} \\ \mu^{-1}\,\mathrm{rot} & 0 \end{pmatrix} u$$

is unbounded, linear and selfadjoint.

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Idea: approximating ω by a sequence $(\omega_n)_n \subset \mathbb{C}^+$.

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We say u *solves* $\mathrm{Max}(f, \Lambda, \omega)$*, iff*

(S1) $u \in \mathbf{R}_{<-\frac{1}{2}, \Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2}, \Gamma_2}(\Omega)$,

(S2) $(M - \omega)\, u = f$,

(S3) $\left(\Lambda_0 + \sqrt{\varepsilon_0 \mu_0}\, \Xi\right) u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)$,

where

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Remark:
Because of

$$\left(\Lambda_0 + \sqrt{\varepsilon_0 \mu_0}\, \Xi\right) u = \Lambda_0 \left(E + \sqrt{\frac{\mu_0}{\varepsilon_0}} \xi \times H, H - \sqrt{\frac{\varepsilon_0}{\mu_0}} \xi \times E \right),$$

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Theorem (Generalized Fredholm alternative)

Let $\omega \in \mathbb{R} \setminus \{0\}$, $\Omega \subset \mathbb{R}^3$ an exterior weak Lipschitz domain and ε, μ κ-admissible with $\kappa > 1$. Then we have:

(1) *For* $\mathrm{N}_{\mathrm{gen}}(\mathcal{M} - \omega) := \{\, u \mid u$ *is sol. to* $\mathrm{Max}(0, \Lambda, \omega) \,\}$ *we have*

$$\mathrm{N}_{\mathrm{gen}}(\mathcal{M} - \omega) \subset \bigcap_{t \in \mathbb{R}} \Big(\mathbf{R}_{\mathrm{t},\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}_{\mathrm{t},\Gamma_2}(\Omega) \Big) \times \Big(\mathbf{R}_{\mathrm{t},\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathbf{D}_{\mathrm{t},\Gamma_1}(\Omega) \Big)$$

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(2) $\dim \Big(\mathrm{N}_{\mathrm{gen}}(\mathcal{M} - \omega) \Big) < \infty$.

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(3) *For* $\sigma_{\mathrm{gen}}(\mathcal{M}) := \big\{ \omega \mid \mathrm{Max}(0, \Lambda, \omega)$ *has non-trivial sol.* $\big\}$ *we have:*

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$$\forall\, v \in \mathrm{N_{gen}}(\mathcal{M} - \omega) : \quad < f, v >_{\mathsf{L}^2_\Lambda(\Omega)} = 0. \qquad \text{(C1)}$$

Moreover we can choose u such that

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(5) $\forall\, t < -\frac{1}{2}$ *the solution operator*

$$\mathcal{L}_\omega : \Big(\mathsf{L}^2_{\mathrm{s}}(\Omega) \times \mathsf{L}^2_{\mathrm{s}}(\Omega) \Big) \cap \mathrm{N}(\mathcal{M} - \omega)^{\perp_\Lambda} \longrightarrow \Big(\mathsf{R}_{t,\Gamma_1}(\Omega) \times \mathsf{R}_{t,\Gamma_2}(\Omega) \Big) \cap \mathrm{N}(\mathcal{M} - \omega)^{\perp_\Lambda}$$

defined by (4) *is continuous.*

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$$\mathcal{L}_\omega : \Big(\mathsf{L}^2_{\mathrm{s}}(\Omega) \times \mathsf{L}^2_{\mathrm{s}}(\Omega) \Big) \cap \mathrm{N}(\mathcal{M} - \omega)^{\perp_\Lambda} \longrightarrow \Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega) \Big) \cap \mathrm{N}(\mathcal{M} - \omega)^{\perp_\Lambda}$$

defined by (4) is continuous.

Polynomial weighted Sobolev Spaces

With $\rho = \rho(x) := \left(1 + |x|^2\right)^{1/2}$, $x \in \mathbb{R}^3$ we define for $s \in \mathbb{R}$:

$$\mathsf{L}^2_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \mid \rho^s u \in \mathsf{L}^2(\Omega) \right\},$$

$$\mathbf{H}^m_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \partial^\alpha u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \ \forall\, |\alpha| \leq m \right\},$$
$$\mathsf{H}^m_s(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \partial^\alpha u \in \mathsf{L}^2_{\mathrm{s}+|\alpha|}(\Omega) \ \forall\, |\alpha| \leq m \right\},$$

$$\mathbf{R}_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \right\}, \quad \mathsf{R}_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{rot} u \in \mathsf{L}^2_{\mathrm{s}+1}(\Omega) \right\},$$
$$\mathbf{D}_{\mathrm{s}}(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{div} u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \right\}, \quad \mathsf{D}_s(\Omega) := \left\{ u \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \mid \operatorname{div} u \in \mathsf{L}^2_{\mathrm{s}+1}(\Omega) \right\},$$

and to formulate boundary conditions: $\left(\mathrm{V} \in \left\{\mathsf{H}, \mathbf{H}, \mathsf{R}, \mathbf{R}, \mathsf{D}, \mathbf{D}\right\}\right)$

$$\mathrm{V}_{s,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathrm{V}_s(\Omega)}}, \qquad i = 1, 2$$

where

$$\mathsf{C}^\infty_{\Gamma_i}(\Omega) := \left\{ u_{|\Omega} \mid u \in \mathsf{C}^\infty(\mathbb{R}^3), \operatorname{supp} u \text{ cpt.}, \operatorname{dist}\left(\operatorname{supp} u, \Gamma_i\right) > 0 \right\}.$$

Weck's local selection theorem

Definition

Let $\gamma \in \{\varepsilon, \mu\}$ be κ-admissible with $\kappa \geq 0$. A domain $\Omega \subset \mathbb{R}^3$ satisfies

* „*Weck's selection theorem*" (WST), iff the imbedding

$$\mathsf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \text{ is compact.}$$

* „*Weck's local selection theorem*" (WLST), iff the imbedding

$$\mathsf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \text{ is compact.}$$

Remarks:

(1) (cf. Bauer, Pauly, Schomburg 2016) :

$$\Omega \text{ bounded, weak Lipschitz} \implies \Omega \text{ satisfies (WST)}.$$

direct consequence:

$$\Omega \text{ ext. domain, weak Lipschitz} \implies \Omega \text{ satisfies (WLST)}.$$

(2) An ext. domain Ω satisfies (WLST), iff for all $s, t \in \mathbb{R}$ with $t < s$ the imbedding

$$\mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_{t}(\Omega) \text{ is compact.}$$

Definition (Time-harmonic solutions)

Let $\omega \in \mathbb{R} \setminus \{0\}$*,* $(\varepsilon, \mu) = (\varepsilon_0, \mu_0) \cdot \mathbb{I} + (\hat{\varepsilon}, \hat{\mu})$ κ*-admissible,* $\kappa \geq 0$ *and* $f \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \times \mathsf{L}^2_{\mathrm{loc}}(\Omega)$*. We say* u *solves* $\mathrm{Max}(f, \Lambda, \omega)$*, iff*

(S1) $u \in \mathbf{R}_{<-\frac{1}{2},\Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2},\Gamma_2}(\Omega)$,

(S2) $(M - \omega)\, u = f$,

(S3) $\left(\Lambda_0 + \sqrt{\varepsilon_0 \mu_0}\, \Xi\right) u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)$,

where

$$\Xi := \begin{pmatrix} 0 & -\xi\times \\ \xi\times & 0 \end{pmatrix}, \quad \xi := \frac{x}{|x|}, \quad \Lambda_0 = \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix}$$

Remark:
Because of

$$\left(\Lambda_0 + \sqrt{\varepsilon_0 \mu_0}\, \Xi\right) u = \Lambda_0 \left(E + \sqrt{\frac{\mu_0}{\varepsilon_0}} \xi \times H, H - \sqrt{\frac{\varepsilon_0}{\mu_0}} \xi \times E \right),$$

(S3) is just the classical Silver-Müller-radiation condition.

Lemma (A-priori estimate for Maxwell, Pauly 2007)

Let $t < -1/2$, $s \in (1/2, 1)$, $J \Subset \mathbb{R} \setminus \{0\}$ an interval and ε, μ κ-admissible, $\kappa > 1$. Then there exist $c, \delta > 0$ and $\hat{t} > -1/2$ s. t. for all $\omega \in \mathbb{C}^+$ with $\omega^2 = \lambda^2 + i\sigma\lambda$, $\lambda \in J$, $\sigma \in (0, (\sqrt{\varepsilon_0\mu_0})^{-1}]$ and for all $f \in \mathsf{L}^2_{\mathrm{s}}(\Omega) \times \mathsf{L}^2_{\mathrm{s}}(\Omega)$ we have

$$\| \mathcal{L}_\omega f \|_{\mathsf{R}_{\mathrm{t}}(\Omega)} + \| (\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\ \Xi)\mathcal{L}_\omega f \|_{\mathsf{L}^2_{\hat{\mathrm{t}}}(\Omega)} \leq c \cdot \Big\{ \| f \|_{\mathsf{L}^2_{\mathrm{s}}(\Omega)} + \| \mathcal{L}_\omega f \|_{\mathsf{L}^2(\Omega(\delta))} \Big\}$$

„direct“ consequence of

Theorem (A-priori-estimate for Helmholtz, Vogelsang 1975)

Let $t < -1/2$, $s \in (1/2, 1)$ and $J \Subset \mathbb{R} \setminus \{0\}$ an interval. Then there exists $c > 0$, s. t. for all $\nu \in \mathbb{C}^+$ with $\nu^2 = \kappa^2 + i\kappa\sigma$, $\kappa \in J$, $\sigma \in (0, 1]$ and for all $g \in \mathsf{L}^2_s(\mathbb{R}^3)$ we have:

$$\| (\Delta + \nu^2)^{-1} g \|_{\mathsf{L}^2_{\mathrm{t}}(\mathbb{R}^3)} + \| \exp(-i\kappa r)(\Delta + \nu^2)^{-1} g \|_{\mathsf{H}^1_{s-2}(\mathbb{R}^3)} \leq \| g \|_{\mathsf{L}^2_{\mathrm{s}}(\mathbb{R}^3)}.$$

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Theorem (Generalized Fredholm alternative)

Let $\omega \in \mathbb{R} \setminus \{0\}$, $\Omega \subset \mathbb{R}^3$ an exterior weak Lipschitz domain and ε, μ κ-admissible with $\kappa > 1$. Then we have:

(1) $$\mathrm{N}_{\mathrm{gen}}(\mathcal{M} - \omega) \subset \bigcap_{t \in \mathbb{R}} \Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathbf{D}_{t,\Gamma_2}(\Omega)\Big) \times \Big(\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathbf{D}_{t,\Gamma_1}(\Omega)\Big)$$

(2) $\dim\Big(\mathrm{N}_{\mathrm{gen}}(\mathcal{M} - \omega)\Big) < \infty$.

(3) $\sigma_{\mathrm{gen}}(\mathcal{M}) \subset \mathbb{R} \setminus \{0\}$ *has no accumulation point in* $\mathbb{R} \setminus \{0\}$.

(4) $\forall\, f \in \mathsf{L}^2_{>\frac{1}{2}}(\Omega) \times \mathsf{L}^2_{>\frac{1}{2}}(\Omega)$ *there exists a solution u of* $\mathrm{Max}(f, \Lambda, \omega)$, *iff*

$$\forall\, v \in \mathrm{N}_{\mathrm{gen}}(\mathcal{M} - \omega): \quad < f, v >_{\mathsf{L}^2_\Lambda(\Omega)} = 0. \tag{C1}$$

Moreover we can choose u such that

$$\forall\, v \in \mathrm{N}_{\mathrm{gen}}(\mathcal{M} - \omega): \quad < u, v >_{\mathsf{L}^2_\Lambda(\Omega)} = 0. \tag{C2}$$

Then u is uniquely determined.

(5) $\forall\, t < -\frac{1}{2}$ *the solution operator*

$$\mathcal{L}_\omega : \Big(\mathsf{L}^2_{\mathrm{s}}(\Omega) \times \mathsf{L}^2_{\mathrm{s}}(\Omega)\Big) \cap \mathrm{N}(\mathcal{M} - \omega)^{\perp_\Lambda} \longrightarrow \Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\Big) \cap \mathrm{N}(\mathcal{M} - \omega)^{\perp_\Lambda}$$

defined by (4) *is continuous.*

Lemma (Polynomial decay for Maxwell, Pauly 2007)

Let $1/2 < s \in \mathbb{R} \setminus \mathbb{I}$, $\omega \in J \Subset \mathbb{R} \setminus \{0\}$ *an interval and* ε, μ κ*-admissible,* $\kappa > 1$*. If* $u \in \mathsf{R}_{>-\frac{1}{2}}(\Omega) \times \mathsf{R}_{>-\frac{1}{2}}(\Omega)$ *satisfies*

$$(\mathcal{M} - \omega)u \in \mathsf{L}^2_s(\Omega) \times \mathsf{L}^2_s(\Omega),$$

then $u \in \mathsf{R}_{s-1}(\Omega)$ *and* $\exists\, c, \delta > 0$ *(independent of* u, f, ω*) s. t.*

$$\| u \|_{\mathsf{R}_{s-1}(\Omega)} \leq c \cdot \Big\{ \| (\mathcal{M} - \omega)u \|_{\mathsf{L}^2_s(\Omega)} + \| u \|_{\mathsf{L}^2(\Omega(\delta))} \Big\}.$$

„direct“ consequence of

Theorem (Polynomial decay for Helmholtz, Weck & Witsch 1997)

Let $t > -1/2$, $s \geq t$, $J \Subset \mathbb{R} \setminus \{0\}$ *an interval and* $\nu \in J$*. If* $u \in \mathsf{L}^2_t(\mathbb{R}^3)$ *and* $(\Delta + \nu^2)u \in \mathsf{L}^2_{s+1}(\mathbb{R}^3)$*, then* $u \in \mathsf{H}^2_s(\mathbb{R}^3)$ *and* $\exists\, c = c(s, J) > 0$*, s. t.*

$$\| u \|_{\mathsf{H}^2_s(\mathbb{R}^3)} \leq c \cdot \Big\{ \| (\Delta + \nu^2)u \|_{\mathsf{L}^2_{s+1}(\mathbb{R}^3)} + \| u \|_{\mathsf{L}^2_{s-1}(\mathbb{R}^3)} \Big\}.$$

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$$(\mathcal{M} - \omega)u \in \mathsf{L}^2_s(\Omega) \times \mathsf{L}^2_s(\Omega),$$

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$$\| u \|_{\mathbf{R}_{s-1}(\Omega)} \leq c \cdot \Big\{ \| (\mathcal{M} - \omega)u \|_{\mathsf{L}^2_s(\Omega)} + \| u \|_{\mathsf{L}^2(\Omega(\delta))} \Big\}.$$

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If $u \in \mathbf{R}_{>-\frac{1}{2}}(\Omega) \times \mathbf{R}_{>-\frac{1}{2}}(\Omega)$ *satisfies*

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Corollary

Let $\omega \in \mathbb{R} \setminus \{0\}$*,* ε, μ κ*-admissible with* $\kappa > 1$ *and* u *a solution of* $\mathrm{Max}(0, \Lambda, \omega)$*.*

$$\Longrightarrow \quad u \in \bigcap_{t \in \mathbb{R}} \Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega) \Big)$$

$\omega = 0$ - the static case

For $\omega = 0$ the system decouples:

$$-\operatorname{rot} E = F \quad \text{on} \quad \Omega, \qquad \operatorname{rot} H = G \quad \text{on} \quad \Omega,$$
$$n \times E = 0 \quad \text{on} \quad \Gamma_1, \qquad n \times H = 0 \quad \text{on} \quad \Gamma_2.$$

$\Longrightarrow$ add two equations

$$\operatorname{div} \varepsilon E = f, \qquad \operatorname{div} \mu H = g.$$

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$$\operatorname{div} \varepsilon E = f, \qquad \operatorname{div} \mu H = g.$$

$\Longrightarrow$ boundary conditions on the other part of the boundary

$$n \cdot \varepsilon E = 0 \quad \text{on} \quad \Gamma_2, \qquad n \cdot \mu H = 0 \quad \text{on} \quad \Gamma_1.$$

$\Longrightarrow$ static problem has non trivial kernel

$$_{\varepsilon}\mathcal{H}_s(\Omega) := {}_0\mathrm{R}_{s,\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathrm{D}_{s,\Gamma_2}(\Omega)$$
$$_{\mu}\mathcal{H}_s(\Omega) := {}_0\mathrm{R}_{s,\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathrm{D}_{s,\Gamma_1}(\Omega)$$

(Dirichlet-Neumann-fields)

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(Dirichlet-Neumann-fields)

Results for $\omega = 0$ - the static case

Theorem

Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain, ε κ-admissible with $\kappa > 0$ and additionally $\varepsilon \in C^1(\Sigma)$ for an exterior domain $\Sigma \subset \Omega$ with $\partial_j \hat{\varepsilon} = \mathcal{O}(r^{-1-\kappa})$ for $r \longrightarrow \infty$. Then for any

$$F \in \operatorname{rot} \mathsf{R}_{-1,\Gamma_1}(\Omega), \qquad f \in \operatorname{div} \mathsf{D}_{-1,\Gamma_2}(\Omega) = \mathsf{L}^2(\Omega)$$

we have a unique

$$E \in \mathsf{R}_{-1,\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathsf{D}_{-1,\Gamma_2}(\Omega) \cap {}_{\varepsilon}\mathcal{H}_{-1}(\Omega)^{\perp_{-1,\varepsilon}}$$

with

$$\operatorname{rot} E = f, \qquad \operatorname{div} \varepsilon E = g,$$

E depends continuously on the data.

Results for $\omega = 0$ - the static case

Theorem

Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain, μ κ-admissible with $\kappa > 0$ and additionally $\mu \in C^1(\Sigma)$ for an exterior domain $\Sigma \subset \Omega$ with $\partial_j \hat{\mu} = \mathcal{O}(r^{-1-\kappa})$ for $r \longrightarrow \infty$. Then for any

$$G \in \operatorname{rot} \mathsf{R}_{-1,\Gamma_2}(\Omega), \qquad g \in \operatorname{div} \mathsf{D}_{-1,\Gamma_1}(\Omega) = \mathsf{L}^2(\Omega)$$

we have a unique

$$H \in \mathsf{R}_{-1,\Gamma_2}(\Omega) \cap \mu^{-1}\mathsf{D}_{-1,\Gamma_1}(\Omega) \cap {}_{\mu}\mathcal{H}_{-1}(\Omega)^{\perp_{-1,\mu}}$$

with

$$\operatorname{rot} H = G, \qquad \operatorname{div} \mu H = g,$$

H depends continuously on the data.

Thank you

for your attention!