Some remarks about the Maxwell equations

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1. Introduction

$\mathcal{E}$	electrical field ,	$\mathcal{H}$	magnetic field ,
$\mathcal{D}$	electric flux ,	$\mathcal{B}$	magnetic flux ,
$\mathcal{J}$	current density ,	ϱ	charge density .

$$\begin{array}{llll}
(1) & \mathbf{curl}\,\mathcal{E} & = -\frac{\partial \mathcal{B}}{\partial t}\,, & (Faraday) \\
(2) & \mathbf{curl}\,\mathcal{H} & = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}\,, & (Ampère - Maxwell) \\
(3) & \mathrm{div}\ \mathcal{D} & = \varrho\,, & (Gauss - Weber) \\
(4) & \mathrm{div}\ \mathcal{B} & = 0\,, & (Gauss - Weber).
\end{array}$$

Constitutive equations:

$$(5)\ \mathcal{D} = \varepsilon\mathcal{E}\,,\ \mathcal{B} = \mu\mathcal{H}\,,\ \mathcal{J} = \sigma\mathcal{E} + \mathcal{J}_0\,,$$

$\varepsilon \geq 0$ electrical permittivity, $\mu > 0$ magnetic permeability,
$\sigma \geq 0$ conductivity; for simplicity let them be constants.

Energy convolution: Ω bounded domain or $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$:

(6) $\quad \mathbb{E}(t) = \frac{1}{2} \int\limits_{\Omega} (\varepsilon |\mathcal{B}|^2 + \mu |\mathcal{D}|^2) d\mathbf{x}$

Then $\quad \frac{d\mathbb{E}}{dt} = -\frac{\sigma}{\varepsilon} \int\limits_{\Omega} |\mathcal{D}|^2 d\mathbf{x} + \int\limits_{\Gamma} \mathbf{n} \cdot (\mathcal{B} \times \mathcal{D}) ds$; $\mathcal{D} \times \mathcal{B}$ pointing vector.

> **Lemma 1:** If $\mathbf{n} \cdot (\mathcal{B} \times \mathcal{D})|_{\Gamma} = \mathbf{0}$ and $\mathcal{B}(0) = \mathcal{D}(0) = \mathbf{0}$,
> Then $\mathcal{B}(t) = \mathcal{D}(t) = \mathbf{0}$

Proof: $0 \le \frac{1}{2} \int\limits_{\Omega} \mu |\mathcal{D}|^2 d\mathbf{x} \le \mathbb{E}(t) - \mathbb{E}(0) = -\frac{\sigma}{\varepsilon} \int\limits_{0} \int\limits_{\Omega} |\mathcal{D}|^2 d\mathbf{x} dt \le 0$

Hence $\mathcal{D}(\mathbf{x}, t) = \mathbf{0} \overset{(5)}{\Rightarrow} \mathcal{E} = \mathbf{0} \overset{(1)}{\Rightarrow} \frac{\partial \mathcal{B}}{\partial t} = \mathbf{0} \Rightarrow \mathcal{B} = \mathbf{0}$. $\qquad \square$

Lemma 2:
Let $\mathbf{B}$ denote the Bogovskii operator in $\mathbb{R}^3$, $a\ \psi do_{-1}$,
let $\varrho_0 \in L_2(\mathbb{R}^3)$ have compact support
and $\mathbf{J}_0 = \mathbf{0}, \sigma > 0, \varepsilon > 0$. Then

$$(7) \quad \mathcal{D}(\mathbf{x}, t) = e^{-\frac{\sigma t}{\varepsilon}} \mathbf{B} \varrho_0 \quad \text{for } t \geq 0.$$

Proof: div of (2) with (3) and (5) gives

$$\frac{\partial \varrho}{\partial t} + \frac{\sigma}{\epsilon} \varrho = 0 \Rightarrow \varrho = e^{-\frac{\sigma t}{\varepsilon}} \varrho_0 \operatorname{div} \mathcal{D}. \qquad \square$$

$\mathbf{B}$ is a right inverse to div. [M. Costabel + A. McIntosh 2010]

Electrostatic fields: [see E. Martensen 1968, R. Kress 1970]

Time derivatives zero. Bounded $\Omega = \bigcup_{\mu=1}^{\widehat{m}} \Omega_\mu$, $\overline{\Omega}_\mu \cap \overline{\Omega}_\nu = \emptyset$ for $\mu \neq \nu$ with $\Gamma_\mu = \partial\Omega_\mu \in C^{1,\alpha}$, $0 < \alpha \leq 1$, Lyapounov.

$\widehat{\Omega} := \mathbb{R}^3 \setminus \overline{\Omega} = \bigcup_{i=0}^{m} \widehat{\Omega}^i$, $\widehat{\Omega}^0$ unbounded $\overline{\widehat{\Omega}}^i \cap \overline{\widehat{\Omega}}^j = \emptyset$ for $i \neq j$,
$\widehat{\Omega}^i$ bounded: $i, j = 1, \dots, m$.

$$(8) \quad \begin{array}{llll} \mathbf{curl}\,\mathbf{E} & = \mathbf{0}\,, \ \mathbf{curl}\,\mathbf{H} & = \sigma\mathbf{E} + \mathbf{J_0} \\ \mathbf{div}\,\varepsilon\mathbf{E} & = \varrho\,, \ \mathbf{div}\,\mu\mathbf{H} & = 0. \end{array}$$

Continuity equation: $\frac{\varrho}{\varepsilon} = -\frac{1}{\sigma}\,\mathrm{div}\ \mathbf{J}_0$

Electrical boundary condition: Given $\mathbf{f}$ on $\Gamma = \bigcup_{\mu=1}^{\widehat{m}} \Gamma_\mu$ and $\frac{\varrho}{\varepsilon}$ in Ω,

(9) $-[\mathbf{n} \times \mathbf{E}]|_\Gamma = \mathbf{f}$ and $\mathrm{div}\ \mathbf{E}|_\Gamma = \eta$ on Γ;

(10) $\int_\Gamma \widehat{\mathbf{n}} \cdot \mathbf{E}\ \widehat{z}^i ds_{\mathbf{y}} = E^i$, productivities, $i = 1 \dots, m$, where

$$\widehat{z}^i := \begin{cases} 1 \text{ in } \overline{\widehat{\Omega}}^i \\ 0 \text{ in } \widehat{\Omega} \setminus \widehat{\Omega}^i. \end{cases}$$

$$\mathbf{grad}\,\mathrm{div}\,\mathbf{E} - \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \Delta\mathbf{E} \Rightarrow$$

(11) $\Delta\mathbf{E} = \mathbf{grad}\,\frac{\varrho}{\varepsilon}$ in $\Omega \setminus \Gamma$ and $\Omega^c \setminus \Gamma$, respectively .

Ansatz: The solution of the electrostatic boundary value problem can be represented as

$$(12)\quad \begin{aligned}&\mathbf{E}(\mathbf{x}) = -\frac{1}{4\pi}\int\limits_\Gamma \frac{1}{r}\mathbf{n}(\mathbf{y}) \times \mathbf{h}^*(\mathbf{y}) ds_{\mathbf{y}} - \mathbf{grad}_{\mathbf{x}} \frac{1}{4\pi}\int\limits_\Gamma \frac{1}{r} e^*(\mathbf{y}) ds_{\mathbf{y}} \\ &+\mathbf{curl}_{\mathbf{x}} \frac{1}{4\pi}\int\limits_\Gamma \frac{1}{r}\mathbf{f}(\mathbf{y}) ds_{\mathbf{y}} - \frac{1}{4\pi}\int\limits_\Omega \frac{1}{r}(\mathbf{grad}_{\mathbf{y}} \frac{\varrho}{\varepsilon}) d\mathbf{y} + \frac{1}{4\pi}\int\limits_\Gamma \frac{1}{r}\eta(\mathbf{y})\mathbf{n}(\mathbf{y}) ds_{\mathbf{y}}.\end{aligned}$$

where $\mathbf{h}^* \in TC^\beta(\Gamma)$ and $e^* \in C^\beta(\Gamma)$ are to be determined, $r = |\mathbf{x} - \mathbf{y}|$.

{If Ω unbounded:
(13) $\mathbf{E}(\mathbf{x}) = O(|\mathbf{x}|^{-1})$, $\mathrm{div}\,\mathbf{E}(\mathbf{x}) = O(|\mathbf{x}|^{-2})$, $\mathbf{curl}\,\mathbf{E}(\mathbf{x}) = O(|\mathbf{x}|^{-2})$.}

Boundary integral operators
Double layer potentials: $0 \le \beta < \alpha$.

$$
(14)\quad \left.\begin{aligned} K\varphi(\mathbf{x}) &= -\frac{1}{2\pi}\int_\Gamma (\partial_{\mathbf{n},\mathbf{y}}\frac{1}{r}\varphi(\mathbf{y})ds_{\mathbf{y}} \\ K^*\psi(\mathbf{x}) &= -\frac{1}{2\pi}\int_\Gamma (\partial_{\mathbf{n},\mathbf{x}}\frac{1}{r}\psi(\mathbf{y})ds_{\mathbf{y}} \end{aligned}\right\} : C^\beta(\Gamma) \to C^\alpha(\Gamma) \overset{\text{comp}}{\hookrightarrow} C^\beta(\Gamma),
$$

Electrical field boundary integral operators:

$$
(15)\quad \left.\begin{aligned} (\mathbf{A}\mathbf{g})(\mathbf{x}) &:= \frac{1}{2\pi}\int_\Gamma \mathbf{n}(\mathbf{x}) \times \left\{(\mathbf{grad}_{\mathbf{y}}\frac{1}{r}) \times \mathbf{g}(\mathbf{y})\right\}ds_{\mathbf{y}} \\ (\mathbf{A}^*\mathbf{h})(\mathbf{x}) &:= \frac{1}{2\pi}\mathbf{n}(\mathbf{x}) \times \int_\Gamma \mathbf{n}(\mathbf{x}) \times \left\{(\mathbf{grad}_{\mathbf{y}}\frac{1}{r}) \times \big(\mathbf{n}(\mathbf{y}) \times \mathbf{h}(\mathbf{y})\big\}ds_{\mathbf{y}} \end{aligned}\right\} :
$$

$$
TC^\beta(\Gamma) \to TC^\alpha(\Gamma) \overset{\text{comp}}{\hookrightarrow} TC^\beta(\Gamma).
$$

The system of boundary integral equations [R. Kress 1970]

$$
(16)\ \mathbf{h}^* - \mathbf{A}^*\mathbf{h}^* = \frac{1}{2\pi}\mathbf{n}(\mathbf{x}) \times \Big\{\mathbf{n}(\mathbf{x}) \times \mathbf{curl}_{\mathbf{x}}\Big(\int_\Omega (\mathbf{grad}\,\frac{\varrho}{\varepsilon})\frac{1}{r}d\mathbf{y} - \int_\Gamma \frac{1}{r}\mathbf{n}(\mathbf{y})\eta(\mathbf{y})ds_{\mathbf{y}}\Big) - \mathbf{grad}_{\mathbf{x}}\int_\Gamma \frac{1}{r}\operatorname{div}_{\Gamma,\mathbf{y}}\mathbf{f}(\mathbf{y})ds_{\mathbf{y}}\Big\}
$$

$$
(17)\ e^* + K^*e^* - \frac{1}{2\pi}\mathbf{n}(\mathbf{x})\cdot\int_\Gamma \frac{1}{r}\mathbf{n}(\mathbf{y}) \times \mathbf{h}^*(\mathbf{y})ds_{\mathbf{y}} = \frac{1}{2\pi}\mathbf{n}(\mathbf{x})\cdot\Big\{\int_\Omega \frac{1}{r}\mathbf{grad}_{\mathbf{y}}\frac{\varrho}{\varepsilon}d\mathbf{y} - \int_\Gamma \frac{1}{r}\mathbf{n}(\mathbf{y})\eta(\mathbf{y})ds_{\mathbf{y}} - \mathbf{curl}_{\mathbf{x}}\int_\Gamma \frac{1}{r}\mathbf{f}(\mathbf{y})ds_{\mathbf{y}}\Big\}
$$

The system (16), (17) defines a Fredholm mapping of index zero. But

Theorem: Dirichlet fields: (see [R. Kress 1969])
$\langle K\varphi, \psi\rangle = \langle \varphi, K^*\psi\rangle$ adjoint w.r.t. to $\langle\cdot,\cdot\rangle := (\cdot,\cdot)_{L_2(\Gamma)}$.
-1 m–fold eigenvalue of K and K^*, $K\widehat{z}^i = -\widehat{z}^i$, $\langle \varphi_i, \widehat{z}^k\rangle = \delta_i^k$,
$i, k = 1, \ldots, m$;
$\mathbf{w}_i(\mathbf{x}) := \mathbf{grad}_{\mathbf{x}} \frac{1}{4\pi} \int\limits_\Gamma \frac{1}{r}\varphi_i(\mathbf{y})ds_{\mathbf{y}}$
linear independent basis of Dirichlet vector fields and
$\varphi_i(\mathbf{x}) = -\mathbf{n}(\mathbf{x}) \cdot \mathbf{w}_i(\mathbf{x})$, productivities $\int\limits_\Gamma \widehat{\mathbf{n}}(\mathbf{y}) \cdot \mathbf{w}_i(\mathbf{y})\widehat{z}^k(\mathbf{y})ds_{\mathbf{y}} = \delta_i^k$.
Note that $\mathbf{n}(\mathbf{x}) \times \mathbf{w_i}(\mathbf{x}) = \mathbf{0}$ on Γ. [Martensen 1968]

Necessary compatibility conditions for given $\frac{\varrho}{\varepsilon}$ and η:
(18) $\int\limits_\Omega (\mathbf{grad}_{\mathbf{y}} \frac{\varrho}{\varepsilon}) \cdot \mathbf{w}_i(\mathbf{y})d\mathbf{y} = \int\limits_\Gamma \eta(\mathbf{y})\mathbf{n}(\mathbf{y}) \cdot \mathbf{w}_i(\mathbf{y})ds_{\mathbf{y}}\,,\ i = 1, \ldots, m.$

Hence, we also can require that with m given productivities $E^i \in \mathbb{R}$, in addition to (16), (17):
(19) $\int\limits_\Gamma e^*\widehat{z}^i ds_{\mathbf{y}} = E^i$.

Theorem: [R. Kress 1970]

(20) $\langle \mathbf{A}\mathbf{g}, \mathbf{h}\rangle = \langle \mathbf{g}, \mathbf{A}^*\mathbf{h}\rangle \ \forall\ \mathbf{g}, \mathbf{h} \in TC^\beta(\Gamma)$.

-1 is an n–fold eigenvalue of $\mathbf{A}$ and $\mathbf{A}^*$ with n linearly independent eigensolutions

$\mathbf{h}^i = \widehat{\mathfrak{z}}^i$ of $\mathbf{A}^*$ and $\mathbf{g}_i$ of $\mathbf{A}$, $\langle \mathbf{g}_i, \mathbf{h}^k\rangle = \delta_i^k$, $i, k = 1, \dots, n$.

The fields

(21) $\mathfrak{z}_i(\mathbf{x}) := \mathbf{curl}_{\,\mathbf{x}} \int\limits_\Gamma \frac{1}{r}\mathbf{g}_i(\mathbf{y}) ds_{\mathbf{y}}$

define a basis of linearly independent **Neumann fields** and

$$\mathbf{g}_i(\mathbf{x}) = -\mathbf{n}(\mathbf{x}) \times \mathfrak{z}_i(\mathbf{x}) \quad \text{on } \Gamma.$$

Additional linear algebraic equations:

$$(22) \quad \int\limits_\Gamma \mathbf{h}^* \cdot \big(\mathbf{n}(\mathbf{y}) \times \mathcal{A}_\mu(\mathbf{y})\big) ds_{\mathbf{y}} = \int\limits_\Omega \mathbf{grad}_{\,\mathbf{y}} \frac{\rho}{\epsilon} \cdot \mathcal{A}_\mu(\mathbf{y}) d\mathbf{y} - \int\limits_\Gamma \{\mathbf{f} \cdot \mathfrak{z}_\mu + \eta \mathbf{n}(\mathbf{y}) \cdot \mathcal{A}_\mu(\mathbf{y})\} ds_{\mathbf{y}}, \ \mu = 1, \dots, n;$$

Here $\mathcal{A}_\mu := \frac{1}{4\pi} \int\limits_\Gamma \frac{1}{r} \mathfrak{z}_\mu(\mathbf{y}) \times \mathbf{n}(\mathbf{y}) ds_{\mathbf{y}} = \frac{1}{4\pi} \mathbf{curl}_{\,\mathbf{x}} \int\limits_\Omega \frac{1}{r} \mathfrak{z}_\mu(\mathbf{y}) d\mathbf{y}$.

Main Theorem: [R. Kress 1970]
The system of integral equations (16), (17) together with the $m+n$ linear algebraic equations (19), (22) has exactly one solution $\mathbf{h}^* \in TC^\beta(\Gamma)\,,\, e^* \in C^\beta(\Gamma)$ defining the unique solution $\mathbf{E}$ in (12) of the Maxwell equations with electrical boundary conditions (9), (10) satisfying the compatibility conditions (18).

Characterization of the Neumann vector fields: [Martensen 1968]

Lemma:
Let Ω be bounded and simply connected. Determine U satisfying

$$\Delta U = 0 \quad \text{in } \Omega\,,\ \frac{\partial U}{\partial n}\Big|_{\Gamma} = 0$$

Then U is unique modulo real constants. Let $\mathfrak{C}$ be a C^1–curve in Ω with τ the arc length parameter from $\mathbf{x} \in \overline{\Omega}$ to $\mathbf{x}_0 \in \Omega$.
Then U is uniquely determined by $U(x_0) = 0$ and

$$U(\mathbf{x}) = -\int_{\mathfrak{C}} \mathbf{t} \cdot \mathbf{grad}\, U d\tau \quad \text{for every differentiable } \mathfrak{C},$$

where $\mathbf{t}$ is the unit tangent of $\mathfrak{C}$ in the direction of $\mathbf{x}_0$. Then

$$\mathfrak{z}(\mathbf{x}) := -\mathbf{grad}\, U(\mathbf{x})$$

is called **Neumann field** satisfying $\Delta \mathfrak{z} = \mathbf{0}$ in Ω and $\mathfrak{z} \cdot \mathbf{n}|_{\Gamma} = 0$ on Γ.

Basis of Neumann vector fields in $\Omega \bigcup\limits_{\mu=1}^{\widehat{m}} \Omega_\mu$:

Determination of $\mathfrak{z}_\mu$:

- For Ω_μ simply connected $\mathfrak{z}_\mu := \mathbf{0}$.
- For Ω_μ multiply connected let n_μ denote the genus of Ω_μ , $n_\mu \geq 1$.

Neumann vector field $\mathfrak{z}_\kappa$ for Ω_μ with $n_\mu \geq 1$:

$S_1, \ldots, S_{n_\Gamma}$ cuts of Ω_μ to make $\Omega_\mu \setminus \bigcup\limits_{\kappa=1}^{n_\Gamma} S_\kappa$ simply connected.

$\mathfrak{C}^\kappa := \partial S_\kappa$ on $\partial\Omega_\mu$, $\kappa = 1, \ldots, n_\mu$. Determine $U, \Delta U = 0$ in $\Omega_\mu \setminus \bigcup\limits_\kappa S_\kappa$,

$Z^\kappa := (U^+ - U^-)|_{S_\kappa}$. Compute Neumann fields $\mathfrak{z}_\kappa$ in Ω_μ with $\mathfrak{z}_\kappa \cdot \mathbf{n}|_{\partial\Omega_\mu} = 0$ satisfying the circulation conditions

$$\int\limits_{\mathfrak{C}^\kappa} \mathbf{t} \cdot \mathfrak{z}_\kappa d\tau = Z^\kappa. \quad \text{(See Martensen p.229)}$$

Then the collection of $\mathfrak{z}_\kappa$ for Ω_μ and all Ω_μ with $n_\mu \geq 1$ define a basis of Neumann vector fields of dimension $n = \sum\limits_\mu n_\mu$.

Time harmonic fields:

$$\begin{array}{llll} \mathcal{E}(\mathbf{x},t) = \mathrm{Re}\left\{\mathbf{E}(\mathbf{x})e^{-i\omega t}\right\}, & \mathcal{D}(\mathbf{x},t) = \mathrm{Re}\left\{\mathbf{D}(\mathbf{x})e^{-i\omega t}\right\}, \\ \mathcal{H}(\mathbf{x},t) = \mathrm{Re}\left\{\mathbf{H}(\mathbf{x})e^{-i\omega t}\right\}, & \mathcal{B}(\mathbf{x},t) = \mathrm{Re}\left\{\mathbf{B}(\mathbf{x})e^{-i\omega t}\right\}, \\ \mathcal{J}(\mathbf{x},t) = \mathrm{Re}\left\{\mathbf{J}(\mathbf{x})e^{-i\omega t}\right\}, & \varrho(\mathbf{x},t) = \mathrm{Re}\left\{\varrho(\mathbf{x})e^{-i\omega t}\right\}. \end{array}$$

Maxwell equations:

(23) $\quad \mathbf{curl}\,\mathbf{E}(\mathbf{x}) = i\omega\mu\mathbf{H}(\mathbf{x})\,,\ \mathbf{curl}\,\mathbf{H}(\mathbf{x}) = (-i\omega\varepsilon + \sigma)\mathbf{E}(\mathbf{x}) + \mathbf{J}_0,$

(24) $\mathrm{div}\ (\varepsilon\,\mathbf{E})(\mathbf{x}) = \varrho(\mathbf{x})\,,\ \mathrm{div}\ (\mu\mathbf{H})(\mathbf{x}) = 0$

(25) $\quad \mathbf{J}(\mathbf{x}) = \sigma\mathbf{E}(\mathbf{x}) + \mathbf{J}_0(\mathbf{x}),$

Continuity equation:

(26) $\quad \varrho(\mathbf{x}) = \mathrm{div}\ \mathbf{J}_0(\mathbf{x})(i\omega - \sigma/\varepsilon)^{-1}.$

2. Boundary value problems:

2.1 The cavity problem in $\Omega \subset \mathbb{R}^3$, bounded Lipschitz domain:
Given $\mathbf{J}_0$ and ϱ in Ω satisfying the continuity equation (26), and $\mathbf{f}$ (or $\mathbf{g}$) on $\Gamma = \partial\Omega$.
Solve (23)-(25) in Ω satisfying the boundary conditions

(27) $\quad \mathbf{n} \times \mathbf{E}|_\Gamma = \mathbf{f}\,,\ \mathrm{div}\ \mathbf{E}|_\Gamma = \eta$ on Γ. and productivities

$\int\limits_\Gamma \widehat{\mathbf{n}}\cdot\mathbf{E}\,\widehat{z}^i ds_y = E^i$ for $i = 1, \ldots, m$ with given $E^1, \ldots, E^m$.

Equations based on Kress's approach:

Ansatz: Domain integral equation for $\mathbf{E}(\mathbf{x})\,,\ \mathbf{x} \in \Omega$:

(28)
$$\begin{aligned}\mathbf{E}(\mathbf{x}) = &-\tfrac{1}{4\pi}\int\limits_\Gamma \tfrac{1}{r}\mathbf{n}(\mathbf{y}) \times \mathbf{h}^*(\mathbf{y})ds_y - \mathbf{grad}_{\mathbf{x}}\tfrac{1}{4\pi}\int\limits_\Gamma \tfrac{1}{r}e^*(\mathbf{y})ds_y \\ &+ \mathbf{curl}_{\mathbf{x}}\tfrac{1}{4\pi}\int\limits_\Gamma \tfrac{1}{r}\mathbf{f}(\mathbf{y})ds_y + \tfrac{1}{4\pi}\int\limits_\Gamma \tfrac{1}{r}\eta(\mathbf{y})\mathbf{n}(\mathbf{y})ds_y \\ &+ \tfrac{1}{4\pi}\int\limits_\Gamma \tfrac{1}{r}\big(i\omega\mu\mathbf{J}_0(\mathbf{y}) - \mathbf{grad}_{\mathbf{y}}\tfrac{\varrho}{\varepsilon}(\mathbf{y})\big)d\mathbf{y} + \tfrac{1}{4\pi}\int\limits_\Omega \tfrac{1}{r}\big\{k^2\mathbf{E}(\mathbf{y})\big\}d\mathbf{y}.\end{aligned}$$

Boundary integral equations for h^* and e^* on Γ:

$$(29)\quad \mathbf{h}^*(\mathbf{x}) - (\mathbf{A}^*\mathbf{h}^*)(\mathbf{x})$$

$$= \frac{1}{2\pi}\mathbf{n}(\mathbf{x}) \times \Big\{\mathbf{n}(\mathbf{x}) \times \mathbf{curl}_{\mathbf{x}}\Big(\int_\Omega \frac{1}{r}\Big[\mathbf{grad}_{\mathbf{y}}\frac{\varrho}{\varepsilon}(\mathbf{y}) - i\omega\mu\mathbf{J}_0(\mathbf{y}) - k^2\mathbf{E}(\mathbf{y})\Big]d\mathbf{y}$$

$$- \int_\Gamma \frac{1}{r}\eta(\mathbf{y})\mathbf{n}(\mathbf{y})ds_y\Big) - \mathbf{grad}_{\mathbf{x}}\int_\Gamma \frac{1}{r}\operatorname{div}_{\Gamma,y}\mathbf{f}(\mathbf{y})ds_y\Big\},$$

$$(30)\quad e^*(\mathbf{x}) + (K^*e^*)(\mathbf{x}) - \frac{1}{2\pi}\mathbf{n}(\mathbf{x}) \cdot \int_\Gamma \frac{1}{r}\mathbf{n}(\mathbf{y}) \times \mathbf{h}^*(\mathbf{y})ds_y$$

$$= \frac{1}{2\pi}\mathbf{n}(\mathbf{x}) \cdot \Big\{\int_\Gamma \frac{1}{r}\Big[\mathbf{grad}_{\mathbf{y}}\frac{\varrho}{\eta}(\mathbf{y}) - i\omega\mu\mathbf{J}_0(\mathbf{y}) - k^2\mathbf{E}(\mathbf{y})\Big]d\mathbf{y}$$

$$- \int_\Gamma \frac{1}{r}\eta(\mathbf{y})ds_y - \mathbf{curl}_{\mathbf{x}}\int_\Gamma \frac{1}{r}\mathbf{f}(\mathbf{y})ds_y\Big\}, \ \mathbf{x} \in \Gamma.$$

where $r = |\mathbf{x} - \mathbf{y}|$ and $\varrho(\mathbf{x}) = \operatorname{div} \mathbf{J}_0(\mathbf{x})\big(i\omega - \frac{\sigma}{\varepsilon}\big)^{-1}$.

Algebraic equations:

(31) **Productivities:** $\int\limits_\Gamma \widehat{\mathbf{n}}\cdot\mathbf{E}\,\widehat{z}^i ds_y = E^i$ for $i = 1, \dots, m$; and

(32) $\int\limits_\Gamma \mathbf{h}^*\cdot\big(\big(\mathbf{n}(\mathbf{y}) \times \mathbf{A}_\mu(\mathbf{y})\big)ds_y$

$$= \int\limits_\Omega \big(\mathbf{grad}_{\,\mathbf{y}} \tfrac{\varrho}{\varepsilon} - i\omega\mu\mathbf{J}_0(\mathbf{y}) - k^2\mathbf{E}(\mathbf{y})\big)d\mathbf{y}$$

$$- \int\limits_\Gamma \big\{\mathbf{f}(\mathbf{y})\mathfrak{z}_\mu(\mathbf{y}) + \eta(\mathbf{y})\mathbf{n}(\mathbf{y}) \cdot \mathcal{A}_\mu(\mathbf{y})\big\}ds_y, \mu = 1, \dots, n.$$

Theorem: There exists $k_0 > 0$ such that for $|k| < k_0$ and also for k not a ficticious eigenvalue, the system of equations (28)–(32) is uniquely solvable.

(See also [R. Hiptmair 2003])

2.2 Exterior scattering problem in $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$:

(See also [D. Colton, R. Kress 1983])
Given $\mathbf{J}_0$ and ϱ in Ω satisfying (26) and $\mathbf{f} = -\mathbf{n} \times \mathbf{E}^{inc}|_\Gamma$ (or $\mathbf{g} = \mathbf{n} \times \mathbf{H}^{inc}|_\Gamma$) on $\Gamma = \partial\Omega^c$ Lipschitz.
Solve (23)-(25) in Ω^c satisfying the boundary condition

$$(33) \quad \mathbf{n} \times \mathbf{E}|_\Gamma = \mathbf{f} = -\mathbf{n} \times \mathbf{E}^{inc}|_\Gamma (\text{or } \mathbf{n} \times \mathbf{H}|_\Gamma = \mathbf{g}, \text{respectively},) \text{ on } \Gamma;$$

and the Silver-Müller radiation conditions

$$(34) \quad \lim_{|\mathbf{x}| \to \infty} \left\{ \left(\mathbf{x} \times (\mathbf{curl}\, \mathbf{E}(\mathbf{x}) \right) - ik|\mathbf{x}|\mathbf{H}(\mathbf{x}) \right\} = 0,$$

$$(35) \quad \lim_{|\mathbf{x}| \to \infty} \left\{ \left(\mathbf{x} \times (\mathbf{curl}\, \mathbf{H}(\mathbf{x}) \right) + ik|\mathbf{x}|\mathbf{E}(\mathbf{x}) \right\} = 0,$$

$$(36) \quad k^2 = \omega^2 \mu\varepsilon + i\omega\mu\sigma \,,\ k = +\sqrt{k^2} \quad \text{with } \operatorname{Re} k > 0 \,,\ \operatorname{Im} k \geq 0.$$

2.3 Transmission scattering problem in $\Omega \dot{\cup} \Omega^c$ with $\Gamma = \partial\Omega \cap \partial\Omega^c$ Lipschitz:

(See also [X. Claeys, R. Hiptmair 2012, 2013])

Given $\mathbf{J}_0$ and ϱ satisfying (26) in Ω and Ω^c and $\mathbf{f}$, and $\mathbf{g}$ in $T(\Gamma)$.
Find $(\mathbf{E}, \mathbf{H})$ in Ω and $(\mathbf{E}^c, \mathbf{H}^c)$ in Ω^c satisfying (23)-(25) in Ω and Ω^c, respectively, and the transmission conditions

$$(37) \quad \mathbf{n} \times (\mathbf{E} - \mathbf{E}^c)|_\Gamma = \mathbf{f} \quad \text{and } \mathbf{n} \times (\mathbf{H} - \mathbf{H}^c)|_\Gamma = \mathbf{g} \quad \text{on } \Gamma$$

and the Silver-Müller radiation conditions (34), (35) for $(\mathbf{E}^c, \mathbf{H}^c)$.

2.4 Eddy current problem in Ω:

Let Ω be a bounded Lipschitz domain and Ω_c with $\overline{\Omega}_c \subset \Omega$ be a conductor, $\sigma_c > 0$;
$\Omega_I := \Omega \setminus \overline{\Omega}_c$ be a dielectric medium, $\sigma_I = 0$. Given $\mathbf{J}_0$ in Ω_I satisfying $\mathrm{div}\ \mathbf{J}_0 = 0$ in Ω_I. Find $(\mathbf{E}_c, \mathbf{H}_c)$ satisfying

(38) $\quad \mathbf{curl}\,\mathbf{H}_c - \sigma_c \mathbf{E}_c = \mathbf{0}$ and $\mathbf{curl}\,\mathbf{E}_c - i\omega\mu_c \mathbf{H}_c = \mathbf{0}$ in Ω_c,

and $(\mathbf{E}, \mathbf{H})$ satisfying

(39) $\quad \mathbf{curl}\,\mathbf{H} = \mathbf{J}_0\,,\ \mathbf{curl}\,\mathbf{E} - i\omega\mu\mathbf{H} = \mathbf{0}$ and $\mathrm{div}\,(\varepsilon\mathbf{E}) = 0$ in Ω_I;

the transmission conditions

(40) $\quad \mathbf{n} \times \mu_c \mathbf{H}_c|_\Sigma - \mathbf{n} \times \mu\mathbf{H}|_\Sigma = \mathbf{0}$ and $\mathbf{n} \times \mathbf{E}_c|_\Sigma - \mathbf{n} \times \mathbf{E}|_\Sigma = \mathbf{0}$ on $\Sigma := \partial\Omega_c \cap \partial\Omega_I$;

and the boundary conditions

(41) $\quad \mathbf{n} \times \mathbf{E} = \mathbf{0}$ on $\partial\Omega$.

If $\Omega = \mathbb{R}^3$, then (41) is to be replaced by the Silver-Müller radiation conditions (34), (35). (See also [R. Hiptmair 2002],[A.A. Rodriguez, R. Hiptmair, A. Valli 2004]).

Theorem: Let $\sigma > 0$. Then the cavity problems, the exterior scattering problems, the transmission scattering problem and the eddy curent problem, respectively, all have at most one solution. [C. Müller 1957, E. Meister 1996]

Proof for the cavity problems:

Set $\mathbf{J}_0, \varrho$ and $\mathbf{f} = \mathbf{0}$. Then

$$\int_\Gamma (\mathbf{n} \times \overline{\mathbf{E}}) \cdot \mathbf{H} ds = \int_\Gamma \mathbf{n} \cdot (\overline{\mathbf{E}} \times \mathbf{H}) ds = 0$$

and

$$0 = \int_\Omega \operatorname{div} (\overline{\mathbf{E}} \times \mathbf{H}) d\mathbf{x} = \int_\Omega \mathbf{H} \cdot \mathbf{curl}\, \overline{\mathbf{E}} d\mathbf{x} - \int_\Omega \overline{\mathbf{E}} \cdot \mathbf{curl}\, \mathbf{H} d\mathbf{x}$$

$$\overset{(23)}{=} i\omega \int_\Omega (\varepsilon |\mathbf{E}|^2 - \mu |\mathbf{H}|^2) d\mathbf{x} - \int_\Omega \sigma |\mathbf{E}|^2 d\mathbf{x}.$$

$$\Rightarrow \mathbf{E} = \mathbf{0} \overset{(23)}{\Rightarrow} \mathbf{H} = \mathbf{0}.$$

□

Remark: Let $\sigma = 0$, $\varrho = 0$, $\mathbf{J}_0 = \mathbf{0}$ for the cavity problem. Then there exists a countable sequence of eigenfrequencies $\{\omega_n\}_{n\in\mathbb{N}}$ with $0 < \omega_1 \leq \omega_2 \leq \dots$ of finite multiplicity and $\lim_{n\to\infty} \omega_n = \infty$; and corresponding electromagnetic eigenfields $(\mathbf{E}_n, \mathbf{H}_n)$ satisfying the Maxwell system

$$\text{curl}\ \mathbf{E}_n = i\omega_n \mu \mathbf{H}_n\,,\ \text{curl}\ \mathbf{H}_n = -i\omega_n \varepsilon \mathbf{E}_n \quad \text{in } \Omega.$$

[H. Weyl 1952, C. Müller + H. Niemeyer 1961].

3. Variational Formulations:

3.1 Picard's system: [R. Picard 1984, R. Picard + D. McGhee 2011]

Extended Maxwell–Heaviside system

(42) $$\mathbf{U} := (\varphi, \mathbf{H}, \psi, \mathbf{E})^\top \in H_0^1(\Omega) \times \mathbf{H}(\mathbf{curl}\,, \Omega) \times H_0^1(\Omega) \times \mathbf{H}(\mathbf{curl}\,, \Omega) =: \mathfrak{X}(\Omega)$$

where

$$\mathbf{H}(\mathbf{curl}\,, \Omega) = \{\mathbf{H} \in \mathbf{L}_2(\Omega) \wedge \operatorname{div} \mathbf{H} = 0 \wedge \mathbf{curl}\,\mathbf{H} \in \mathbf{L}_2(\Omega)\}$$

$$\overset{\circ}{\mathbf{H}}(\mathbf{curl}\,, \Omega) := \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}\,, \Omega) \wedge \mathbf{n} \times \mathbf{E}|_\Gamma = \mathbf{0}\}.$$

$$(\!(\mathbf{U}, \mathbf{V})\!)_{\mathfrak{X}(\Omega)} := (\mathbf{U}, \mathbf{V})_{L_2(\Omega)} + (u_1, v_1)_{H^1(\Omega)} + (u_3, v_3)_{H^1(\Omega)} + (\mathbf{curl}\,\mathbf{u}_2, \mathbf{curl}\,\mathbf{v}_2)_{L_2(\Omega)} + (\mathbf{curl}\,\mathbf{u}_4, \mathbf{curl}\,\mathbf{v}_4)_{L_2(\Omega)}$$

(43) $(i\mathfrak{I} + \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3)\mathbf{U} = (-\frac{\varrho}{\varepsilon}, 0, 0, \mathbf{J}_0)^\top$
where

$$\mathfrak{I} := \omega \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (\varepsilon + i\frac{\sigma}{\omega}) \end{pmatrix}, \qquad \mathfrak{A}_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{curl} \\ 0 & 0 & 0 & 0 \\ 0 & \mathbf{curl} & 0 & 0 \end{pmatrix},$$

$$\mathfrak{A}_2 := \begin{pmatrix} 0 & 0 & 0 & -\operatorname{div} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathbf{grad} & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{A}_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{grad} & 0 \\ 0 & \operatorname{div} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Equation (43) becomes the Maxwell system (23), (24) for $\varphi = \psi = 0$.

Note: $\mathbf{grad}^* = -\operatorname{div}$, $\operatorname{div}^* = -\mathbf{grad}$.

With

$$\mathfrak{I}^{-1} = \frac{1}{\omega}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (\varepsilon + i\frac{\sigma}{\omega})^{-1} \end{pmatrix},$$

and the properties

(44) $\mathfrak{A}_j \circ \mathfrak{A}_k = \mathbf{0}$ for $j \neq k$,

the system (37) can also be written as

$$-\mathfrak{I}^{-2}(i\mathfrak{I} + \mathfrak{A}_1)(i\mathfrak{I} + \mathfrak{A}_2)(i\mathfrak{I} + \mathfrak{A}_3)\mathbf{U} = (-\tfrac{\varrho}{\varepsilon}, 0, 0, \mathbf{J}_0)^\top$$

or

$$(45) \quad (\mathfrak{I} - i\mathfrak{A}_1)(\mathfrak{I} - i\mathfrak{A}_2)(\mathfrak{I} - i\mathfrak{A}_3)\mathbf{U} = i\mathfrak{I}^2(-\tfrac{\varrho}{\varepsilon}, 0, 0, \mathbf{J}_0)^\top.$$

Theorem: For the bilinear equations: find $\mathbf{U} \in \mathfrak{X}(\Omega)$ with

$$\mathbf{n} \times \mathbf{E}|_\Gamma = \mathbf{n} \times \mathbf{B}(\tfrac{\varrho}{\varepsilon})|_\Gamma + \mathbf{f}\,,\ \mathbf{f} \in TH^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma),$$

(46) $\quad (\!(\mathbf{U}^t, (\mathfrak{I} - i\mathfrak{A}_k)(\mathbf{U} + \mathbf{U}_1))\!)_{\mathfrak{X}(\Omega)} = (\!(\mathbf{U}^t, \mathbf{F})\!)_{\mathfrak{X}(\Omega)}$ for all $\mathbf{U}^t \in \mathfrak{X}_0(\Omega)$ where

$\mathbf{U}_1 = \big(0, 0, 0, -\mathbf{B}(\tfrac{\varrho}{\varepsilon})\big)^\top$, and $\mathfrak{X}_0(\Omega) = \{\mathbf{V} \in \mathfrak{X}(\Omega) \wedge \mathbf{n} \times \mathbf{E}|_\Gamma = \mathbf{0}\}$

with $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{A}_3$ skew selfadjoint one has

(47) $\quad \omega\|\mathbf{U}^t\|^2_{\mathfrak{X}(\Omega)} \leq \operatorname{Re}(\!(\overline{\mathbf{U}^t}, (\mathfrak{I} - i\mathfrak{A}_k)\mathbf{U}^t)\!)_{\mathfrak{X}(\Omega)}$ where

$$(\!(\mathbf{V}, \mathbf{U})\!)_{\mathfrak{X}(\Omega)} := (v_1, u_1)_{H^1(\Omega)} + \mu(\mathbf{v}_2, \mathbf{u}_2)_{\mathbf{H}(\mathbf{curl}\,,\Omega)} + (v_3, u_3)_{H^1(\Omega)} + \varepsilon(\mathbf{v}_4, \mathbf{u}_4)_{\mathbf{H}(\mathbf{curl}\,,\Omega)}$$

Hence, with the Lax–Milgram Lemma we find that the inverses

$$(\mathfrak{I} - i\mathfrak{A}_k)^{-1} \quad \text{exist for } k = 1, 2, 3.$$

To prove (47) is a calculation exploiting the properties of $\mathfrak{A}_j$, $j = 1, 2, 3$. The coerciveness inequalities (47) are also satisfied if the function space $\mathfrak{X}(\Omega)$ in (46) and (47) is **reduced** to the subspace $\mathfrak{X}_M(\Omega)$ where $\varphi = \psi = 0$, and therefore the inverses $(\mathfrak{I} - i\mathfrak{A}_k)_M^{-1}$ also exist on $\mathfrak{X}_M(\Omega)$ and

$$\begin{aligned} \mathbf{U}_M &= (0, \mathbf{H}, 0, \mathbf{E})^\top \\ &= \operatorname{Re}(\mathfrak{I} - i\mathfrak{A}_3)_M^{-1} \circ (\mathfrak{I} - i\mathfrak{A}_2)_M^{-1} \circ (\mathfrak{I} - i\mathfrak{A}_1)_M^{-1} \big(i\mathfrak{I}^2(\tfrac{\varrho}{\varepsilon}, 0, 0, \mathbf{J}_0)^\top\big).\Box \end{aligned}$$

For coercive combined field integral equations see [A. Buffa, R. Hiptmair 2004].

Example cavity problem:

Applying $\mathbf{curl}$ to (23) gives

(48) $\quad \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\mathbf{E} = i\omega\mu\mathbf{J}_0$ where $k^2 = \omega^2\mu\varepsilon + i\omega\mu\sigma$.

The variational formulation:

Given $\mathbf{m} \subset \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathbf{J}_0 \in \mathbf{H}^1(\Omega)$ with $\varrho \in L_2(\Omega)$ satisfying (26).
Find $\mathbf{E} \in \mathbf{H}(\mathbf{curl}\,,\Omega)$ satisfying

$$\mathbf{n} \times \mathbf{E}|_\Gamma = \mathbf{m}\,,\ \mathrm{div}\ \mathbf{E} = \tfrac{1}{\varepsilon}\varrho$$

and the variational equation

$$(49)\quad \mathfrak{A}(\mathbf{E},\mathbf{E}^t) = \int\limits_\Omega \{\mathbf{curl}\,\mathbf{E}) \cdot \overline{(\mathbf{curl}\,\mathbf{E}^t)} - k^2\mathbf{E}\cdot\overline{\mathbf{E}^t}\}d\mathbf{y}$$

$$= \int\limits_\Omega (i\varepsilon\omega - \sigma)^{-1}\mathbf{J}_0 \cdot \overline{\mathbf{E}^t}d\mathbf{y} + \int\limits_\Gamma \mathbf{m}\cdot\overline{\mathbf{E}^t}ds_\mathbf{y} =: \mathcal{L}(\mathbf{E}^t)$$

for all $\mathbf{E}^t \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)$;

$$\mathbf{H}_0(\mathbf{curl}\,,\Omega) = \{\mathbf{E}^t \in \mathbf{H}(\mathbf{curl}\,,\Omega) \wedge \mathrm{div}\ \mathbf{E}^t = 0 \wedge \mathbf{n}\times\mathbf{E}^t|_\Gamma = \mathbf{0}\}.$$

Helmholtz decomposition: $\mathbf{H}(\mathbf{curl}\,,\Omega) = \mathbf{M}(\Omega) \oplus \mathbf{P}(\Omega)$;

$\mathbf{M}(\Omega) := \{\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega) \wedge \int\limits_\Omega \mathbf{u} \cdot \mathbf{grad}\, p d\mathbf{x} = 0 \quad \text{for all } p \in H_0^1(\Omega)\}$,

$\mathbf{P}(\Omega) := \{\mathbf{v} = \mathbf{grad}\, p \wedge p \in H_0^1(\Omega)\}$;

$$
(50) \quad \begin{aligned} (\!(\mathbf{u}, \mathbf{v})\!) &:= \int\limits_\Omega \mathbf{curl}\,\mathbf{u} \cdot \overline{\mathbf{curl}\,\mathbf{v}}\, d\mathbf{y} + \int\limits_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\, d\mathbf{y}, \\ (\!(\mathbf{u}, \mathbf{v})\!) &= 0 \quad \text{for } \mathbf{u} \in \mathbf{M}(\Omega) \quad \text{and } \mathbf{v} \in \mathbf{P}(\Omega). \end{aligned}
$$

Theorem:
$\mathbf{M}(\Omega) \overset{\text{comp}}{\hookrightarrow} \mathbf{L}_2(\Omega)$ and it holds Gårding's inequality

$$
(51) \quad \begin{aligned} &\operatorname{Re} \mathfrak{A}(\mathbf{u} + \mathbf{grad}\, p, \mathbf{u} + \mathbf{grad}\, p) \\ &\quad \geq \|\mathbf{u} + \mathbf{grad}\, p\|_\Omega^2 - (1 + |k^2|)\|\mathbf{u} + \mathbf{grad}\, p\|_{L_2(\Omega)}^2 \end{aligned}
$$

where $\|\mathbf{u} + \mathbf{grad}\, p\|_\Omega := (\!(\mathbf{u} + \mathbf{grad}\, p, \mathbf{u} + \mathbf{grad}\, p)\!)^{\frac{1}{2}}$.

Consequence: Let $\sigma > 0$. Then to any given $\mathbf{m} \in T\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $\mathbf{J}_0 \in \mathbf{H}^1(\Omega)$ there exists exactly one solution $\mathbf{E} \in \mathbf{H}(\mathbf{curl}\,,\Omega)$ of (49) because of Gårding's inequality, Lax–Milgram and uniqueness, due to the Fredholm alternative.

4. Boundary integral equations:

4.1 Stratton–Chu representation formulae: [J. Stratton, I. Chu 1939]

Helmholtz fundamental solution: $G_k(\mathbf{z}) = \frac{1}{4\pi}\frac{\exp(ik|\mathbf{z}|)}{|\mathbf{z}|}$.

$$\begin{aligned}\mathbf{E}(\mathbf{x}) = &\int_\Omega [G_k(\mathbf{x}-\mathbf{y})i\omega\mu\mathbf{J}_0(\mathbf{y}) + \tfrac{1}{\varepsilon}\varrho(\mathbf{y})\mathbf{grad}_y G_k(\mathbf{x}-\mathbf{y})]d\mathbf{y}\\ &+\int_\Gamma \big(\operatorname{div}(\mathbf{n}\times\mathbf{H}) + \mathbf{n}\cdot\mathbf{J}_0\big)(-i\omega\varepsilon+\sigma)^{-1}\mathbf{grad}_y G_k(\mathbf{x}-\mathbf{y})ds_y\\ &-\int_\Gamma\{i\omega\mu G_k(\mathbf{x}-\mathbf{y})\big(\mathbf{n}(\mathbf{y})\times\mathbf{H}(\mathbf{y})\big)\\ &\quad+\big(\mathbf{n}(\mathbf{y})\times\mathbf{E}(\mathbf{y})\big)\mathbf{grad}_y G_k(\mathbf{x}-\mathbf{y})\}ds_y \quad \text{for any } \mathbf{x}\in\Omega,\end{aligned} \tag{52}$$

$$\begin{aligned}\mathbf{H}(\mathbf{x}) = &\int_\Omega \mathbf{J}_0(\mathbf{y})\times\mathbf{grad}_y G_k(\mathbf{x}-\mathbf{y})d\mathbf{y}\\ &-\int_\Gamma\Big\{(-i\omega\varepsilon+\sigma)G_k(\mathbf{x}-\mathbf{y})\big(\mathbf{n}(\mathbf{y})\times\mathbf{E}(\mathbf{y})\big)\\ &\quad+\tfrac{i}{\omega\mu}\operatorname{div}\big(\mathbf{n}(\mathbf{y})\times\mathbf{E}(\mathbf{y})\big)\mathbf{grad}_y G_k(\mathbf{x}-\mathbf{y})\\ &\quad+\big(\mathbf{n}(\mathbf{y})\times\mathbf{H}(\mathbf{y})\big)\times\mathbf{grad}_y G_k(\mathbf{x}-\mathbf{y})\Big\}ds_y \quad \text{for any } \mathbf{x}\in\Omega.\end{aligned} \tag{53}$$

4.2 Boundary potentials

$$(54) \quad \begin{aligned} (\mathfrak{V}\boldsymbol{\lambda})(\mathbf{x}) \quad &:= -i\omega\mu \int_\Gamma G_k(\mathbf{x}-\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})ds_{\mathbf{y}} \\ &\quad + \int_\Gamma (-i\omega\varepsilon+\sigma)^{-1}\big(\mathbf{grad}_{\mathbf{y}} G_k(\mathbf{x}-\mathbf{y})\big)\big(\operatorname{div}_{\Gamma,\mathbf{y}} \boldsymbol{\lambda}(\mathbf{y})\big)ds_{\mathbf{y}} \end{aligned}$$

$$(55) \quad \mathfrak{W}\boldsymbol{\varphi}(x) := \int_\Gamma \boldsymbol{\varphi}(\mathbf{y}) \times \mathbf{grad}_{\mathbf{y}} G_k(\mathbf{x}-\mathbf{y})ds_{\mathbf{y}}.$$

$$(56) \quad \begin{aligned} \mathfrak{V} \,:\, \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) \quad &\to \mathbf{H}(\mathbf{curl}\,, \Omega) \quad \text{and} \\ \mathfrak{W} \,:\, \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) \quad &\to \mathbf{H}(\mathbf{curl}\,, \Omega) \end{aligned}$$

4.3 Boundary integral equations for the Dirichlet problem in the cavity Ω:

Representation of $\mathbf{E}(\mathbf{x})$ in (37) where $\mathbf{m}(\mathbf{x}) = \mathbf{n} \times \mathbf{E}|_\Gamma$ is given and $\mathbf{n} \times \mathbf{H}|_\Gamma =: \boldsymbol{\lambda}$ is unknown. Then on Γ:

$$
\begin{aligned}
\mathbf{m}_1(\mathbf{x}) &:= \mathbf{m}(\mathbf{x}) \\
&\quad -\mathbf{n}(\mathbf{x}) \times \int_\Omega \Big\{ G_k(\mathbf{x}-\mathbf{y}) i\omega\mu \mathbf{J}_0(\mathbf{y}) + \tfrac{1}{\varepsilon}\varrho(y)\,\mathbf{grad}_{\mathbf{y}} G_k(\mathbf{x}-\mathbf{y}) \Big\} d\mathbf{y} \\
&\quad -i\omega\mu\mathbf{n}(\mathbf{x}) \times (\mathbf{V}_k\boldsymbol{\lambda})(\mathbf{x}) \\
&\quad -(-i\omega\varepsilon+\sigma)^{-1}\mathbf{n}(\mathbf{x}) \times \big(\mathbf{grad}_{\mathbf{x}} V_k \operatorname{div}_\Gamma \boldsymbol{\lambda}\big)(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma, \qquad (57) \\
\mathbf{V}_k\boldsymbol{\lambda} &:= \int_\Gamma G_k(\mathbf{x}-\mathbf{y})\boldsymbol{\lambda}(\mathbf{y}) ds_{\mathbf{y}}
\end{aligned}
$$

Weak variational form on Γ: $\langle \cdot, \cdot \rangle_\Gamma$ denotes $L_2(\Gamma)$–duality:

(58) $\langle \mathbf{V}_k\boldsymbol{\lambda}, \overline{\boldsymbol{\lambda}^t}\rangle_\Gamma - \frac{1}{k^2}\langle V_k \operatorname{div}_\Gamma \boldsymbol{\lambda}, \operatorname{div}_\Gamma \overline{\boldsymbol{\lambda}^t}\rangle_\Gamma = \frac{-i}{\omega\mu}\langle \mathbf{n}\times\mathbf{m}_1, \boldsymbol{\lambda}^t\rangle_\Gamma$

for all $\boldsymbol{\lambda}^t \in T\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$.

Hodge decomposition on Γ, the Rumsey principle:
[V.H. Rumsey 1954, A. Bendali 1984] (also J.C. Nedelec, Springer 2001)

$T\mathbf{H}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) = \mathbf{grad}_\Gamma H^{3/2}(\Gamma)/\mathbb{C} \oplus \mathbf{curl}_\Gamma \mathbf{H}^{1/2}(\Gamma)/\mathbb{C}$

$\mathbf{grad}_\Gamma H^{3/2}(\Gamma)/\mathbb{C}$ orthogonal to $\mathbf{curl}_\Gamma T\mathbf{H}^{1/2}(\Gamma)/\mathbb{C}$ with respect to $\langle \cdot, \cdot \rangle_\Gamma$.

$\boldsymbol{\lambda}(\mathbf{x}) := \mathbf{grad}_\Gamma \varphi + \mathbf{curl}_\Gamma \boldsymbol{\psi}\,,\ \varphi \in H^{3/2}(\Gamma)\,,\ \boldsymbol{\psi} \in T\mathbf{H}^{1/2}(\Gamma).$

Boundary integral equation: Find $\varphi \in H^{3/2}(\Gamma)$ and $\boldsymbol{\psi} \in T\mathbf{H}^{1/2}(\Gamma)$ such that

$$\begin{aligned}\mathcal{A}_\Gamma(\varphi, \boldsymbol{\psi}; \varphi^t, \boldsymbol{\psi}^t) &:= \langle V_k \Delta_\Gamma \varphi, \overline{\Delta_\Gamma \varphi^t} \rangle_\Gamma + \langle \mathbf{V}_k \mathbf{curl}_\Gamma \boldsymbol{\psi}, \overline{\mathbf{curl}_\Gamma \boldsymbol{\psi}^t} \rangle_\Gamma \\ &\quad - k^2 \langle \mathbf{V}_k \mathbf{grad}_\Gamma \varphi, \overline{\mathbf{grad}_\Gamma \varphi^t} \rangle_\Gamma \\ &= \frac{i}{\omega\mu} \langle \mathbf{n} \times \mathbf{m}_1, (k^2 \overline{\mathbf{grad}_\Gamma \boldsymbol{\varphi}^t} - \overline{\mathbf{curl}_\Gamma \boldsymbol{\psi}^t} \rangle_\Gamma\end{aligned}$$

Theorem: Gårding's inequality holds with $c_0 > 0$ and $c_1 \geq 0$.

$$\begin{aligned}\operatorname{Re}\mathcal{A}_\Gamma(\varphi,\boldsymbol{\psi};\varphi,\boldsymbol{\psi}) \geq c_0\{\|\varphi\|^2_{H^{3/2}} + \|\boldsymbol{\psi}\|^2_{H^{1/2}(\Gamma)}\} \\ - c_1(\|\varphi\|^2_{L_2(\Gamma)} + \|\boldsymbol{\psi}\|^2_{L_2(\Gamma)})\end{aligned}$$

Consequence: There exists exactly one solution to the Dirichlet problem in the cavity.

Remarks: Similar results hold for all the interior and exterior boundary value problem as well as for the transmission and eddy current problems. A different approach is given by R. McCamy and E.P. Stephan 1984 and by O. Sterz and C. Schwab where the Hodge decomposition is not needed.

For an introduction to Maxwell's Equations see e.g. [R. Hiptmair 2015].

Thank you for your attention!

References:

A. Bendali: Numerical analysis of the exterior boundary value problem for the time harmonic Maxwell equations by a boundary finite element method. Part I: The contiuous problen. Math. Comp. **43** (1984) 29–46.

A. de la Bourdonnaye: Some formulations coupling finite element and integral equation methods for Helmholtz equation and electromagnetism. Numer. Math. **69** (1995) 257–268.

A. Buffa & R. Hiptmair: A coercive combined field integral equation for electromagnetic scattering. SIAM J. Numer. Anal. **42** (2) (2004) 621–640.

X Claeys & R. Hiptmair: Electromagnetic scattering at composite objects. A novel multi–trace boundary integral formulation. Math. Modelling Numer. Anal. **46** (6) (2012) 1421–1445.

X Claeys & R. Hiptmair: Multi–trace boundary integral formulation for acoustic scattering by composite structures. Pure Applied Math. **66** (8) (2013) 1163–1201.

D. Colton & R. Kress: Inverse Acoustic and Electromagnetic Scattering Theory. Springer–Verlag Berlin 1992 & 1998.

D. Colton & R. Kress: Integral Equation Methods in Scattering Theory. John Wiley & Sons, Inc. New York 1983.

M. Costabel: A coercive bilinear form for Maxwell's equations. J. Math. Anal. Appl. **157** (1991) 527–541.

M. Costabel & A. McIntosh: On Bogovskii and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. Math. Z. **265** (2010) 297–320.

R. Hiptmair: Symmetric coupling for eddy current problems. SIAM J. Numer. Anal. **40** (1) (2002) 41–65.

R. Hiptmair: Coercive combined field integral equations. J. Numer. Anal. **11** (2) (2003) 115–133.

R. Hiptmair: Maxwell's Equations: Continuous and Discrete. Lecture Notes in Mathematics **2148** (2015) pp. 1–58.

A. Kirsch & F. Hettlich: The Mathematical Theory of Time–Harmonic Maxwell Equations. Springer, Heidelberg 2015.

R. Kress: Über die Integralgleichung des Pragerschen Problems, Archive Rat. Mech. Analysis **30** (1968), 381–400.

R. Kress: Potentialtheoretische Randwertprobleme bei Tensorfeldern beliebiger Dimension und beliebigen Ranges. Methoden Verf. Math. Physik **2** (1969) 49–83.

R. Kress: Die Behandlung zweier Randwertprobleme für die vektorielle Poissongleichung nach einer Integralgleichungsmethode. Archive Rat. Mech. Analysis **39** (1970) 206–226.

R. Kress: Ein kombiniertes Dirichlet–Neumannsches Randwertproblem bei harmonischen Vektorfeldern, Archive Rat. Mech. Analysis **42** (1971) 40–49.

R. Kress: Potentialtheoretische Randwertprobleme bei Tensorfeldern beliebiger Dimension und beliebigen Ranges. Archive Rat. Mech. Analysis **47** (1972) 59–80.

R. Kress: Robinsches und Pragersches Problem der Potentialtheorie bei beliebiger Dimension. Appl. Analysis **5** (1976) 265–276.

R. Kress: On the existence of a solution to a singular integral equation in electromagnetic reflection. J. Math. Anal. Appl. **77** (1980) 555-566.

R. Kress: On the Fredholm alternative. Integral Eqns. Operator Theory **6** (1983) 453–457.

R. Kress: Linear Integral Equations. Springer–Verlag New York 1989 & 1999.

R. Kress: Fast 100 Jahre Fredholmsche Alternative. In: Jahrbuch, Überblicke Mathematik **1994**. Vieweg Stuttgart, 14–27.

S. Kurz & B. Auchmann: Differential forms and boundary integral equations for Maxwell–type problems. In: Fast Boundary Element Methods in Engineering and Industrial Applications (U. Langer et al. eds.) Springer–Verlag Heidelberg 2012, 1–62.

R.C. MacCamy & E. Stephan: A simple layer potential method for three–dimensional eddy current problems. In: Lecture Notes in Mathematics (A. Dold & B. Eckmann eds.) Springer–Verlag Berlin (1982), 477–484.

R.C. MacCamy & E. Stephan: A boundary element method for an exterior problem for three–dimensional Maxwell equations. Appl. Analysis **157** (1983) 141–163.

R.C. MacCamy & E. Stephan: Solution procedures for three–dimensional eddy current problems. J. Math. Analysis Appl. **101** (1984) 348–379.

E. Martensen: Potentialtheorie. B.G. Teubner, Stuttgart 1968.

E. Meister: Partielle Differentialgleichungen. Akademie–Verlag Berlin 1996.

C. Müller: Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen. Springer–Verlag Berlin 1957.

C. Müller & H. Niemeyer: Greensche Tensoren und asymptotische Gesetze der elektromagnetischen Hohlraumschwingungen. Archive Rat. Mech. Analysis **7** (1961) 305–348.

J.-C. Nedelec: Acoustic and Electromagnetic Equations. Springer–Verlag New York 2001.

R. Picard: On a structural observation in generalized electromagnetic theory. J. Math. Anal. Appl. **110** (1985) 247–264.

R. Picard & Des McGhee: Partial Differential Equations. Walter de Gruyter Berlin 2011.

A.A. Rodriguez, R. Hiptmair & A. Valli: Mixed finite element approximation of eddy current problems. IMA J. Numer. Anal. **24** (2) (2004) 255–271.

O. Sterz & C. Schwab: A scalar boundary integrodifferential equation for eddy current problems using an impedance boundary condition. In: Lecture Notes Comput. Sci. Eng. **18** (2001) Springer Berlin 129–136.

J.A. Stratton: Electromagnetic Theory. McGraw–Hill, New York 1941.

J.A. Stratton & I.J. Chu: Diffraction theory of electromagnetic waves. Physical Review **56** (1939) 99–107.

V.H. Rumsey; Reaction concept in electromagnetic theory. Physical Review **94** (1954) 1438–1491.