A new approach for Kirchhoff-Love plates and shells

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Outline

Kirchhoff plate bending problem

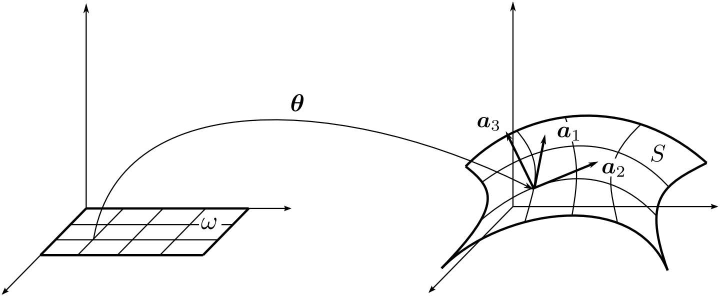


Figure: Geometry of a shell

Unknown: covariant components of the displacement

$$\boldsymbol{u} = (u_i)$$

Kirchhoff plate bending problem

plate: S is flat, for simplicity: $\theta = \mathrm{Id}$

pure bending: $u_3 = w$.

dimensional reduction of 3D elasticity for $\Omega = \omega \times (-t/2, t/2)$

find $w \in W \subset H^2(\omega)$ which minimizes

$$J(w) = \frac{1}{2}\int_\omega \mathcal{C}\nabla^2 w : \nabla^2 w \, dx - \int_\omega g\, w \, dx$$

with

$$\mathcal{C}\varepsilon = 2\mu \left(\varepsilon + \frac{\nu}{1-\nu} (\operatorname{tr} \varepsilon) \boldsymbol{I} \right)$$

Kirchhoff plate bending problem

(kinematic) boundary conditions:

$$\partial\omega = \gamma_c \cup \gamma_s \cup \gamma_f$$

	clamped	on γ_c	
if the plate is	simply supported	on γ_s	then
	free	on γ_f	

$$W = \{v \in H^2(\omega) : \; v = 0, \; \partial_n v = 0 \text{ on } \gamma_c, \quad v = 0 \text{ on } \gamma_s\}.$$

variational formulation: find $w \in W$ such that

$$\int_\omega \mathcal{C}\nabla^2 w : \nabla^2 v \, dy = \int_\omega g \, v \, dx \quad \text{for all } v \in W,$$

Kirchhoff plate bending problem

quantities of interest: w vertical deflection
$\boldsymbol{M} = -\mathcal{C}\nabla^2 w$ bending moments

Mixed variational formulation: Find $\boldsymbol{M} \in \boldsymbol{V}$ and $w \in Q$ such that

$$\int_\omega \mathcal{C}^{-1}\boldsymbol{M} : \boldsymbol{L}\, dy + \int_\omega \boldsymbol{L} : \nabla^2 w\, dy = 0 \qquad \text{for all } \boldsymbol{L} \in \boldsymbol{V},$$

$$\int_\omega \boldsymbol{M} : \nabla^2 v\, dy = -\int_\omega g\, v\, dx \qquad \text{for all } v \in Q,$$

e.g., with the function spaces

$$\boldsymbol{V} = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}, \quad Q = W.$$

Kirchhoff plate bending problem

An different choice for Q with less smoothness:

if

$$W = \{v \in H^2(\omega) : \; v = 0, \; \partial_n v = 0 \text{ on } \gamma_c, \quad v = 0 \text{ on } \gamma_s\},$$

then

$$Q = \{v \in H^1(\omega) : \; v = 0, \text{ on } \gamma_c, \quad v = 0 \text{ on } \gamma_s\} = H^1_{0,\gamma_c \cup \gamma_s}(\omega).$$

The choice for $\boldsymbol{V}$: transposition method

$$b(v, \boldsymbol{L}) = \int_\omega \boldsymbol{L} : \nabla^2 v \, dy \equiv \langle Bv, \boldsymbol{L} \rangle \longrightarrow \langle B^* \boldsymbol{L}, v \rangle,$$

where $\langle B^* \boldsymbol{L}, v \rangle$ is well-defined for $v \in Q$.

Kirchhoff plate bending problem

The linear operator B:

$$B\colon D(B) \subset Q \longrightarrow \boldsymbol{Y}^*,$$

here with

$$\text{domain } D(B) = W \quad \text{and} \quad \boldsymbol{Y} = \boldsymbol{L}^2(\Omega)_{\mathrm{sym}},$$

is a densely defined linear operator and unbounded operator in Q.

Its adjoint operator

$$B^*\colon D(B^*) \subset \boldsymbol{Y} \longrightarrow Q^*$$

is given by the identity

$$\langle B^* \boldsymbol{L}, v\rangle = \langle Bv, \boldsymbol{L}\rangle \quad \text{for all } v \in W$$

and all $\boldsymbol{L}$ from the domain

$$D(B^*) = \{\boldsymbol{L} \in \boldsymbol{Y}\colon |\langle Bv, \boldsymbol{L}\rangle| \le c\,\|v\|_Q \text{ for all } v \in D(B)\}$$

Kirchhoff plate bending problem

standard integration by parts:

$$\int_\omega \boldsymbol{L} : \nabla^2 v \, dy$$

$$= -\int_\omega \operatorname{Div} \boldsymbol{L} \cdot \nabla v \, dy + \int_{\partial\omega} (\boldsymbol{L}n) \cdot \nabla v \, ds$$

$$= -\int_\omega \operatorname{Div} \boldsymbol{L} \cdot \nabla v \, dy + \int_{\partial\omega} \boldsymbol{L}_{nt} \, \partial_t v \, ds + \int_{\partial\omega} \boldsymbol{L}_{nn} \, \partial_n v \, ds$$

? ? $\boldsymbol{L}_{nn} = 0 \text{ on } \gamma_s \cup \gamma_f$

$(\boldsymbol{L}_{nn} = \boldsymbol{L}n \cdot n, \quad \boldsymbol{L}_{nt} = \boldsymbol{L}n \cdot t)$

$\partial\omega = \gamma_s$:

$$D(B^*) = \{\boldsymbol{L} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} : \operatorname{div} \operatorname{Div} \boldsymbol{L} \in H^{-1}(\omega)\}$$

Kirchhoff plate bending problem

Lemma

Let $\boldsymbol{L} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} \cap \boldsymbol{C}^1(\overline{\omega})$*. Then* $\boldsymbol{L} \in D(B^*)$ *if and only if*

$$\boldsymbol{L}_{nn} = 0 \quad \text{on } \gamma_s \cup \gamma_f$$

and

$$[\![\boldsymbol{L}_{nt}]\!]_x = 0 \quad \text{on interior corner points } x \text{ of } \gamma_f.$$

Kirchhoff plate bending problem

Notations for $B = \nabla^2 = \operatorname{Grad}\operatorname{grad}$:

$$B^* = \operatorname{div}\operatorname{Div}, \quad D(B^*) = \boldsymbol{H}(\operatorname{div}\operatorname{Div}, \omega; Q^*)_{\mathrm{sym}}.$$

with

$$\begin{aligned}&\boldsymbol{H}(\operatorname{div}\operatorname{Div}, \omega; Q^*)_{\mathrm{sym}} \\ &= \{\boldsymbol{L} \in \boldsymbol{L}^2(\Omega)_{\mathrm{sym}} : \left|\int_\omega \boldsymbol{L} : \nabla^2 v \, dy\right| \le c \, \|v\|_{H^1(\omega)} \text{ for all } v \in W\}\end{aligned}$$

Observe

$$\boldsymbol{H}_0^1(\omega)_{\mathrm{sym}} \subset \boldsymbol{H}(\operatorname{div}\operatorname{Div}, \omega; Q^*)_{\mathrm{sym}} \subset \boldsymbol{L}^2(\Omega)_{\mathrm{sym}}$$

Bernardi/Girault/Maday (1992), Z. (2015), Pechstein/Schöberl (2011), Rafetseder/Z. (2017)

Kirchhoff plate bending problem

Mixed variational formulation of

Find $\boldsymbol{M} \in \boldsymbol{V}$ and $w \in Q$ such that

$$\int_\omega \mathcal{C}^{-1} \boldsymbol{M} : \boldsymbol{L} \, dy + \langle \operatorname{div} \operatorname{Div} \boldsymbol{L}, w \rangle = 0 \qquad \text{for all } \boldsymbol{L} \in \boldsymbol{V},$$

$$\langle \operatorname{div} \operatorname{Div} \boldsymbol{M}, v \rangle = -\int_\omega g \, v \, dx \qquad \text{for all } v \in Q,$$

with the function spaces

$$\boldsymbol{V} = \boldsymbol{H}(\operatorname{div} \operatorname{Div}, \omega; Q^*)_{\mathrm{sym}}, \quad Q = H^1_{0,\gamma_c \cup \gamma_s}(\omega),$$

equipped with norms $\|v\|_Q = \|v\|_1$ and $\|\boldsymbol{L}\|_{\boldsymbol{V}} = \|\boldsymbol{L}\|_{\operatorname{div} \operatorname{Div}; Q^*}$, given by

$$\|\boldsymbol{L}\|^2_{\operatorname{div} \operatorname{Div}; Q^*} = \|\boldsymbol{L}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div} \operatorname{Div} \boldsymbol{L}\|^2_{Q^*}.$$

Kirchhoff plate bending problem

Theorem

The mixed problem is well-posed in

$$\boldsymbol{V} \times Q = \boldsymbol{H}(\operatorname{div}\operatorname{Div}, \omega; Q^*)_{\mathrm{sym}} \times H^1_{0,\gamma_c \cup \gamma_s}(\omega).$$

Corollary

The primal variational problem and the mixed problem are equivalent:

- *If* $w \in W$ *solves the primal problem,*
  then $\boldsymbol{M} = -\mathcal{C}\nabla^2 w \in \boldsymbol{V}$ *and* $(\boldsymbol{M}, w)$ *solves the mixed problem.*
- *If* $(\boldsymbol{M}, w) \in \boldsymbol{V} \times Q$ *solves the mixed problem,*
  then $w \in W$ *and* w *solves the primal problem.*

Kirchhoff plate bending problem

Theorem

Let ω *be simply connected.* $\boldsymbol{M} \in \boldsymbol{H}(\operatorname{div}\operatorname{Div}, \omega; Q^*)_{\mathrm{sym}}$ *if and only if*

$$\boldsymbol{M} = p\,\boldsymbol{I} + \operatorname{symCurl}\phi$$

with

$$p \in Q = H^1_{0,\gamma_c \cup \gamma_s}(\omega) \quad \textit{and} \quad \phi \in \Psi_p$$

with

$$\Psi_p = \left\{ \psi \in (H^1(\omega))^2 : \langle \partial_t \psi, \nabla v \rangle_\Gamma = -\int_\Gamma p\,\partial_n v\,ds \quad \textit{for all } v \in W \right\}$$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \operatorname{sym}\operatorname{Curl}\phi = \begin{pmatrix} \partial_2\phi_1 & \frac{1}{2}(\partial_2\phi_2 - \partial_1\phi_1) \\ \frac{1}{2}(\partial_2\phi_2 - \partial_1\phi_1) & -\partial_1\phi_2 \end{pmatrix}$$

Krendl/Rafetseder/Z. (2014,2016), Rafetseder/Z. (2017)

Kirchhoff plate bending problem

Equivalent variational formulation in

$$p \in Q = H^1_{0,\gamma_c \cup \gamma_s}(\omega), \quad \phi \in \Psi_p \subset \left(H^1(\omega)\right)^2, \quad w \in Q = H^1_{0,\gamma_c \cup \gamma_s}(\omega).$$

1. Find $p \in Q$ such that

$$\int_\omega \nabla p \cdot \nabla v \, dy = \int_\omega g \, v \, dx \quad \text{for all } v \in Q.$$

2. For given $p \in Q$, find $\phi \in \Psi_p$ such that

$$\int_\omega \mathcal{C}^{-1} \operatorname{symCurl} \phi : \operatorname{symCurl} \psi \, dy = \langle G[p], \psi \rangle \quad \text{for all } \psi \in \Psi_0.$$

3. For given $\boldsymbol{M} = p\boldsymbol{I} + \operatorname{symCurl} \phi$, find $w \in Q$ such that

$$\int_\omega \nabla w \cdot \nabla q \, dy = \langle H[\boldsymbol{M}], q \rangle \quad \text{for all } q \in Q.$$

Decomposition of the problem

2. for $\phi^\perp = (-\phi_2, \phi_1)^T$ we obtain

$$\int_\omega \hat{\mathcal{C}}^{-1} \varepsilon(\phi^\perp) : \varepsilon(\psi) \, dy = \langle \hat{G}[p], \psi \rangle \quad \text{for all } \psi \in \Psi_0^\perp$$

with an appropriately rotated material tensor $\hat{\mathcal{C}}^{-1}$ and

$$\varepsilon_{\alpha\beta}(\psi) = \frac{1}{2}(\partial_\beta \psi_\alpha + \partial_\alpha \psi_\beta)$$

decomposition of the Kirchhoff model into

1. Poisson problem
2. plane linear elasticity problem
3. Poisson problem

Mindlin-Reissner plates

Mindlin-Reissner model

$$J = \frac{1}{2} \int_\omega \left\{ \mathcal{C}\varepsilon(\theta) : \varepsilon(\theta) + \mu\, t^{-2}\, \|\nabla w - \theta\|^2 \right\}\, dx - \int_\omega g\, w\, dx \;\longrightarrow \min$$

Kirchhoff model: $\theta = \nabla w$.

$$J = \frac{1}{2} \int_\omega \left\{ \mathcal{C}\nabla^2 w : \nabla^2 w \right\}\, dx - \int_\omega g\, w\, dx \;\longrightarrow \min$$

decomposition of the Mindlin-Reisser model into

1. Poisson problem
2. Stokes-like problem
3. Poisson problem

Brezzi/Fortin 1986

Kirchhoff-Love shells

variational formulation of the linearized model:

find $\boldsymbol{u} \in \boldsymbol{W}$ such that

$$\int_\omega \left[t\, \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) + \frac{t^3}{12} \mathcal{C}\boldsymbol{\kappa}(\boldsymbol{u}) : \boldsymbol{\kappa}(\boldsymbol{v}) \right] \sqrt{a}\, dy = \langle F, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in \boldsymbol{W}$$

with

$$\boldsymbol{W} \subset H^1(\omega) \times H^1(\omega) \times \boxed{H^2(\omega)},$$

the membrane strain

$$\varepsilon_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}\, u_3,$$

the bending strain

$$\begin{aligned} \kappa_{\alpha\beta}(\boldsymbol{u}) &= u_{3|\alpha\beta} - b_\alpha^\sigma b_{\sigma\beta} u_3 + b_\alpha^\sigma u_{\sigma|\beta} + b_\beta^\tau u_{\tau|\alpha} + b_{\beta|\alpha}^\tau u_\tau \\ &= \boxed{\partial_{\alpha\beta} u_3} + \ldots . \end{aligned}$$

Kirchhoff-Love shells

quantities of interest: $\boldsymbol{u}$ displacement

$\boldsymbol{M} = \frac{t^3}{12}\, \mathcal{C}\boldsymbol{\kappa}(\boldsymbol{u})\sqrt{a}$ bending moments

$= \hat{\mathcal{C}}\boldsymbol{\kappa}(\boldsymbol{u})$

Mixed variational formulation: Find $\boldsymbol{M} \in \boldsymbol{V}$ and $\boldsymbol{u} \in Q$ such that

$$\int_\omega \hat{\mathcal{C}}^{-1}\boldsymbol{M} : \boldsymbol{L}\, dy \qquad -\langle \operatorname{div}\operatorname{Div}\boldsymbol{L}, u_3\rangle - \int_\omega \boldsymbol{L} : \boldsymbol{\kappa}^1(\boldsymbol{u})\, dy = 0$$

$$-\langle \operatorname{div}\operatorname{Div}\boldsymbol{M}, v_3\rangle - \int_\omega \boldsymbol{M} : \boldsymbol{\kappa}^1(\boldsymbol{v})\, dy - \int_\omega t\, \mathcal{C}\epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v})\, \sqrt{a}\, dy \qquad = -\langle F, \boldsymbol{v}\rangle$$

for all $\boldsymbol{L} \in \boldsymbol{V}$ and $\boldsymbol{v} \in Q$, with the function spaces

$$\boldsymbol{V} = \boldsymbol{H}(\operatorname{div}\operatorname{Div}, \omega; Q_3^*)_{\mathrm{sym}},$$
$$Q = Q_1 \times Q_2 \times Q_3,$$

where $Q_i \subset H^1(\omega)$.

Numerical results

benchmark examples of the shell obstacle course:

- Scordelis-Lo roof
- pinched cylinder
- pinched hemisphere

midsurface represented as B-spline surfaces

approximation spaces:

- equal-order approximation for u_i, p, ϕ
- single patch: B-splines of maximum smoothness
- multipatch: C^0

implementation in G+Smo (C++ library for IgA)

Example: Scordelis-Lo roof

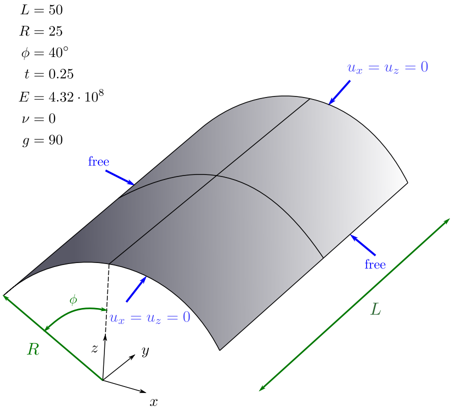
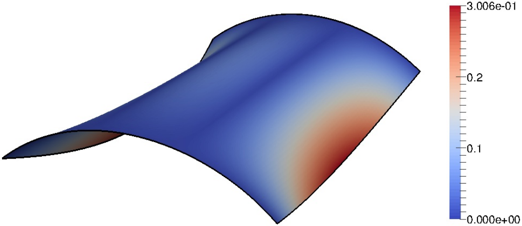


Figure: Geometry and load

Numerical results

Example: Scordelis-Lo roof



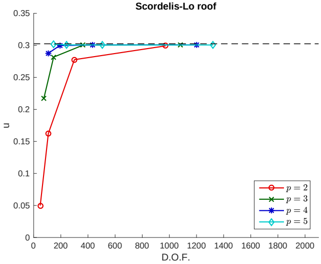
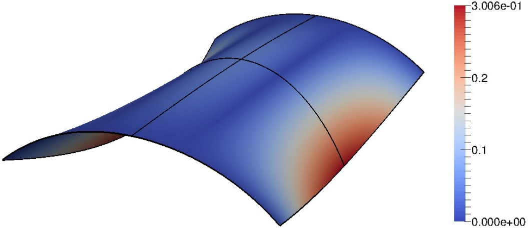


Figure: Results, 1 patch

Example: Scordelis-Lo roof



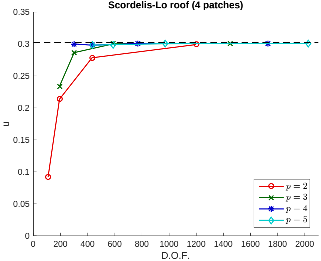


Figure: Results, 4 patches

Numerical results

Example: pinched Cylinder

$$L = 600$$
$$R = 300$$
$$t = 3$$
$$E = 3 \cdot 10^6$$
$$\nu = 0.3$$
$$F = 1$$

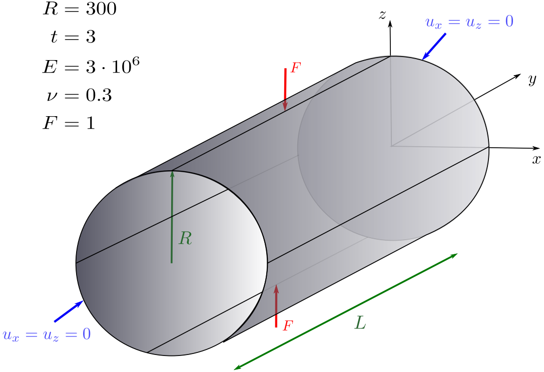


Figure: Geometry and load

Numerical results

Example: pinched Cylinder

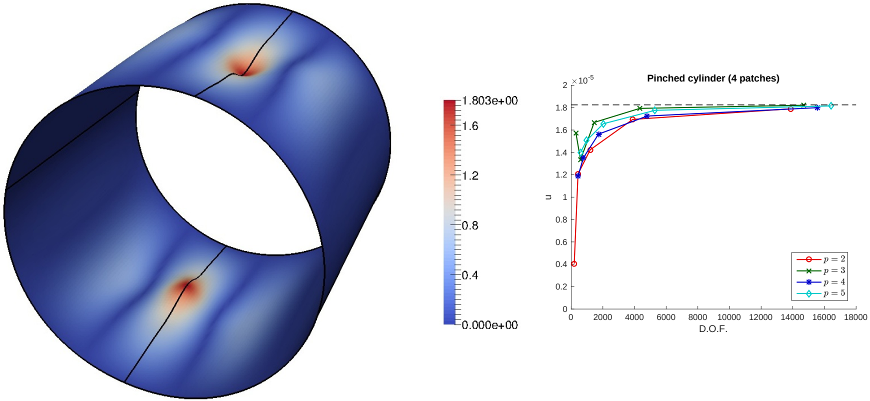
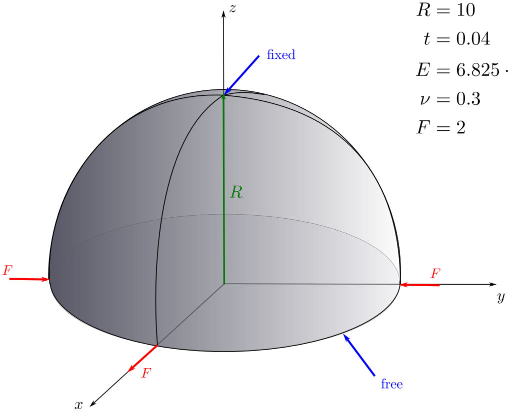


Figure: Results, 4 patches

Example: pinched Hemisphere



$R = 10$

$t = 0.04$

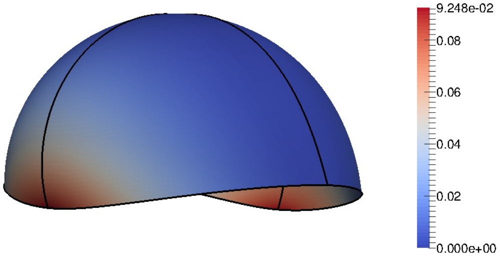
$E = 6.825 \cdot 10^7$

$\nu = 0.3$

$F = 2$

Figure: Geometry and load

Example: pinched Hemisphere



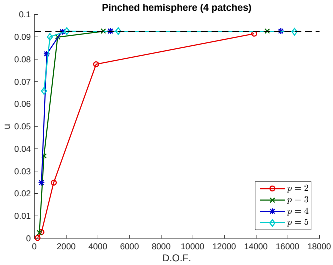


Figure: Results, 4 patches

Conclusion and outlook

- Kirchhoff plate bending problems and similar 4-th order problems can be decomposed in three (consecutively to solve) second-order problems.
- extension to Kirchhoff-Love shell: formulation in H^1 spaces.
- work in progress:
  - extend the numerical analysis from Kirchhoff plates to Kirchhoff-Love shells.
  - behavior of equal-order discretization method w.r.t. membrane locking
  - extension to Mindlin-Reissner plates and shells

References

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