A Posteriori Estimates for a coupled piezoelectric model

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03. Oktober 2017

Outline

Introduction to piezoelectric problem

The system of equations describing deformation of a piezoelectric body in Ω

$$\begin{aligned} \mathrm{Div}\, \sigma(\mathbf{u}, \varphi) &= f, \\ -\mathrm{div}\, D(\mathbf{u}, \varphi) &= g, \end{aligned}$$

for the elastic displacement $\mathbf{u} : \Omega \to \mathbb{R}^d$, and the electric potential field $\varphi : \Omega \to \mathbb{R}$, where the symmetric stress tensor σ and the electric displacement D are coupled via the linear piezoelectric material law:

$$\begin{aligned} \sigma(\mathbf{u}, \varphi) &= \mathbb{L}\varepsilon(\mathbf{u}) + B \cdot \nabla\varphi \\ -D(\mathbf{u}, \varphi) &= K \cdot \nabla\varphi - B^T : \varepsilon(\mathbf{u}) \end{aligned}$$

with the divergence operators for the tensor and vector valued functions, respectively:

$$\mathrm{Div}\, \mathbf{a} = \nabla \cdot \mathbf{a} = a_{ij,j} \quad \text{and} \quad \mathrm{div}\, \mathbf{q} = \nabla \cdot \mathbf{q} = q_{i,i}$$

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and the strain tensor ε is the symmetric part of the displacement gradient:

$$\varepsilon(\mathbf{u}) = 0.5\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

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$\mathbb{L} = (L_{ijkl})$ - linear-elastic material tensor, $c_1^2 \, |\varepsilon|^2 \leq \mathbb{L}\,\varepsilon : \varepsilon \leq c_2^2 \, |\varepsilon|^2, \quad \forall \varepsilon \in \mathbb{M}^{d \times d}_{sym},$

$$\mathbb{L}_{ijkm} = \mathbb{L}_{jikm} = \mathbb{L}_{kmij} \in L^\infty(\Omega), \quad i, j, k, m = 1, ..., d.$$

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$K=(K_{ij})$ - dielectric material tensor with $\gamma_1^2\,|\zeta|^2 \le K\,\zeta\cdot\zeta \le \gamma_2^2\,|\zeta|^2, \quad \forall\zeta\in\mathbb{R}^d$

$$K_{ij}=K_{ji}\in L^\infty(\Omega), \quad i,j,s=1,...,d.$$

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$B = (B_{ijs})$ - piezoelectric tensor

$$b_{ijs} \in L^\infty(\Omega), \quad i,j,s = 1,...,d.$$

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f is the body force vector, g is the electrical charge density

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with the Dirichlét boundary conditions

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \Gamma_{D,\mathbf{u}}, \qquad \varphi = \varphi_0 \quad \text{on} \quad \Gamma_{D,\varphi}$$

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with the Neumann boundary conditions on the remaining part of the boundary

$$\sigma \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{N,\mathbf{u}}, \qquad\qquad D \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{N,\varphi}.$$

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The generalized solution $(\mathbf{u}, \varphi)$ is defined by

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\begin{aligned}
c(\mathbf{u}, \mathbf{w}) + b(\varphi, \mathbf{w}) &= F(\mathbf{w}), \quad \forall w \in V_0, \\
-b(\eta, \mathbf{u}) + k(\varphi, \eta) &= G(\eta), \quad \forall \eta \in M_0,
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$$\begin{aligned} c(\mathbf{u}, \mathbf{w}) &:= \int_\Omega \varepsilon(\mathbf{u}) : \mathbb{L} : \varepsilon(\mathbf{w}) \, \mathrm{d}\mathbf{x}, \qquad b(\varphi, \mathbf{w}) := \int_\Omega \varepsilon(\mathbf{w}) : B \cdot \nabla\varphi \, \mathrm{d}\mathbf{x} \\ k(\varphi, \eta) &:= \int_\Omega \nabla\varphi \cdot K \cdot \nabla\eta \, \mathrm{d}\mathbf{x}, \quad F(\mathbf{w}) := \int_\Omega f \cdot \mathbf{w} \, \mathrm{d}\mathbf{x}, \quad G(\eta) := \int_\Omega g \, \eta \, \mathrm{d}\mathbf{x} \end{aligned}$$

exists and is unique

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$$\mathbf{u} \in V_0 + \mathbf{u}_0, \;\; V_0 := \left\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^d) \,\middle|\, \mathbf{v}|_{\Gamma_{D,\mathbf{u}}} = 0 \right\}$$

and

$$\varphi \in M_0 + \varphi_0, \;\; M_0 := \left\{ \psi \in H^1(\Omega) \,\middle|\, \psi|_{\Gamma_{D,\varphi}} = 0 \right\}.$$

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$$|[\mathbf{u}, \varphi|]^2 := \|\varepsilon(\mathbf{u})\|_{\mathbb{L}}^2 + \|\nabla\varphi\|_K^2 = c(\mathbf{u}, \mathbf{u}) + k(\varphi, \varphi)$$

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Aim: Find the upper bound of

$$|[\mathbf{u} - \mathbf{v}, \varphi - \psi|]^2 := \|\varepsilon(\mathbf{u} - \mathbf{v})\|_{\mathbb{L}}^2 + \|\nabla(\varphi - \psi)\|_K^2.$$

Theorem

i) *For any* $\mathbf{v} \in V_0 + \mathbf{u_0}$ *and* $\psi \in M_0 + \phi_0$ *the combined error norm meets the est.*

$$|[\mathbf{u} - \mathbf{v},\ \varphi - \psi]|^2 \leq \mathcal{M}_1^2(\mathbf{v}, \psi, \boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v}, \psi, \boldsymbol{q}), \tag{1}$$

$$\mathcal{M}_1(\mathbf{v}, \psi, \boldsymbol{\tau}) := \|\boldsymbol{\tau} - \mathbb{L}\,\varepsilon(\mathbf{v}) - B\cdot\nabla\psi\|_{\mathbb{L}^{-1}} + \mu_F(\mathbb{L}, \Omega, \Gamma_{D,\mathbf{u}})\|f + \mathrm{Div}\boldsymbol{\tau}\|$$

$$\mathcal{M}_2(\mathbf{v}, \psi, q) := \|\boldsymbol{q} - K\nabla\psi + B^T : \varepsilon(\mathbf{v})\|_{K^{-1}} + \mu_F(K, \Omega, \Gamma_{D,\varphi})\|g + \mathrm{div}\boldsymbol{q}\|,$$

where μ_F *are constants in the Friedrichs type inequalities,*

$$\boldsymbol{\tau} \in H^+(\Omega, \mathrm{Div}) := \{\mathbf{a} \in H(\Omega, \mathrm{Div}) \mid \int_\Omega (\mathbf{a} : \nabla\mathbf{w} + \mathrm{Div}\mathbf{a}\cdot\mathbf{w})\,dx = 0\ \forall\mathbf{w} \in V_0,\}$$

$$\boldsymbol{q} \in H^+(\Omega, \mathrm{div}) := \{v \in H(\Omega, \mathrm{div}) \mid \int_\Omega (q\cdot\nabla\psi + \mathrm{div}q\psi)\,dx = 0\ \ \forall\psi \in M_0\},$$

ii) *The right-hand side of (3) vanishes if and only if*

$$\mathbf{v} = \mathbf{u},\ \psi = \varphi,\ \boldsymbol{\tau} = \mathbb{L}\,\varepsilon(\mathbf{u}) + B\cdot\nabla\psi \text{ and } \boldsymbol{q} = K\,\nabla\varphi - B^T : \varepsilon(\mathbf{u}).$$

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Voigt notation

Due to symmetries in tensors Voigt notation can be used.

2D-Case:

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\,\varepsilon_{12} \end{pmatrix}, \; \mathbb{L} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & b_{12} \\ 0 & b_{22} \\ b_{31} & 0 \end{pmatrix}, \; K = \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix}.$$

Voigt notation

3D-Case:

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\,\varepsilon_{12} \\ 2\,\varepsilon_{23} \\ 2\,\varepsilon_{13} \end{pmatrix}, \quad \mathbb{L} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & b_{33} \\ 0 & 0 & 0 \\ 0 & b_{52} & 0 \\ b_{61} & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{pmatrix}.$$

Material constants

For the material $PZT4_b$ the material constants are

$$\mathbb{L} = \begin{pmatrix} 13.9 & 7.43 & 0 \\ 7.43 & 11.3 & 0 \\ 0 & 0 & 2.56 \end{pmatrix} \cdot 10^{10} \left[\frac{N}{m^2}\right],$$

$$K = \begin{pmatrix} 6 & 0 \\ 0 & 5.47 \end{pmatrix} \cdot 10^{-9} \left[\frac{F}{m}\right], \quad B = \begin{pmatrix} 0 & -6.98 \\ 0 & 13.84 \\ 13.44 & 0 \end{pmatrix} \left[\frac{C}{m^2}\right].$$

Normalization of material coefficients

Functions $\mathbf{u}$ and φ are measured in physical quantities with corresponding units

$$< U > \qquad \text{and} \qquad < \Phi >$$

We can change them and use other units:

$$< U' > = a < U > \qquad \text{and} \qquad < \Phi' > = b < \Phi > .$$

Then the coefficients of equations changes:

$$c'(\mathbf{u}', \mathbf{w}') = \frac{1}{a^2} c(\mathbf{u}, \mathbf{w}), \qquad \mathbb{L}' = \frac{1}{a^2}\mathbb{L}, \qquad \text{and} \qquad K' = \frac{1}{b^2}K, \qquad B' = \frac{1}{ab}B.$$

By changing the corresponding units of variables, we change the coefficients and can balance them properly. Certainly F and G should be also scaled accordingly.

Numerical tests $PZT4_b$

Let $\Omega = (0,1)^2$, $\Gamma_{N,\mathbf{u}} = \Gamma_{N,\varphi} = \{0 \leq x_1 \leq 1,\ x_2 = 0\}$, $\Gamma_{D,\mathbf{u}} = \Gamma_{D,\varphi} = \partial\Omega \setminus \Gamma_{N,\mathbf{u}}$

Right-hand side f and g in Ω are chosen such that the exact solution is given by

$$(\mathbf{u},\ \varphi) = \left(\begin{pmatrix} 3x_1 x_2^2 \\ x_2^2 \end{pmatrix},\ x_1^2 x_2^2 - 2x_2^2 \right)$$

The normalization constants $a = 10^5$ and $b = 10^{-4}$ are used and

$$\mathbb{L} = \begin{pmatrix} 13.9 & 7.43 & 0 \\ 7.43 & 11.3 & 0 \\ 0 & 0 & 2.56 \end{pmatrix}, \quad K = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.547 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -0.698 \\ 0 & 1.384 \\ 1.344 & 0 \end{pmatrix}.$$

Numerical tests $PZT4_b$

Test 1:
We assume $\mathbf{v} = \mathbf{u} + (2.5x_1^2 x_1, 0.1x_2^2)^T$, $\psi = \varphi - (0.1x_1^2 x_2^2 + 0.2x_2^6)$ and chose

$$\tau = \mathbb{L}\,\varepsilon(\mathbf{v}) + B\nabla\psi, \quad \text{and} \quad q = K\,\nabla\psi \; - B^T\,\varepsilon(\mathbf{v})$$

. In this case

$$\frac{(\mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v},\psi,\boldsymbol{q}))^{1/2}}{\|[\mathbf{u}-\mathbf{v},\,\varphi-\psi]\|} = 2.1059$$

Test 2:
We assume $\mathbf{v} = \mathbf{u} + (2.5x_1^2 x_1, 0.1x_2^2)^T$, $\psi = \varphi - (0.1x_1^2 x_2^2 + 0.2x_2^6)$ and chose

$$\tau = \mathbb{L}\,\varepsilon(\mathbf{u}) + B\nabla\varphi, \quad \text{and} \quad q = K\,\nabla\varphi \; - B^T\,\varepsilon(\mathbf{u})$$

. In this case $\|f + \mathrm{Div}\tau\| = 0$, $\|g + \mathrm{div}q\| = 0$ and

$$\frac{(\mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v},\psi,\boldsymbol{q}))^{1/2}}{\|[\mathbf{u}-\mathbf{v},\,\varphi-\psi]\|} = 1.0229$$

Numerical tests $PZT4_b$

With the normalization constants $a = 10^5$ and $b = 10^{-5}$ we obtain

$$\mathbb{L} = \begin{pmatrix} 13.9 & 7.78 & 7.43 & 0 & 0 & 0 \\ 7.78 & 13.9 & 7.43 & 0 & 0 & 0 \\ 7.43 & 7.43 & 11.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.06 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.56 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.56 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & -6.98 \\ 0 & 0 & -6.98 \\ 0 & 0 & 13.84 \\ 0 & 0 & 0 \\ 0 & 13.44 & 0 \\ 13.44 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 67.9 & 0 & 0 \\ 0 & 67.9 & 0 \\ 0 & 0 & 67.5 \end{pmatrix}.$$

Numerical tests *PLZT – Ceramics*

Let $\Omega = (0,1)^3$,

$$\Gamma_{N,\mathbf{u}} = \Gamma_{N,\varphi} = \{0 \leq x_1 \leq 1,\ 0 \leq x_3 \leq 1,\ x_2 = 0\},\ \Gamma_{D,\mathbf{u}} = \Gamma_{D,\varphi} = \partial\Omega \setminus \Gamma_{N,\mathbf{u}}$$

Right-hand side f and g in Ω are chosen such that the exact solution is given by

$$(\mathbf{u},\ \varphi) = \left(\begin{pmatrix} 3x_1x_2^2 \\ x_2^2 \\ x_3x_2^2 \end{pmatrix},\ x_1^2x_2^2 - 2x_2^2 + \sin(x_3)x_2^2 \right)$$

Numerical tests $PZT4_b$

Test 1:

We assume
$\mathbf{v} = \mathbf{u} + ((2.5x_1^2 x_1, 0.1x_2^2, 0.1x_3x_2^2)^T,\ \psi = \varphi - (0.1x_1^2x_2^2 + 0.2x_2^6 + 0.1\sin(x_3)x_2^2)$ and chose

$$\tau = \mathbb{L}\,\varepsilon(\mathbf{v}) + B\nabla\psi, \quad \text{and} \quad q = K\,\nabla\psi\ - B^T\,\varepsilon(\mathbf{v})$$

. In this case

$$\frac{(\mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v},\psi,\boldsymbol{q}))^{1/2}}{\|[\mathbf{u}-\mathbf{v},\ \varphi-\psi]\|} = 2.7211$$

Test 2:

We assume
$\mathbf{v} = \mathbf{u} + (2.5x_1^2 x_1, 0.1x_2^2, 0.1x_3x_2^2)^T,\ \psi = \varphi - (0.1x_1^2x_2^2 + 0.2x_2^6 + 0.1\sin(x_3)x_2^2)$ and chose

$$\tau = \mathbb{L}\,\varepsilon(\mathbf{u}) + B\nabla\varphi, \quad \text{and} \quad q = K\,\nabla\varphi\ - B^T\,\varepsilon(\mathbf{u})$$

. In this case $\|f + \mathrm{Div}\tau\| = 0,\ \|g + \mathrm{div}q\| = 0$ and

$$\frac{(\mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v},\psi,\boldsymbol{q}))^{1/2}}{\|[\mathbf{u}-\mathbf{v},\ \varphi-\psi]\|} = 1.0374$$

Numerical tests *PLZT – Ceramics*

With the normalization constants $a = 10^5$ and $b = 10^{-5}$ we obtain

$$\mathbb{L} = \begin{pmatrix} 70.43 & 62.40 & -76.55 & 0 & 0 & 0 \\ 62.40 & 70.43 & -76.55 & 0 & 0 & 0 \\ -76.55 & -76.55 & 103.86 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.015 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.525 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8.525 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 23.478 \\ 0 & 0 & 23.478 \\ 0 & 0 & -19.585 \\ 0 & 0 & 0 \\ 0 & 8.785 & 0 \\ 8.785 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 67.9 & 0 & 0 \\ 0 & 67.9 & 0 \\ 0 & 0 & 67.5 \end{pmatrix}.$$

Numerical tests *PLZT – Ceramics*

Let $\Omega = (0,1)^3$,

$$\Gamma_{N,\mathbf{u}} = \Gamma_{N,\varphi} = \{0 \leq x_1 \leq 1,\ 0 \leq x_3 \leq 1,\ x_2 = 0\},\ \Gamma_{D,\mathbf{u}} = \Gamma_{D,\varphi} = \partial\Omega \setminus \Gamma_{N,\mathbf{u}}$$

Right-hand side f and g in Ω are chosen such that the exact solution is given by

$$(\mathbf{u},\ \varphi) = \left(\begin{pmatrix} 3x_1x_2^2 \\ x_2^2 \\ x_3x_2^2 \end{pmatrix},\ x_1^2x_2^2 - 2x_2^2 + \sin(x_3)x_2^2 \right)$$

Numerical tests $PZT4_b$

Test 1:

We assume $\mathbf{v} = \mathbf{u} + ((2.5x_1^2 x_1, 0.1x_2^2, 0.1x_3 x_2^2)^T$, $\psi = \varphi - (0.1x_1^2 x_2^2 + 0.2x_2^6 + 0.1\sin(x_3)x_2^2)$ and chose

$$\tau = \mathbb{L}\,\varepsilon(\mathbf{v}) + B\nabla\psi, \quad \text{and} \quad q = K\,\nabla\psi\ - B^T\,\varepsilon(\mathbf{v})$$

. In this case

$$\frac{(\mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v},\psi,\boldsymbol{q}))^{1/2}}{\|[\mathbf{u}-\mathbf{v},\ \varphi-\psi]\|} = 4.1123$$

Test 2:

We assume $\mathbf{v} = \mathbf{u} + (2.5x_1^2 x_1, 0.1x_2^2, 0.1x_3 x_2^2)^T$, $\psi = \varphi - (0.1x_1^2 x_2^2 + 0.2x_2^6 + 0.1\sin(x_3)x_2^2)$ and chose

$$\tau = \mathbb{L}\,\varepsilon(\mathbf{u}) + B\nabla\varphi, \quad \text{and} \quad q = K\,\nabla\varphi\ - B^T\,\varepsilon(\mathbf{u})$$

. In this case $\|f + \mathrm{Div}\tau\| = 0$, $\|g + \mathrm{div} q\| = 0$ and

$$\frac{(\mathcal{M}_1^2(\mathbf{v},\psi,\boldsymbol{\tau}) + \mathcal{M}_2^2(\mathbf{v},\psi,\boldsymbol{q}))^{1/2}}{\|[\mathbf{u}-\mathbf{v},\ \varphi-\psi]\|} = 1.0511$$

Thank you

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