Regularization of An Optimal Control Problem With BV-Functions

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Problem

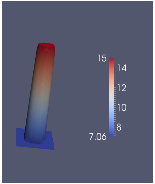
Optimal Control Problem

$$\min_{u \in BV(\Omega)} J(y, u) := \frac{1}{2}\|y - y_\Omega\|^2_{L^2(\Omega)} + \beta|u|_{BV(\Omega)}, \qquad \text{(OC)}$$

$$\text{such that } \begin{cases} -\Delta y &= u \text{ in } \Omega, \\ y|_{\partial\Omega} &= 0. \end{cases} \qquad \text{(PDE)}$$

Here we have:

- $\Omega \subset \mathbb{R}^N$ a $C^{1,1}$-domain, $N \leq 3$,
- $y_\Omega \in L^2(\Omega)$,
- $S : H_0^1(\Omega)^* \to H_0^1(\Omega)$, continuous, linear, injective, solves the PDE,
- $\beta > 0$.

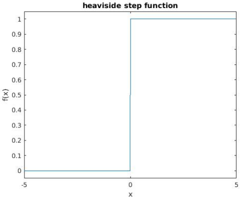


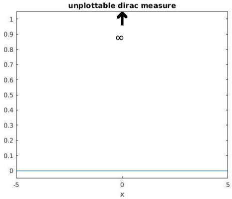
Motivation

(reduced) Optimal Control Problem

$$\min_{u\in BV(\Omega)} j(u) := \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \beta|u|_{BV(\Omega)} \qquad \text{(OC)}$$

- $BV(\Omega) = \left\{u \in L^1(\Omega) : \text{ in the weak sense } \nabla u \in M(\Omega)^N\right\}$,
- $BV(\Omega) \supset W^{1,1}(\Omega)$, allows jump discontinuities,
- $|u|_{BV(\Omega)} = \int_\Omega d|\nabla u|$, total variation.





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$\rightarrow$ we penalize the variation of the control,
$\rightarrow$ we expect the control to be piecewise constant.

Existence of solutions

Optimal Control Problem

$$\min_{u \in BV(\Omega)} j(u) := \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \beta|u|_{BV(\Omega)} \qquad \text{(OC)}$$

Existence and Uniqueness

(OC) has a unique solution $\bar{u} \in BV(\Omega)$.

- Existence: utilizes $N \leq 3$ and lower semi continuity of $|\cdot|_{BV(\Omega)}$ for $L^p(\Omega)$-norms,
- Uniqueness: uses injectivity of S.

Optimality Conditions

Optimal Control Problem

$$\min_{u \in BV(\Omega)} j(u) := \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \beta|u|_{BV(\Omega)} \qquad \text{(OC)}$$

Optimality Conditions [Bredies, Holler 2012]

If $\bar{u} \in BV(\Omega)$ is optimal for (OC) then there exists a $\bar{h} \in W^{\text{div},\infty}(\Omega) \cap W_0^{\text{div},\frac{N}{N-1}}(\Omega)$ with

- $\operatorname{div} \bar{h} = S^*(S\bar{u} - y_\Omega) =: \bar{p}$,
- $\bar{h} = \beta \frac{\nabla \bar{u}_a}{|\nabla \bar{u}_a|}$ $\mathcal{L}^N$-almost everywhere in $\{x \in \Omega : \nabla \bar{u}_a(x) \neq 0\}$, Here $\nabla \bar{u}_a$ is the part of $\nabla \bar{u}$ absolutely continuous $\mathcal{L}^N$.

Demanding more for $\bar{h}$ yields equivalence.

Issues

Optimal Control Problem

$$\min_{u \in BV(\Omega)} j(u) := \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \beta |u|_{BV(\Omega)} \qquad \text{(OC)}$$

Challenges

- Problem is not smooth: $\int_\Omega d|\nabla u|$!
- $BV(\Omega)^*$ is challenging,
- $u \in BV(\Omega)$ is not smooth (finite elements? discretization?).

Ways out

Possible ways out:

- Regularize the $|\cdot|_{BV(\Omega)}$-term (and discretize) ([Acar/Vogel 1994], [Chan/Chan/Zhou 1995]),
- Discretize very specifically on a very particular mesh ([Casas/Kunisch/Pola 1999]),
- Look at the predual problem ([Clason/Kunisch 2011),
- Smoothen u to $H^1(\Omega)$-regularity ([Casas/Kruse/Kunisch 2017] for 1Dish case).

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$\rightarrow$ Try two at once!

Regularization

Optimal Control Problem

$$\min_{u\in H^1(\Omega)} j_{\gamma,\delta}(u) := \frac{1}{2}\|Su - y_\Omega\|^2_{L^2(\Omega)} + \beta\psi_\delta(u) + \frac{\gamma}{2}\|u\|^2_{H^1(\Omega)}. \qquad \text{(ROC)}$$

- $\gamma, \delta > 0$,
- $\psi_\delta(u) = \int_\Omega \sqrt{\delta + |\nabla u|^2}\, dx$.

Existence and Uniqueness

(ROC) has a unique solution $\bar{u}_{\gamma,\delta} \in H^1(\Omega)$.

Path

Notation:

- $\bar{y} = S\bar{u}$, $\bar{y}_{\gamma,\delta} = S\bar{u}_{\gamma,\delta}$,
- $\bar{p} = S^*(\bar{y} - y_\Omega)$, $\bar{p}_{\gamma,\delta} = S^*(\bar{y}_{\gamma,\delta} - y_\Omega)$.

One can show:

- $\|\bar{u}_{\gamma,\delta} - \bar{u}\|_{L^1(\Omega)} + \left||\bar{u}_{\gamma,\delta}|_{BV(\Omega)} - |\bar{u}|_{BV(\Omega)}\right| \xrightarrow{\gamma,\delta\to 0} 0$,
- $\|\bar{y}_{\gamma,\delta} - \bar{y}\|_{H^1(\Omega)}, \|\bar{p}_{\gamma,\delta} - \bar{p}\|_{H^2(\Omega)} \xrightarrow{\gamma,\delta\to 0} 0$,
- $j_{\gamma,\delta}(\bar{u}_{\gamma,\delta}) \xrightarrow{\gamma,\delta\to 0} j(\bar{u})$.

(Stronger results are available)

Optimality conditions for (ROC)

Turns out $j_{\gamma,\delta}$ is once continously Frechét-differentiable $H^1(\Omega) \to \mathbb{R}$.

Sufficient and Necessary Optimality Condition for $\bar{u}_{\gamma,\delta}$

$\bar{u}_{\gamma,\delta}$ is optimal for (ROC) iff $(\bar{u}_{\gamma,\delta}, \bar{y}_{\gamma,\delta}, \bar{p}_{\gamma,\delta})$ satisfies

$$\int_\Omega \left(\gamma + \frac{\beta}{\sqrt{\delta + |\nabla \bar{u}_{\gamma,\delta}|^2}}\right) \nabla \bar{u}_{\gamma,\delta} \cdot \nabla v + \gamma \bar{u}_{\gamma,\delta} v \; dx = \int_\Omega -\bar{p}_{\gamma,\delta} v \; dx \quad \forall v \in H^1(\Omega).$$

Optimizers of (ROC) thus satisfy:

$$\begin{cases} -\operatorname{div}\left(\left(\gamma + \frac{\beta}{\sqrt{\delta+|\nabla \bar{u}_{\gamma,\delta}|^2}}\right) \nabla \bar{u}_{\gamma,\delta}\right) + \gamma \bar{u}_{\gamma,\delta} & = -\bar{p}_{\gamma,\delta} \text{ in } \Omega, \\ \partial_n \bar{u}_{\gamma,\delta} & = 0 \text{ on } \Gamma. \end{cases} \qquad \text{(NHN)}$$

Higher Regularity

$$\begin{cases} -\operatorname{div}\left(\left(\gamma + \frac{\beta}{\sqrt{\delta + |\nabla \bar{u}_{\gamma,\delta}|^2}}\right) \nabla \bar{u}_{\gamma,\delta}\right) + \gamma \bar{u}_{\gamma,\delta} & = -\bar{p}_{\gamma,\delta} \text{ in } \Omega, \\ \partial_n \bar{u}_{\gamma,\delta} & = 0 \text{ on } \Gamma. \end{cases} \tag{NHN}$$

[Fan/Zhao 2000/2006]

Solutions to the quasilinear, homogenous Neumannproblem (*NHN*) are $C^{1,\alpha}(\bar{\Omega})$ regular.

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Solutions to the quasilinear, homogenous Neumannproblem (NHN) are $C^{1,\alpha}(\bar{\Omega})$ regular.

$\rightarrow$ Does this help?

Higher Regularity of $\mathbf{j}_{\gamma,\delta}$

straightforward calculations reveal:

$$\Psi_\delta : W^{1,4}(\Omega) \to \mathbb{R}, \qquad u \mapsto \int_\Omega \sqrt{\delta + |\nabla u|^2}\, dx$$

is twice continously Frechét-differentiable and thus:

C^2-regularity of $j_{\gamma,\delta}$

$j_{\gamma,\delta}|_{W^{1,4}(\Omega)}$ is twice continously Frechét-differentiable. In particular: $j_{\gamma,\delta}|_{C^{1,\alpha}(\bar{\Omega})}$ is twice continously Frechét-differentiable.

Higher Regularity of $\mathbf{j}_{\gamma,\delta}$

The second derivative of $j_{\gamma,\delta}(u)$ for $u \in W^{1,4}(\Omega)$ is given by:

$$j''_{\gamma,\delta}(u)[v,w] =$$
$$(Sv, Sw)_{L^2(\Omega)} + \beta \int_\Omega \frac{\nabla v \cdot \nabla w}{(\delta + |\nabla u|^2)^{\frac{1}{2}}} - \frac{\nabla v \cdot \nabla u \; \nabla w \cdot \nabla u}{(\delta + |\nabla u|^2)^{\frac{3}{2}}} \; dx + \gamma(v,w)_{H^1(\Omega)}$$

- All summands of $j_{\gamma,\delta}$ are convex,
- All summands of $j''_{\gamma,\delta}$ are (at least) positive semi definite,
- For $u \in W^{1,4}(\Omega)$ and $v \in H^1(\Omega)$:

$$j''_{\gamma,\delta}(u)[v,v] \geq \gamma \|v\|^2_{H^1(\Omega)}.$$

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→ Newton's method can now be applied to $j_{\gamma,\delta}|_{C^{1,\alpha}(\bar{\Omega})}$.

Newton's Method in $C^{1,\alpha}(\bar{\Omega})$

Newton's Method (bare bones version)

0 Choose $k = 0$, $u_k = 0 \in C^{1,\alpha}(\bar{\Omega})$,

1 Compute $u_{k+1} = u_k - \left(j''_{\gamma,\delta}(u_k)\right)^{-1} j'_{\gamma,\delta}(u_k)$,

2 $k = k + 1$, goto line 1.

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[Troianello 1987]

For $u \in C^{1,\alpha}(\bar{\Omega})$, $f \in C^{0,\alpha}(\bar{\Omega})^N$, $g \in L^\infty(\Omega)$ is the solution s of

$$\begin{cases} -\operatorname{div}\left(\left(\gamma + \frac{\beta}{\sqrt{\delta + |\nabla u|^2}}\right)\nabla s\right) + \gamma s & = -\operatorname{div} f + g \text{ in } \Omega, \\ \partial_n s & = 0 \text{ on } \Gamma, \end{cases}$$

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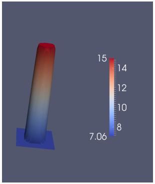
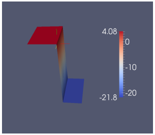
→ one can now show local quadratic convergence, depends on $\gamma, \delta > 0$.

Pathfollowing

$u = 0, \gamma = \gamma_0, \delta = \delta_0, \lambda \in (0,1), \mathrm{NewtonTol} > 0;$
for $n = 1, 2, 3, \ldots$
    $u_{old} = u, \mathrm{MaxNewtonIt} > 0;$
    $u = \mathrm{Newton}(u, j_{\gamma,\delta}, \mathrm{NewtonTol});$
    if(Newton converged)
        $\gamma = \lambda\gamma, \delta = \lambda\delta, \lambda = 0.5\lambda;$
    else
        $\gamma = \frac{1}{\lambda}\gamma, \delta = \frac{1}{\lambda}\delta, \lambda = \frac{1+\lambda}{2};$
        $u = u_{old};$





Outlook/Extensions

- performance enhancehemnt when working with (y, p), not u,
- prove convergence of the pathfollowing algorithm $\rightarrow$ use a modified version of $j_{\gamma,\delta}$,
- use a posteriori regularization error estimators to adapt γ and δ indepently.
- properly analyse the finite element error, use adaptivity there,

Thank You very much!!

Questions, Criticisms, Insights?