Stabilized Space-Time Finite Element Methods for Parabolic Problems

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1. Introduction
   - Notation-Objectives
   - Space-Time forms
2. Stabilized FEM by bubbles
   - The discrete form
   - Error Analysis
3. Numerical Examples

Goal-Objectives

$u_t - \kappa\Delta u = f$ in Q + Initial and BCs

▶ Usually, a separation of the discretizations in space and time is applied, e.g., we discretize in space and then in time by Finite-Difference method.

In this work:

- We formulate the problem in a setting that does not distinguish between time and space, The variational setting is space and time.
- We entirely develop a fully discrete scheme in the finite element frame.
- We stabilize the first-order in time using bubble finite element spaces.

Model Parabolic Problem

$$\overline{Q} = \overline{\Omega} \times [0,T],\ \Sigma = \partial\Omega \times (0,T), \quad \Sigma_T = \Omega \times \{T\}, \quad \Sigma_0 = \Omega \times \{0\}$$

$$u_t - \kappa \Delta u = f \quad \text{in } Q$$

$$u = 0 \quad \text{on} \quad \Sigma, \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{on} \quad \Sigma_0,$$

where $f : Q \to \mathbb{R}$, and $u_0 : \Omega \to \mathbb{R}$, are known, and $u : \overline{Q} \to \mathbb{R}$ is the unknown, $u(x,t)$. Let ℓ, m be positive integers, we define the spaces

$$H^{\ell,m}(Q) = \{u \in L^2(Q) : \partial_x^\alpha u \in L^2(Q) \text{ with } 0 \leq |\alpha| \leq \ell, \text{ and } \partial_t^i u \in L^2(Q),\ i = 1, ..., m\}$$

and the subspaces

$$H_0^{1,0}(Q) = \{u \in L^2(Q) : \nabla_x \in [L^2(Q)]^d,\ u = 0 \text{ on } \Sigma\},$$

$$H_{0,\bar{0}}^{1,1}(Q) = \{u \in L^2(Q) : \nabla_x \in [L^2(Q)]^d,\ \partial u \in L^2(Q), u = 0 \text{ on } \Sigma,\ u = 0 \text{ on } \Sigma$$

Weak formulation

weak solution

$$\tilde{a}(u,v) = l(v), \text{ for all } v \in H^{1,1}_{0,\bar{0}}(Q), \quad \text{with} \tag{1.2a}$$

$$\tilde{a}(u,v) = -\int_Q \partial_t u(x,t) v(x,t)\, dx\, dt + \kappa \int_Q \nabla_x u(x,t) \cdot \nabla_x v(x,t)\, dx\, dt, \tag{1.2b}$$

$$l(v) = \int_Q f(x,t) v(x,t)\, dx\, dt. \tag{1.2c}$$

Weak formulation

Assumption: u belongs to $V_{0,\underline{0}} = H^{1,1}_{0,\underline{0}}(Q) \cap H^{\ell,m}(Q)$ with $\ell \geq 2$ and $m > 1$.

weak solution satisfies the form

$$a(u,v) = l(v), \text{ for all } v \in H^{1,1}_{0,\bar{0}}(Q), \quad \text{with} \tag{1.3a}$$

$$a(u,v) = \int_Q \partial_t u(x,t) v(x,t)\, dx\, dt + \kappa \int_Q \nabla_x u(x,t) \cdot \nabla_x v(x,t)\, dx\, dt, \tag{1.3b}$$

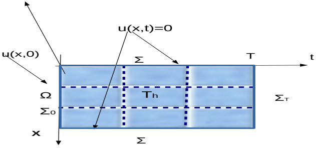
$$l(v) = \int_Q f(x,t) v(x,t)\, dx\, dt. \tag{1.3c}$$

Space-Time $T_h(Q)$ and FE solution

$T_h(Q)$ be a regular partition of Q into triangular/ quadrilateral.

$$V_{h0} = \{v_h \in H^{1,1}_{0,\underline{0}}(Q) : v_h|_E \in \mathbb{P}^1(E), \text{ for every } E \in T_h(Q)\}, \qquad (1.4)$$

$$\boxed{\text{find } u_h \in V_{h0} \text{ such that : } a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}}.$$



For κ is small, the coercivity properties can not ensure that the finite element scheme performs well.

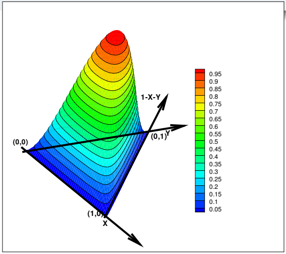
Stabilized spaces

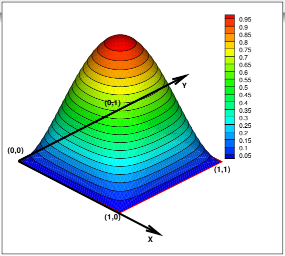
$$u_t - \kappa \Delta u = f \text{ in } Q \quad \text{and} \quad u = 0 \text{ on } \Sigma, \quad u(\cdot, 0) = \text{ on } \Sigma_0, \tag{2.1}$$

The FE spaces

$$V_{h,b} = \{v_h \in V_{0,\underline{0}} : v_h|_E \in \mathbb{P}^1(E) \oplus V_B(E), \text{ for every } E \in T_h(Q)\}, \tag{2.2}$$

$$v_h = v_h^1 + v_h^b, \text{ with } v_h^1 \in V_{h0} \text{ and } v_h^b \in V_B. \tag{2.3}$$





The discrete problem

find $u_h \in V_{h,b}$ $s.t.:$ $a_h(u_h, v_h) := a(u_h, v_h) + b_h(u_h^b, v_h^b) = (f, v_h), \; \forall v_h \in V_{h,b}$

$$a(w, v) = \int_Q \partial_t w \, v \; dx \, dt + \kappa \int_Q \nabla_x w \cdot \nabla_x v \; dx \, dt, \tag{2.4a}$$

$$b_h(w^b, v^b) = \theta h \int_Q \partial_t w^b \, \partial_t v^b \; dx \, dt, \; \theta > 0.$$

$$\|v_h\|_h = \left(\kappa \|\nabla_x v_h\|^2_{L^2(Q)} + \theta h \|\partial_t v_h^b\|^2_{L^2(Q)} + \frac{1}{2}\|v_h\|_{L^2(\Sigma_T)}\right)^{\frac{1}{2}}. \tag{2.5}$$

Classical properties

The form $a_h(\cdot,\cdot) : V_{h,b} \times V_{h,b} \to \mathbb{R}$ is $V_{h,b}$-coercive, i.e.,

$$a_h(v_h, v_h) \geq C_s \, \|v_h\|_h^2, \quad \forall \, v_h \in V_{h,b}. \tag{2.6}$$

Let $v_h \in V_{h,b}$. We have that $v_h(x,0) = 0$ and $n_t|_\Sigma = 0$.

Then, by Green's formula $\int_Q \partial_t v_h \, v_h + v_h \, \partial_t v_h \, dx\, dt = \int_{\partial Q} n_t v_h^2 \, ds,$

....

Classical properties

We define the norms

$$\|v\|_{h,*} = \left(\kappa\|\nabla_x v\|^2_{L^2(Q)} + \theta h\|\partial_t v\|^2_{L^2(Q)} + \frac{1}{2}\|v\|^2_{L^2(\Sigma_T)}\right)^{\frac{1}{2}},$$

$$\|v\|_{h,V} = \left(\kappa\|\nabla_x v\|^2_{L^2(Q)} + \theta h\|\partial_t v\|^2_{L^2(Q)} + \frac{1}{2}\|v\|^2_{L^2(\Sigma_T)} + (\theta h)^{-1}\|v\|^2_{L^2(Q)}\right)^{\frac{1}{2}}.$$

$\|\cdot\|_{h,*}$ is defined for all $v \in V_{0,\underline{0}}$.

Lemma

There is a constant $C_b(\kappa,\theta,h) > 0$ *such that*

$$|a(u,v_h)| \leq C_b(\kappa,\theta,\, h)\,\|u\|_{h,V}\|v_h\|_h, \quad \forall (u,v_h) \in (V_{0,\underline{0}} \times V_{h,b}). \tag{2.7}$$

Weak consistency and convergence

Lemma (weak consistency)

Let u_h *be the FE solution,* u *the continuous, and* $z_h \in V_{h,b}$ *and* $v_h^1 \in V_{h0}^1$,

$$a_h(u_h, z_h) = a(u, z_h), \quad \textit{and} \tag{2.8a}$$

$$a_h(v_h^1, z_h) = a(v_h^1, z_h). \tag{2.8b}$$

Approximation bounds

Theorem

Under regularity assumptions and choosing $\theta \geq h$, *exists* $c_{*,V}$,

$$\|u - u_h\|_{h,*}^2 \leq c_{*,V}^2 \left(\mu_1(\kappa,\theta,h)\|u - z_h^1\|_{h,V}^2 + \mu_2(\kappa,\theta,h)\|u - z_h^1\|_{L^2(Q)}^2\right), \quad fo$$

where $\mu_1(\kappa,\theta,h) = \left(1 + \widetilde{\gamma}^2(\kappa,\theta,h) + \widetilde{\gamma}^2(\kappa,\theta,h)\,\widetilde{\gamma}(\kappa,\theta,h) + \widetilde{\gamma}(\kappa,\theta,h)\right)$ *and* $\mu_2(\kappa,\theta,h) = \widetilde{\gamma}^2(\kappa,\theta,h)\,h^{-2}$, *with* $\widetilde{\gamma}(\kappa,\theta,h) = (\theta h)^{\frac{1}{2}}\,\gamma(\kappa,\theta,h)$ *and* $\gamma(\kappa,\theta,h) = \left(\frac{1}{(\theta\, h)^{\frac{1}{2}}} + \frac{c_{inv}\kappa^{\frac{1}{2}}}{h}\right)$.

Approximation bounds: skeleton of the proof

Let $z_h^1 \in V_{h0}$ and $\sigma_h = (u_h^1 + u_h^b) - z_h^1$.

1. $\|u - u_h\|_{h,*}^2 \leq \ldots \leq \|u - z_h^1\|_{h,V}^2 + \overbrace{(\theta h)\|\partial_t u_h^1 - \partial_t z_h^1\|_{L^2(Q)}^2}^{T_2} + \|\sigma_h\|_h^2,$

2. $\|\partial_t u_h^1 - \partial_t z_h^1\|_{L^2(Q)} \leq \sup_{v_h \in V_{h,b}} \dfrac{\int_Q \partial_t\big((u_h^1 + u_h^b) - z_h^1 - u_h^b\big)\, v_h \; dx\, dt}{\|v_h\|_{L^2(Q)}}$
$\leq \ldots computations \ldots \leq \gamma(\kappa, \theta, h)\big(\|u - z_h^1\|_{h,V} + \|\sigma_h\|_h\big).$

3. $\|\sigma_h\|_h^2 \leq a_h(\sigma_h, \sigma_h) = a_h(u_h, \sigma_h) - a_h(z_h^1, \sigma_h) \overset{(2.8)}{=} a(u, \sigma_h) - a(z_h^1, \sigma_h) =$
$a(u - z_h^1, \sigma_h) \leq \ldots many\, calculations$
$\leq 2\, c_\varepsilon\, (\theta h)^{\frac{1}{2}}\, \gamma(\kappa, \theta, h)\, \|u - z_h^1\|_{h,V}^2 + c_\varepsilon h^{-2} \|u - z_h^1\|_{L^2(Q)}^2 + 2\varepsilon\, C_{inv,\kappa} \|\sigma_h\|_h^2,$

4. $T_2 := \theta h\, \|\partial_t u_h^1 - \partial_t z_h^1\|_{L^2(Q)}^2 \leq 2\, \widetilde{\gamma}^2(\kappa, \theta, h) \left(\|u - z_h^1\|_{h,V}^2 + \|\sigma_h\|_h^2\right)$
$\leq \ldots \leq 2C_\varepsilon \Big(\big(\widetilde{\gamma}^2(\kappa, \theta, h) + \widetilde{\gamma}^2(\kappa, \theta, h)\, \widetilde{\gamma}(\kappa, \theta, h)\big) \|u - z_h^1\|_{h,V}^2 +$
$\widetilde{\gamma}^2(\kappa, \theta, h)\, h^{-2} \|u - z_h^1\|_{L^2(Q)}^2\Big).$

Error estimates

- $|v - \pi_h v|_{H^m(Q)} \leq c_{intp}\, h^{\min(p+1,s)-m}\, \|v\|_{H^s(Q)}$,
- $\|v - \pi_h v\|^2_{h,V} \leq c_2 \big(\kappa\, h^{2r-2} + \theta\, h^{2r-1} + \theta^{-1}\, h^{2r-1} + h^{2r-1}\big) \|v\|^2_{H^s(Q)}$,

Theorem (error estimates)

Let $u \in V := H^{1,1}_{0,\underline{0}}(Q) \cap H^s(Q)$*, with* $s \geq 2$*.The* u_h *satisfies the estimate*

$$\|u - u_h\|_{h,*} \leq c_1\, h\, \|u\|_{H^s(Q)}, \quad \textit{for} \quad \theta \approx h, \tag{2.9}$$

remarks

1. The analysis has been derived without tuning θ with respect to κ,
2. The analysis holds for $\mathbb{P}^{p \geq 2}$.

Smooth Solution $u(x,t) = \sin(2\pi x)\sin(\pi t)$

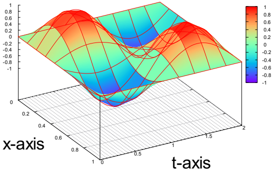
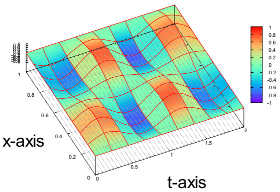


Figure: Example 1: The solution u on Q.

h_s	$p=1,\ \kappa=1$	$p=1,\ \kappa=0.005$
$h_0/2^s$	Convergence rates r	
$s=1$	0.98	1.61
$s=2$	0.99	1.51
$s=3$	0.99	1.20
$s=4$	0.98	1.35
$s=5$	1.00	1.17
$s=6$	0.99	1.05
$s=7$	1.00	1.01

Table: The convergence rates r.

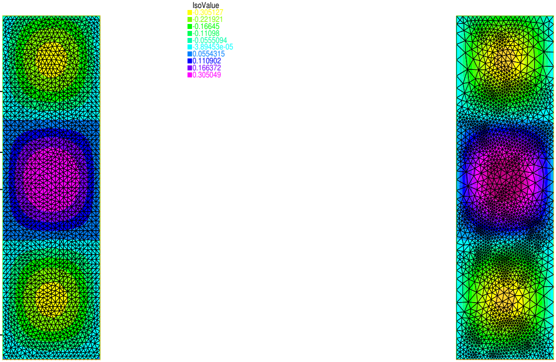
Smooth Solution $u(x,t) = \sin(2\pi x)\sin(2\pi t)$



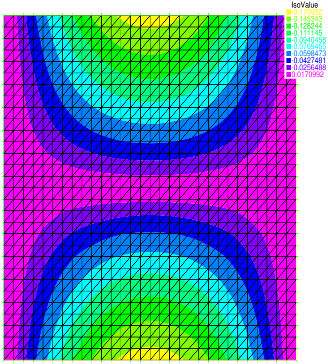
h_s	$p = 2,\ \kappa = 1$	$p = 2,\ \kappa = 0.005$
$h_0/2^s$	Convergence rates r	
$s = 1$	1.73	1.83
$s = 2$	1.98	2.17
$s = 3$	2.01	2.04
$s = 4$	2.00	2.05
$s = 5$	2.00	2.00
$s = 6$	2.00	2.00
$s = 7$	2.00	2.00

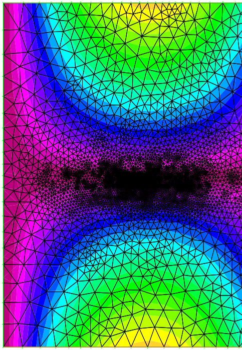
Smooth Solution $u(x,t) = (x^2 - x)\sin(3t)$

$\frac{h}{2^s}$	rates	
	$\kappa = 1$	$\kappa = 0.005$
$s = 4$	0.98	1.35
$s = 5$	1.00	1.17
$s = 6$	0.99	1.05
$s = 7$	1.00	1.01



Singular $u(x,t) = |t - 0.5|^{0.66}(x^2 - x) \in H^{\ell,1.1}(Q)$





Conclusions

- A stabilized FE method presented for parabolic problems.
- Optimal convergence rates for smooth solutions and linear polynomial spaces.

Thank you for your attention

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