Functional type error control for stabilised space-time IgA approximations to I-BVP of parabolic type

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Joint project with U. Langer[2] and S. Repin[3] on
'Fully-adaptive space-time IgA schemes for parabolic problems'

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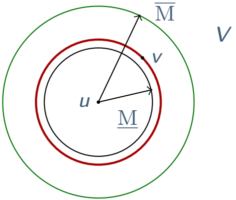
Functional type error estimates (EEs)

For a class of parabolic I-BVP problems

$$\partial_t u + \mathcal{L}\, u = f, \quad u(0) = u_0, \quad \text{in } \Omega \subset \mathbb{R}^d,\ t \in (0, T)$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

with unknown exact solution $u \in V$,
reconstructed approximation $v \in V$, and
problem data $\mathcal{D}$.



Functional a posteriori EEs

$$\underline{\mathrm{M}}(v, \mathcal{D}(\Omega, u_0, f)) \leq \| u - v \| \leq \overline{\mathrm{M}}(v, \mathcal{D}(\Omega, u_0, f)),$$

minorant error majorant

- universal for any $v \in V$,
- computable,
- reliable, i.e., $\| u - v \| \leq \overline{\mathrm{M}}(v, \mathcal{D})$,
- realistic in comparison to error, i.e., $I_{\mathrm{eff}} = \frac{\overline{\mathrm{M}}}{\| u-v \|}$ is close to 1,
- efficient for adaptive algorithms $V_h \to V_{h_{\mathrm{ref}}}$.

Model I-BVP problem

Find $u : \overline{Q} \to \mathbb{R}$ satisfying *linear parabolic initial-boundary value problem (I-BVP)*

$$
\begin{aligned}
\partial_t u - \operatorname{div}_x \mathbf{p} &= f && \text{in } Q,\\
\mathbf{p} &= \nabla_x u && \\
u(x,0) &= u_0 && \text{on } \Sigma_0,\\
u &= 0 && \text{on } \Sigma,
\end{aligned}
$$

$x \in \Omega \subset \mathbb{R}^d$, $d = \{1, 2, 3\}$, $T > 0$
$(x, t) \in Q := \Omega \times (0, T)$
$(x, t) \in \partial Q := \Sigma \cup \overline{\Sigma}_0 \cup \overline{\Sigma}_T$
$\Sigma := \partial\Omega \times (0, T)$
$\Sigma_0 := \Omega \times \{0\}$
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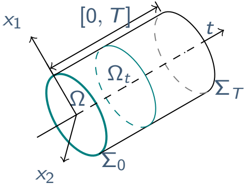

where ∂_t is the time derivative,
div_x and ∇_x are divergence and gradient operators in space, respectively,
$u_0 \in H^1_0(\Sigma_0)$ is a given initial state,
f is a source function in $L^2(Q)$, with $\|u\|_{L^2(Q)} = \|u\|_Q$ induced by $(v, w)_Q =: \int_Q v\, w \,\mathrm{d}x\mathrm{d}t$.

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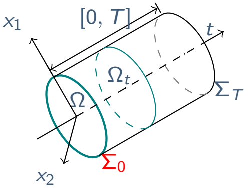

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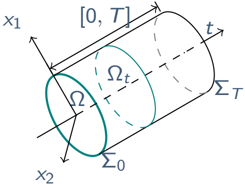

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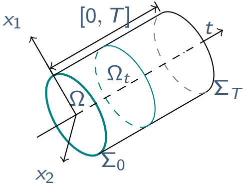

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Solvability results **[Ladyzhenskaya, 1954]**

- **Weak formulation**: if $f \in L^2(Q)$ and $u_0 \in L^2(\Sigma_0)$, find $u \in H_0^{1,0}(Q) := \{ v \in L^2(Q) \,:\, \nabla_x v \in [L^2(Q)]^d,\ v|_{\Sigma} = 0 \}$ satisfying

$$\mathbf{a}(\mathbf{u}, \mathbf{w}) = \mathbf{l}(\mathbf{w}), \quad \forall w \in H_{0,\overline{0}}^{1,1}(Q) = \{ v \in H_0^{1,0}(Q) \,:\, \partial_t v \in L^2(Q),\ v|_{\overline{\Sigma}_T} = 0 \},$$

$$a(u, w) := \big(\nabla_x u, \nabla_x w\big)_Q - \big(u, \partial_t w\big)_Q,$$

$$l(w) := \big(f, w\big)_Q + (u_0, w)_{\Sigma_0}.$$

- **The distance between generalised solution** $u \in H_0^1(Q)$ and **any function** $v \in H_0^1(Q)$ is measured in terms of the norm

$$\| u - v \|^2_{(\nu)} := \nu_{x,Q} \underbrace{\|\nabla_x (u - v)\|^2_Q}_{\text{energy error}} + \nu_{t,\Sigma_T} \underbrace{\| u - v \|^2_{\Sigma_T}}_{\text{error at the final time}}, \nu_{x,Q}, \nu_{t,\Sigma_T} > 0.$$

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Functional a posteriori error analysis for model I-BVP problem

Theorem [Repin, 2002]*

For $\forall v \in H^1_0(Q)$ and $\forall \boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q) := \left\{ \boldsymbol{y} \in [L^2(Q)]^{d+1} : \mathrm{div}_x \boldsymbol{y} \in L^2(Q) \right\}$, we have

$$\| u - v \|^2_{(\nu)} := \nu_{x,Q} \, \|\nabla_x (u-v)\|^2_Q + \nu_{t,\Sigma_T} \, \| u - v \|^2_{\Sigma_T}$$

$$\leq \overline{\mathrm{M}}^{\mathrm{I}}(v, \boldsymbol{y}) := (1 + \beta^{\mathrm{I\!I}}) \underbrace{\|\boldsymbol{y} - \nabla_x v\|^2_Q}_{\text{dual term}} + (1 + \tfrac{1}{\beta^{\mathrm{I\!I}}}) \, C^2_{\mathrm{F}\Omega} \underbrace{\|f + \mathrm{div}_x \boldsymbol{y} - \partial_t v\|^2_Q}_{\text{reliability term}},$$

$\forall w \in H^1_0(Q)$ (additional free-function), we have :

$$\| u - v \|^2_{(\tilde{\nu})} \leq \overline{\mathrm{M}}^{\mathrm{I\!I}}(v, \boldsymbol{y}, w) := \|w - v\|^2_{\Sigma_T} + 2\,\mathcal{F}(v, w_h - v)$$

$$+ (1 + \beta^{\mathrm{I\!I}}) \underbrace{\|\boldsymbol{y} + w - 2\nabla_x v\|^2_Q}_{\text{dual term}} + (1 + \tfrac{1}{\beta^{\mathrm{I\!I}}}) \, C^2_{\mathrm{F}\Omega} \underbrace{\|f + \mathrm{div}_x \boldsymbol{y} - \partial_t w\|^2_Q}_{\text{improved reliability term}},$$

where $\mathcal{F}(v, w - v) := (\nabla_x v, \nabla_x (w - v)) + (\partial_t v - f, w - v)$, and $\beta^{\mathrm{I\!I}}, \beta^{\mathrm{I\!I}} > 0$.

* Numerically tested in [Gaevskaya, Repin, 2005], [Matculevich, Repin, 2014], [Matculevich, Holm, 2017].

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Solvability results [Ladyzhenskaya, 1954]

- **Stronger results**: if $f \in L^2(Q)$ and $u_0 \in H^1_0(\Sigma_0)$, then the I-BVP is **uniquely solvable** in

$$H_0^{\Delta_x,1}(Q) = \{u \in H_0^{1,1}(Q) \,:\, \Delta_x u \in L^2(Q)\}.$$

- For any $v \in H_0^{\Delta_x,1}(Q)$ approximating u, we have **error identity** [Anjam, Pauly 2016]:

$$\begin{aligned}\|\Delta_x(u-v)\|^2_Q + \|\partial_t(u-v)\|^2_Q + \|\nabla_x(u-v)\|^2_{\Sigma_T} \\ =: \|u-v\|^2_{\mathcal{L},Q} \equiv \mathrm{Ed}(v) \\ := \|\nabla_x(u_0-v)\|^2_{\Sigma_0} + \|\Delta_x v + f - \partial_t v\|^2_Q.\end{aligned}$$

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Weak formulation of stabilised parabolic I-BVP

Stabilised weak formulation for $u \in H^1_{0,\underline{0}}(Q) := \{ w \in H^1_0(Q) \,:\, v|_{\overline{\Sigma}_0} = 0 \}$:
we use the upwind test function

$$\lambda\, w + \mu\, \partial_t w, \quad w \in H^{\nabla_x \partial_t, 1}_{0,\underline{0}} := \{ w \in H^{\Delta_x, 1}_{0,\underline{0}} : \nabla_x \partial_t w \in L^2(Q) \}, \quad \lambda, \mu \geq 0 :$$

such that

$$a(u, \lambda\, w + \mu\, \partial_t w) \;=:\; \mathbf{a_s}(\mathbf{u}, \mathbf{w}) = \mathbf{l_s}(\mathbf{w}) \;:=\; l(\lambda\, w + \mu\, \partial_t w)_Q, \quad \forall w \;\in\; H^{\nabla_x \partial_t, 1}_{0,\underline{0}}.$$

For any v and any $u \in H^{\Delta_x, 1}_{0,\underline{0}}$, the error $e = u - v$ is measured in terms of

$$\| \, u - v \, \|^2_{(\boldsymbol{\nu}, s)} := \underbrace{\nu_{x,Q} \, \|\nabla_x (u - v)\|^2_Q + \nu_{t,Q} \, \|\partial_t (u - v)\|^2_Q}_{\text{energy error}}$$

$$+ \underbrace{\nu_{x,\Sigma_T} \| \, \nabla_x (u - v) \, \|^2_{\Sigma_T} + \nu_{t,\Sigma_T} \| \, u - v \, \|^2_{\Sigma_T}}_{\text{error at the final time}},$$

$\nu_{x,Q},\, \nu_{t,Q},\, \nu_{t,\Sigma_T},\, \nu_{t,\Sigma_T} > 0.$

Derivation of advanced form of the majorant

- **[Ladyzhenskaya, 1985]**: $H_{0,\underline{0}}^{\nabla_x \partial_t, 1}$ is **dense** in $H_{0,\underline{0}}^{\Delta_x, 1}$, where

$H_{0,\underline{0}}^{\nabla_x \partial_t, 1}$, $\quad \|w\|^2_{H_{0,\underline{0}}^{\nabla_x \partial_t, 1}} := \sup\limits_{t \in [0,T]} \|\nabla_x w(\cdot, t)\|^2_Q + \|w\|^2_{H_{0,\underline{0}}^{\Delta_x}}$, and

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- For sequence $u_n \in H_{0,\underline{0}}^{\nabla_x \partial_t, 1}$,

$$a_s(u_n, w) = (f_n, \lambda\, w + \mu\, \partial_t w)_Q, \quad \text{where} \quad f_n = (u_n)_t - \Delta_x u_n \in L^2(Q).$$

- Consider the approx. seq. $v_n \in H_{0,\underline{0}}^{\nabla_x \partial_t, 1}$ and subtract $a_s(v_n, w)$, such that

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- Setting $w = e_n = u_n - v_n \in H_{0,\underline{0},1}^{\nabla_x \partial_t}$, we arrive at the so-called 'error-identity'

$$\lambda\, \|\nabla_x e_n\|^2_Q + \mu\, \|\,\partial_t e_n\|^2_Q + \tfrac{1}{2}\,(\mu\, \|\nabla_x e_n\|^2_{\Sigma_T} + \lambda \|e_n\|^2_{\Sigma_T})$$
$$= \lambda\Big((f_n - \partial_t v_n, e_n)_Q - (\nabla_x v_n, \nabla_x e_n)_Q\Big) + \mu\Big((f_n - \partial_t v_n, \partial_t e_n)_Q - (\nabla_x v_n, \nabla_x\, \partial_t e_n)_Q\Big).$$

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Derivation of advanced form of majorant

- Introduce Hilbert spaces for **auxiliary vector-valued functions**

$$\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q) = \{\, \boldsymbol{y} \in [L^2(Q)]^{d+1} \ : \ \mathrm{div}_x \boldsymbol{y} \in L^2(Q) \,\}$$

such that

$$(\mathrm{div}_x \boldsymbol{y}, \lambda\, e + \mu\, \partial_t e)_Q + (\boldsymbol{y}, \nabla_x(\lambda\, e + \mu\, \partial_t e))_Q = 0, \quad \lambda, \mu > 0.$$

- Two forms of the majorants are obtained from

$$\lambda\, \|\nabla_x e_n\|^2_Q + \mu\, \|\partial_t\, e_n\|^2_Q + \tfrac{1}{2}\, (\mu\, \|\nabla_x e_n\|^2_{\Sigma_T} + \lambda \|e\|^2_{\Sigma_T})$$
$$= \lambda\, \big((f + \mathrm{div}_x \boldsymbol{y} - \partial_t v_n, e_n)_Q \quad + (\boldsymbol{y} - \nabla_x v_n, \nabla_x e_n)_Q \big)$$
$$+ \mu\, \big((f + \mathrm{div}_x \boldsymbol{y} - \partial_t v_n, \partial_t e_n)_Q + (\boldsymbol{y} - \nabla_x v_n, \nabla_x \partial_t e_n)_Q \big).$$

Derivation of advanced form of majorant

- Introduce Hilbert spaces for **auxiliary vector-valued functions**

$$\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q) = \left\{ \boldsymbol{y} \in [L^2(Q)]^{d+1} \ : \ \mathrm{div}_x \boldsymbol{y} \in L^2(Q) \right\}$$

such that

$$(\mathrm{div}_x \boldsymbol{y}, \lambda\, e + \mu\, \partial_t e)_Q + (\boldsymbol{y}, \nabla_x (\lambda\, e + \mu\, \partial_t e))_Q = 0, \quad \lambda, \mu > 0.$$

- Two forms of the majorants are obtained from

$$\begin{aligned}\lambda\, \|\nabla_x e_n\|^2_Q + \mu\, \|\partial_t\, e_n\|^2_Q + \tfrac{1}{2}\left(\mu\, \|\nabla_x e_n\|^2_{\Sigma_T} + \lambda \|e\|^2_{\Sigma_T}\right) \\ = \quad \lambda\left((f + \mathrm{div}_x \boldsymbol{y} - \partial_t v_n, e_n)_Q \quad + (\boldsymbol{y} - \nabla_x v_n, \nabla_x e_n)_Q \right) \\ + \mu\left((f + \mathrm{div}_x \boldsymbol{y} - \partial_t v_n, \partial_t e_n)_Q + (\boldsymbol{y} - \nabla_x v_n, \nabla_x \partial_t e_n)_Q \right).\end{aligned}$$

Derivation of advanced form of majorant

- Introduce Hilbert spaces for **auxiliary vector-valued functions**

$$\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q) = \{\, \boldsymbol{y} \in [L^2(Q)]^{d+1} \;:\; \mathrm{div}_x \boldsymbol{y} \in L^2(Q) \,\}$$

  such that

$$(\mathrm{div}_x \boldsymbol{y}, \lambda\, e + \mu\, \partial_t e)_Q + (\boldsymbol{y}, \nabla_x(\lambda\, e + \mu\, \partial_t e))_Q = 0, \quad \lambda, \mu > 0.$$

- Two forms of the majorants are obtained from

$$\begin{aligned} \lambda\, \|\nabla_x e_n\|_Q^2 + \mu\, \|\partial_t\, e_n\|_Q^2 + \tfrac{1}{2}\, (\mu\, \|\nabla_x e_n\|_{\Sigma_T}^2 + \lambda \|e\|_{\Sigma_T}^2) \\ = \quad \lambda \left((f + \mathrm{div}_x \boldsymbol{y} - \partial_t v_n, e_n)_Q \quad + (\boldsymbol{y} - \nabla_x v_n, \nabla_x e_n)_Q \right) \\ + \mu \left((f + \mathrm{div}_x \boldsymbol{y} - \partial_t v_n, \partial_t e_n)_Q + (\boldsymbol{y} - \nabla_x v_n, \nabla_x \partial_t e_n)_Q \right). \end{aligned}$$

1st form of the majorant

Theorem 1 [**Langer, Matculevich, Repin, 2016**]

For $\forall v \in H_0^{\Delta_x,1}(Q)$ and $\forall \mathbf{y} \in H^{\mathrm{div}_x,0}(Q)$, error can be estimated as follows:

$$\lambda \, \|\nabla_x e\|^2_Q + \mu \, \|\partial_t e\|^2_Q + \mu \, \|\nabla_x e\|^2_{\Sigma_T} + \lambda \|e\|^2_{\Sigma_T} =: |\!|\!| e |\!|\!|^2_{(\nu, s)}$$

$$\leq \overline{\mathrm{M}}^{\mathrm{I}}_{s,h}(v, \mathbf{y}; \alpha_i) := \lambda \underbrace{\Big((1 + \alpha_1) \, \|\mathbf{r}_{\mathrm{d}}\|^2_Q + (1 + \tfrac{1}{\alpha_1}) \, C^2_{\mathrm{F}\Omega} \, \|\mathbf{r}_{\mathrm{eq}}\|^2_Q \Big)}_{\overline{\mathrm{M}}^{\mathrm{I}}(v, \mathbf{y})}$$

$$+ \mu \Big((1 + \alpha_2) \, \|\mathrm{div}_x \mathbf{r}_{\mathrm{d}}\|^2_Q + (1 + \tfrac{1}{\alpha_2}) \, \|\mathbf{r}_{\mathrm{eq}}\|^2_Q \Big)$$

$$:= \lambda \, \overline{\mathrm{M}}^{\mathrm{I}}(v, \mathbf{y}) + \mu \Big((1 + \alpha_2) \, \|\mathrm{div}_x \mathbf{r}_{\mathrm{d}}\|^2_Q + (1 + \tfrac{1}{\alpha_2}) \, \|\mathbf{r}_{\mathrm{eq}}\|^2_Q \Big),$$

where

$$\mathbf{r}_{\mathrm{eq}}(v, \mathbf{y}) = f + \mathrm{div}_x \mathbf{y} - \partial_t v, \qquad \Leftarrow \qquad \partial_t u - \mathrm{div}_x \mathbf{p} = f$$

$$\mathbf{r}_{\mathrm{d}}(v, \mathbf{y}) = \mathbf{y} - \nabla_x v, \qquad \Leftarrow \qquad \mathbf{p} = \nabla_x u$$

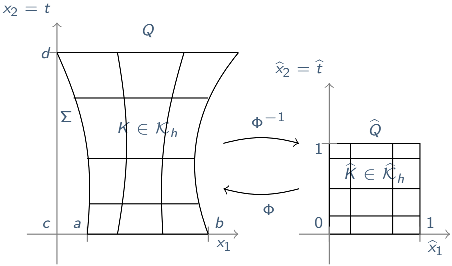
λ, $\mu > 0$, and $\alpha_i, i = 1, 2 > 0$ are auxiliary parameters.

IgA framework

[Hughes, Cottrell, and Bazilevs, 2005],
[Bazilevs, Beirao da Veiga, Cottrell, Hughes, and Sangalli, 2006]:

Physical domain $Q \subset \mathbb{R}^{d+1}$, is defined from
Parametric domain $\widehat{Q} := (0,1)^{d+1}$ by the
Geometrical mapping $\Phi : \widehat{Q} \to Q = \Phi(\widehat{Q}) \subset \mathbb{R}^{d+1}$, $\quad \Phi(\xi) = \sum_{i \in \mathcal{I}} \widehat{B}_{i,p}(\xi)\,\mathbf{P}_i$,

- $\widehat{B}_{i,p}$, $i \in \mathcal{I}$, are the B-Splines, NURBS, THB-splines;
- $\{\mathbf{P}_i\}_{i \in \mathcal{I}} \in \mathbb{R}^{d+1}$ are the control points.



Space-time IgA discrete scheme

Testing parabolic I-BVP by $w = \lambda\, v_h + \mu\, \partial_t v_h$, where $\lambda = 1$ and $\mu = \delta_h$, i.e.,

$$v_h + \delta_h\, \partial_t v_h, \quad \delta_h = \theta h, \quad \theta > 0, \quad v_h \in V_{0h} \subset H^1_{0,\underline{0}}(Q),$$

where h is the global size of the mesh $\mathcal{K}_h$, we arrive at

space-time IgA discrete formulation [Langer, Moore, Neumüller, 2016]

find $u_h \in V_{0h} \subset H^1_{0,\underline{0}}(Q)$ satisfying

$$a_{s,h}(u_h, v_h) = l_{s,h}(v_h), \quad \forall v_h \in V_{0h},$$

$$a_{s,h}(u_h, v_h) := \big(\partial_t u_h, v_h + \delta_h\, \partial_t v_h\big)_Q + \big(\nabla_x u_h, \nabla_x (v_h + \delta_h\, \partial_t v_h)\big)_Q,$$

$$l_{s,h}(v_h) := (f, v_h + \delta_h\, v_h)_Q,$$

with the corresponding norm

$$\|v_h\|^2_{s,h} := \|\nabla_x v_h\|^2_Q + \delta_h\, \|\partial_t v_h\|^2_Q + \|v_h\|^2_{\Sigma_T} + \delta_h\, \|\nabla_x v_h\|^2_{\Sigma_T}.$$

Two forms of majorant for space-time IgA scheme ($\lambda = 1$ and $\mu = \delta_h$)

Corollary 1 [Langer, Matculevich, Repin, 2016]

For all $v \in H_0^{\Delta_x,1}(Q)$ and $\boldsymbol{y} \in H^{\mathrm{div}_x,0}(Q)$, we have:

$$\|\nabla_x e\|^2_Q + \delta_h \|\partial_t e\|^2_Q + \delta_h \|\nabla_x e\|^2_{\Sigma_T} + \|e\|^2_{\Sigma_T} \leq \overline{\mathrm{M}}^{\mathrm{I}}_{s,h}(v, \boldsymbol{y}; \alpha_i)$$

$$:= \underbrace{(1+\alpha_1)\,\|\mathbf{r}_{\mathrm{d}}\|^2_Q + (1+\tfrac{1}{\alpha_1})\, C^2_{\mathrm{F}\Omega}\, \|\mathbf{r}_{\mathrm{eq}}\|^2_Q}_{\overline{\mathrm{M}}^{\mathrm{I}}(v, \boldsymbol{y}; \alpha_i)}$$

$$+ \delta_h \left((1+\alpha_2)\, \|\mathrm{div}_x \mathbf{r}_{\mathrm{d}}\|^2_Q + (1+\tfrac{1}{\alpha_2})\, \|\mathbf{r}_{\mathrm{eq}}\|^2_Q \right)$$

$$:= \overline{\mathrm{M}}^{\mathrm{I}}(v, \boldsymbol{y}; \alpha_i) + \delta_h \left((1+\alpha_2)\, \|\mathrm{div}_x \mathbf{r}_{\mathrm{d}}\|^2_Q + (1+\tfrac{1}{\alpha_2})\, \|\mathbf{r}_{\mathrm{eq}}\|^2_Q \right)$$

Here, $\mathbf{r}_{\mathrm{d}}$ and $\mathbf{r}_{\mathrm{eq}}$ are *residuals* following from the problem statement, $\delta_h = \theta h$, $\theta > 0$, is a parameter of the scheme, and $\gamma \in [\frac{1}{2}, +\infty)$, $\alpha_i > 0$, $i = 1, 2$.

Majorant for heat equation with Dirichlet BVP

on $Q := \Omega \times (0, T)$: find $u \in H_0^{\Delta_x,1}(Q)$

$$u_t - \mathrm{div}_x(\nabla_x u) = f \in L^2(\Omega) \text{ in } \Omega, \; u = u_D \in H^1(\partial\Omega) \text{ on } \partial\Omega, \; u(0,x) = u_0 \text{ on } \Sigma_0.$$

For $\forall v, w \in H_0^{\Delta_x,1}(Q)$, $\forall \boldsymbol{y} \in H(\Omega, \mathrm{div}_x)$, and $\forall \beta^{\mathrm{I}} > 0, \beta^{\mathrm{II}} > 0$,
we have error estimates

$$|\!|\!| e |\!|\!|^2 := \overbrace{\sum_{K \in \mathcal{K}_h} \|\nabla_x e\|_K^2}^{e_{\mathrm{d}}^2} + \|e\|_{\Sigma_T}^2 \leq \overline{\mathrm{M}}^{\mathrm{I}}(v, \boldsymbol{y}, \beta^{\mathrm{I}}) := (1 + \beta^{\mathrm{I}}) \overbrace{\sum_{K \in \mathcal{K}_h} \|\boldsymbol{y} - \nabla_x v\|_K^2}^{\text{error indicator } \overline{\mathrm{m}}_{\mathrm{d},K}^{\mathrm{I}}} + \left(1 + \tfrac{1}{\beta^{\mathrm{I}}}\right) C_{\mathrm{F}\Omega}^2 \|f + \mathrm{div}_x \boldsymbol{y} - \partial_t v\|_\Omega^2,$$

$$|\!|\!| e |\!|\!|_{s,h}^2 \leq \overline{\mathrm{M}}_{s,h}^{\mathrm{I}}(v, \boldsymbol{y}, \beta^{\mathrm{I}}),$$

$$\sum_{K \in \mathcal{K}_h} \|\nabla_x e\|_K^2 \leq \overline{\mathrm{M}}^{\mathrm{II}}(v, \boldsymbol{y}, w, \beta^{\mathrm{II}}),$$

and error identity

$$|\!|\!| e |\!|\!|_{\mathcal{L},Q}^2 \equiv \mathrm{E\!d}(v) := \|\nabla_x(u_0 - v)\|_{\Sigma_0}^2 + \|f + \Delta_x v - \partial_t v\|_Q^2.$$

Reconstruction of efficient $\overline{\mathrm{M}}^{\mathrm{I}}$

Solve $\{\boldsymbol{y}_{\min}, \beta^{\mathrm{I}}_{\min}\} := \arg \inf\limits_{\beta^{\mathrm{I}}>0} \inf\limits_{\boldsymbol{y}\in H(\Omega,\mathrm{div}_x)} \overline{\mathrm{M}}(v, \boldsymbol{y}; \beta^{\mathrm{I}})$, where

$$\overline{\mathrm{M}}^{\mathrm{I}}(v, \boldsymbol{y}; \beta^{\mathrm{I}}) := (1+\beta^{\mathrm{I}}) \underbrace{\|\boldsymbol{y} - \nabla_x v\|^2_\Omega}_{\overline{\mathrm{m}}^2_{\mathrm{d}}} + \Big(1 + \tfrac{1}{\beta^{\mathrm{I}}}\Big) C^2_{\mathrm{F}\Omega} \underbrace{\|f + \mathrm{div}_x \boldsymbol{y} - \partial_t v\|^2_\Omega}_{\overline{\mathrm{m}}^2_{\mathrm{eq}}}$$
$$= (1+\beta^{\mathrm{I}})\, \overline{\mathrm{m}}^2_{\mathrm{d}} + \Big(1 + \tfrac{1}{\beta^{\mathrm{I}}}\Big) C^2_{\mathrm{F}\Omega}\, \overline{\mathrm{m}}^2_{\mathrm{eq}} :$$

- the variation problem for the optimal $\boldsymbol{y}_{\min}$, i.e.,

$$\frac{C^2_{\mathrm{F}\Omega}}{\beta^{\mathrm{I}}_{\min}} (\mathrm{div}_x \boldsymbol{y}_{\mathbf{min}}, \mathrm{div}_x \boldsymbol{\eta})_\Omega + (\boldsymbol{y}_{\mathbf{min}}, \boldsymbol{\eta})_\Omega = -\frac{C^2_{\mathrm{F}\Omega}}{\beta^{\mathrm{I}}_{\min}} (f - \partial_t v, \mathrm{div}_x \boldsymbol{\eta})_\Omega + (\nabla_x v, \boldsymbol{\eta})_\Omega,$$

- where the optimal $\beta^{\mathrm{I}}_{\min} := {}^{C_{\mathrm{F}\Omega}\, \overline{\mathrm{m}}_{\mathrm{eq}}}/_{\overline{\mathrm{m}}_{\mathrm{d}}}$.

IgA spaces for u_h approximation

$$\widehat{V}_h \equiv \widehat{\mathcal{S}}_h^{p,p} := \mathrm{span}\,\{\widehat{B}_{i,p}\},$$

$$u_h \in V_h \equiv \mathcal{S}_h^{p,p} := \{\widehat{V}_h \circ \Phi^{-1}\} \cap H^1_{u_D}(\Omega) := \mathrm{span}\,\{\phi_{h,i} := \widehat{B}_{i,p} \circ \Phi^{-1}\}_{i \in \mathcal{I}} \cap H^1_{u_D}(\Omega).$$

Generated approximation u_h is presented as

$$u_h(x) = \sum_{i \in \mathcal{I}} \underline{\mathrm{u}}_i \, \phi_{h,i}(x), \quad \underline{\mathrm{u}}_h := [\mathrm{u_i}]_{\mathrm{i} \in \mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|},$$

where $\underline{\mathrm{u}}_h$ is a vector of DOFs defined by a system

$$\mathrm{K}_h \, \underline{\mathrm{u}}_h = \mathrm{f}_h,$$
$$\mathrm{K}_h := \left[(\nabla_x \phi_{h,i}, \nabla_x \phi_{h,j})\right]_{i,j}^{\mathcal{I}},$$
$$\mathrm{f}_h := \left[(f, \phi_{h,i})\right]_i^{\mathcal{I}}.$$

IgA spaces for $\boldsymbol{y}_h$ reconstruction

$$\widehat{Y}_h \equiv \oplus^{d+1} \widehat{\mathcal{S}}_h^{q,q},$$

$$\boldsymbol{y}_h = \begin{bmatrix} y_h^{(1)} \\ \dots \\ y_h^{(d+1)} \end{bmatrix} \in Y_h \equiv \oplus^{d+1} \mathcal{S}_h^{q,q} := \left\{ \widehat{Y}_h \circ \Phi^{-1} \right\} = \mathrm{span}\left\{ \psi_i := [\widehat{B}_{i,q}]^{d+1} \circ \Phi^{-1} \right\}_{i \in \mathcal{I}}$$

Generated reconstruction of $\boldsymbol{y}_h$ is presented as

$$\boldsymbol{y}_h(x) := \sum_{i \in \mathcal{I} \times (d+1)} \underline{\mathbf{y}}_{h,i} \, \psi_{h,i}(x),$$

where $\underline{\mathbf{y}}_h := \left[\underline{\mathbf{y}}_{h,i} \right]_{i \in \mathcal{I} \times (d+1)} \in \mathbb{R}^{(d+1)|\mathcal{I}|}$ is a vector of DOFs of $\boldsymbol{y}_h$ defined by a system

$$\left(C_{\mathrm{F}\Omega}^2 \, \mathrm{Div}_h + \beta \, \mathrm{M}_h \right) \underline{\mathbf{y}}_h = - C_{\mathrm{F}\Omega}^2 \, \mathrm{z}_h + \beta \, \mathrm{g}_h,$$

with

$$\mathrm{Div_h} := \left[(\mathrm{div}_x \psi_i, \mathrm{div}_x \psi_j) \right]_{i,j=1}^{(d+1)|\mathcal{I}|}, \qquad \mathrm{z}_h := \left[(f - \partial_t v, \mathrm{div}_x \psi_j) \right]_{j=1}^{(d+1)|\mathcal{I}|},$$

$$\mathrm{M_h} := \left[(\psi_i, \psi_j) \right]_{i,j=1}^{(d+1)|\mathcal{I}|}, \qquad \mathrm{g}_h := \left[(\nabla_x v, \psi_j) \right]_{j=1}^{(d+1)|\mathcal{I}|}.$$

IgA spaces for w_h reconstruction

$$\widehat{W}_h \equiv \widehat{\mathcal{S}}_h^{r,r} := \mathrm{span}\,\{\widehat{B}_{i,r}\},$$
$$w_h \in W_h \equiv \mathcal{S}_h^{r,r} := \{\widehat{W}_h \circ \Phi^{-1}\} \cap H^1_{u_D}(\Omega) := \mathrm{span}\,\{\chi_{h,i} := \widehat{B}_{i,r} \circ \Phi^{-1}\}_{i\in\mathcal{I}} \cap H^1_{u_D}(\Omega).$$

Generated approximation u_h is presented as

$$w_h(x) = \sum_{i\in\mathcal{I}} \underline{\mathrm{w}}_i\, \chi_{h,i}(x), \quad \underline{\mathrm{w}}_h := [\mathrm{w_i}]_{\mathrm{i}\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|},$$

where $\underline{\mathrm{w}}_h$ is a vector of DOFs defined by a system

$$\mathrm{K}_h^{(r)}\, \underline{\mathrm{w}}_h = \mathrm{f}_h^{(r)},$$
$$\mathrm{K}_h^{(r)} := \left[(\nabla_x \chi_{h,i}, \nabla_x \chi_{h,j})\right]_{i,j}^{\mathcal{I}},$$
$$\mathrm{f}_h^{(r)} := \left[(f, \chi_{h,i})\right]_{i}^{\mathcal{I}}.$$

Reliable u_h approximation (single refinement step)

Input: $\mathcal{K}_h$ {discretization of Ω}, $\mathrm{span}\,\{\phi_{h,i}\}$, $i = 1, ..., |\mathcal{I}|$ {V_h-basis}

APPROXIMATE:

- ASSEMBLE the matrix $\mathrm{K_h}$ and RHS $\mathrm{f_h}$: $t_{\rm as}(u_h)$
- SOLVE $\mathrm{K}_h\, \underline{\mathrm{u}}_h = \mathrm{f}_h$: $t_{\rm sol}(u_h)$
- Approximate $u_h = \sum_{i \in \mathcal{I}} \underline{\mathrm{u}}_i\, \phi_{h,i}(x)$

Evaluate $\|e\|^2$, $\|e\|^2_{s,h}$, and $\|e\|^2_{\mathcal{L}}$: $t_{\rm e/w}(\|e\|) + t_{\rm e/w}(\|e\|_{s,h}) + t_{\rm e/w}(\|e\|_{\mathcal{L}})$

ESTIMATE:

- evaluate $\overline{\mathrm{M}}^{\mathrm{I}}(u_h, \boldsymbol{y}_h)$: $t_{\rm as}(\boldsymbol{y}_h) + t_{\rm sol}(\boldsymbol{y}_h) + t_{\rm e/w}(\overline{\mathrm{M}}^{\mathrm{I}})$
- evaluate $\overline{\mathrm{M}}^{\mathrm{II}}(u_h, \boldsymbol{y}_h, w_h)$: $t_{\rm as}(w_h) + t_{\rm sol}(w_h) + t_{\rm e/w}(\overline{\mathrm{M}}^{\mathrm{II}})$
- evaluate $\overline{\mathrm{M}}^{\mathrm{I}}_{s,h}(u_h, \boldsymbol{y}_h)$: $t_{\rm e/w}(\overline{\mathrm{M}}^{\mathrm{I}}_{s,h})$
- evaluate $\mathbb{E}\mathrm{d}(u_h)$: $t_{\rm e/w}(\mathbb{E}\mathrm{d})$

MARK: Using marking $\mathbb{M}(\psi)$, select elements K of mesh $\mathcal{K}_h$ that must be refined

REFINE: Execute the refinement strategy: $\mathcal{K}_{h_{ref}} = R(\mathcal{K}_h)$

Output: $\mathcal{K}_{h_{ref}}$ {refined discretization of Ω}

ESTIMATE step ($\overline{\mathrm{M}}^{\mathrm{I}}$ reconstruction)

Input: u_h {approximation}, $\mathcal{K}_h$ {disctretization of Ω},
$\mathrm{span}\,\{\psi_{h,i}\}$, $i = 1, ..., (d+1)|\mathcal{I}|$ {Y_h-basis}, $N^{\mathrm{iter}}_{\mathrm{maj}}$ {number of opt. iterations}

ASSEMBLE $\mathrm{Div_h}, \mathrm{M_h} \in \mathbb{R}^{(d+1)|\mathcal{I}| \times (d+1)|\mathcal{I}|}$ and $\mathrm{z}_h, \mathrm{g}_h \in \mathbb{R}^{(d+1)|\mathcal{I}|}$ $\quad :t_{\mathrm{as}}(\boldsymbol{y}_h)$

Set $\beta^{\mathrm{I}} = 1$

for $m = 1$ **to** $N^{\mathrm{iter}}_{\mathrm{maj}}$ **do**

SOLVE $\left(\frac{1}{\beta^{\mathrm{I}}}\, C^2_{\mathrm{F}\Omega} \mathrm{Div_h} + \mathrm{M_h}\right) \underline{\mathbf{y}}_h = -\frac{1}{\beta^{\mathrm{I}}}\, C^2_{\mathrm{F}\Omega} \mathrm{z}_h + \mathrm{g}_h$ $\quad :t_{\mathrm{sol}}(\boldsymbol{y}_h)$

Reconstruct $\boldsymbol{y}_h := \sum_{i \in \mathcal{I}} \underline{\mathbf{y}}_{h,i}\, \psi_{h,i}$

Compute $\overline{\mathrm{m}}^2_{\mathrm{eq}} := \|\, f + \mathrm{div}_x \boldsymbol{y}_h - \partial_t u_h \,\|^2_\Omega$ and $\overline{\mathrm{m}}^2_{\mathrm{d}} := \|\boldsymbol{y}_h - \nabla_x u_h\,\|^2_\Omega$

Update $\beta^{\mathrm{I}} = \frac{C_{\mathrm{F}\Omega}\, \overline{\mathrm{m}}_{\mathrm{eq}}}{\overline{\mathrm{m}}_{\mathrm{d}}}$

end for

Evaluate $\overline{\mathrm{M}}^{\mathrm{I}}(u_h, \boldsymbol{y}_h; \beta^{\mathrm{I}}) := (1 + \beta^{\mathrm{I}})\, \overline{\mathrm{m}}^2_{\mathrm{eq}} + (1 + \frac{1}{\beta^{\mathrm{I}}})\, C^2_{\mathrm{F}\Omega} \overline{\mathrm{m}}^2_{\mathrm{d}}$

Output: $\overline{\mathrm{M}}^{\mathrm{I}}$ {majorants on Ω},

Choice of B-Splines (NURBS) for $\mathbf{y}_h$ and w_h

We use the idea from Kleiss, Tomar (2015):

- $u_h \in V_h \equiv \mathcal{S}_h^{p,p}$
- u_h is approx. on $\mathcal{K}_h$

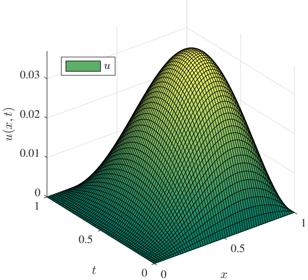
- $\mathbf{y}_h \in Y_{Mh} \equiv \oplus^{d+1} \mathcal{S}_{Mh}^{q,q}$
- $q \gg p$, i.e., $q = p + m$, $m \in \mathbb{N}^+$
- $\mathbf{y}_{Mh}$ is reconstructed on $\mathcal{K}_{Mh}$, $M \in \mathbb{N}^+$
- $M \geq m$

- $w_h \in W_{Lh} \equiv \mathcal{S}_{Lh}^{r,r}$
- $r \gg p$, i.e., $r = p + l$, $l \in \mathbb{N}^+$
- w_{Lh} is reconstructed on $\mathcal{K}_{Lh}$, $L \in \mathbb{N}^+$
- $L \geq l$

Example 1. Polynomial solution

Given data:

- $\Omega = (0, 1),\ T = 1$
- $u = (1-x)\,x^2\,(1-t)\,t$
- $f = -(1-x)\,x^2\,(1-2\,t) - (2-6\,x)\,(1-t)\,t$
- $u_D = 0$



Discretization $q > p$, $M \gg 1$:

- $p = 2 \quad \Rightarrow \quad u_h \in S_h^{2,2}$
- $q = 3,\ M = 6 \quad \Rightarrow \quad \mathbf{y}_h \in S_{6h}^{3,3} \oplus S_{6h}^{3,3}$
- $r = 3,\ M = 6 \quad \Rightarrow \quad w_h \in S_{6h}^{3,3}$

Example 1. Uniform refinement, $u_h \in S_h^{2,2}$, $\boldsymbol{y}_h \in S_{6h}^{3,3} \oplus S_{6h}^{3,3}$, $w_h \in S_{6h}^{3,3}$

# ref.	$\|\nabla_x e\|_Q^2$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	$I_{\rm eff}(\overline{\rm M}^{\rm II})$	$\|\|\|e\|\|\|_{s,h}^2$	$I_{\rm eff}(\overline{\rm M}^{\rm I}_{s,h})$	$\|\|\|e\|\|\|_{\mathcal{L}}^2$	$I_{\rm eff}(\rm Ed)$	e.o.c.
3	6.3789e-4	1.20	1.10	6.3789e-4	2.99	3.9528e-2	1.00	2.71
5	3.9868e-5	1.11	1.05	3.9868e-5	1.76	9.8821e-3	1.00	2.18
7	2.4917e-6	1.16	1.07	2.4917e-6	1.34	2.4705e-3	1.00	2.05
9	1.5573e-7	1.04	1.02	1.5573e-7	1.10	6.1764e-4	1.00	2.01

Comparison estimates' error control efficiency.

# ref.	d.o.f.			$t_{\rm as}$			$t_{\rm sol}$			$t_{\rm e/w}$	
	u_h	$\boldsymbol{y}_h$	w_h	u_h	$\boldsymbol{y}_h$	w_h	u_h	$\boldsymbol{y}_h$	w_h	$\|\|\|e\|\|\|_{\mathcal{L}}^2$	Ed
1	16	50	25	8.66e-4	1.51e-3	9.77e-4	2.60e-5	1.19e-4	5.30e-5	1.20e-3	3.20e-4
3	100	50	25	5.71e-3	1.82e-3	9.80e-4	7.72e-4	1.93e-4	9.40e-5	1.44e-2	4.46e-3
5	1156	50	25	8.13e-2	1.06e-3	8.51e-4	3.27e-2	9.40e-5	5.50e-5	1.47e-1	6.28e-2
7	16900	50	25	1.07e0	1.21e-3	5.37e-4	1.95e0	9.80e-5	6.50e-5	2.27e0	8.55e-1
9	264196	98	49	1.14e1	4.29e-3	2.07e-3	7.84e1	4.97e-4	1.23e-4	3.56e1	1.09e1
				5507 :	2 :	1	637398 :	4 :	1	3.26 :	1

Comparison estimates' time efficiency.

Example 1. Adaptive refinement, $u_h \in S_h^{2,2}$, $\mathbf{y}_h \in S_{6h}^{3,3} \oplus S_{6h}^{3,3}$, $w_h \in S_{6h}^{3,3}$

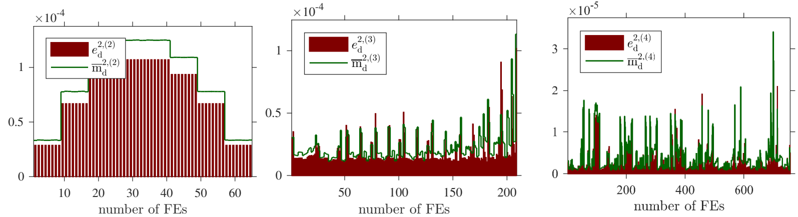
# ref.	$\|\nabla_x e\|_Q^2$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	$I_{\rm eff}(\overline{\rm M}^{\rm II})$	$\|\|\|e\|\|\|_{s,h}^2$	$I_{\rm eff}(\overline{\rm M}^{\rm I}_{s,h})$	$\|\|\|e\|\|\|_{\mathcal{L}}^2$	$I_{\rm eff}(\rm Ed)$	e.o.c.
3	6.37e-4	1.19	1.09	6.3789e-4	2.99	3.9528e-2	1.00	2.71
5	1.30e-4	2.39	1.18	1.3080e-4	2.56	1.5190e-2	1.00	1.60
7	1.0056e-5	1.82	1.28	1.0056e-5	1.94	4.9217e-3	1.00	3.22
9	2.1820e-6	1.86	1.18	2.1820e-6	1.93	2.2159e-3	1.00	2.10

Comparison estimates' error control efficiency.

# ref.	d.o.f.		$t_{\rm as}$		$t_{\rm sol}$		$t_{\rm e/w}$	
	u_h	$\mathbf{y}_h$	u_h	$\mathbf{y}_h$	u_h	$\mathbf{y}_h$	$\|\|\|e\|\|\|_{\mathcal{L}}^2$	Ed
1	16	25	1.08e-2	3.02e-2	2.80e-5	3.72e-4	3.72e-3	3.80e-3
3	100	25	8.80e-2	3.06e-2	5.68e-4	1.62e-4	6.56e-2	6.59e-2
5	761	25	1.14e0	3.51e-2	2.05e-2	2.49e-4	9.64e-1	9.72e-1
7	4222	25	3.24e0	2.92e-2	1.83e-1	1.85e-4	2.73e0	2.48e0
9	24778	49	2.59e1	9.77e-2	2.74e0	4.53e-4	1.78e1	1.55e1
			20 :	1	550 :	1	1.03 :	1

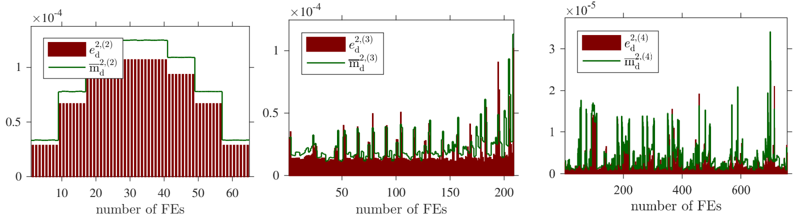
Comparison estimates' time efficiency.

Example 1. Comparison of local distributions

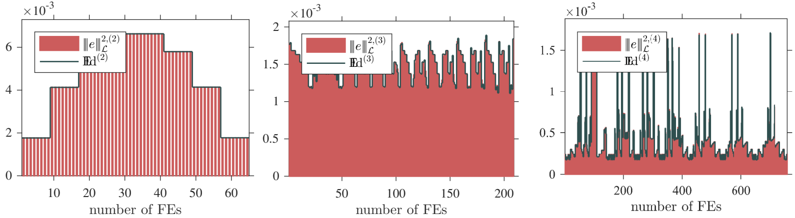


Local distribution $e_d^2 := \|\nabla_x e\|_Q^2$ and $\overline{m}_d := \|\boldsymbol{y}_h - \nabla_x u_h\|_Q^2$

Example 1. Comparison of local distributions



Local distribution $e_{\mathrm{d}}^2 := \|\nabla_x e\|_Q^2$ and $\overline{\mathrm{m}}_{\mathrm{d}} := \|\boldsymbol{y}_h - \nabla_x u_h\|_Q^2$



Local distribution $|||e|||_{\mathcal{L},Q}^2$ and $\mathbb{E}\mathrm{d}$

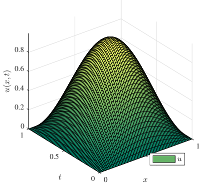
Example 2. Parametrised solution

Given data:

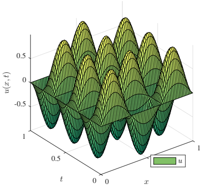
- $\Omega = (0, 1),\ T = 1$
- $u = \sin(k_1\,\pi\,x)\ \sin(k_2\,\pi\,t)$
- $f = \sin(k_1\,\pi\,x)\,(k_2\,\pi\ \cos(k_2\,\pi\,t) + k_1^2\,\pi^2\ \sin(k_2\,\pi\,t))$
- $u_D = 0$

Discretization:

- $u_h \in S_h^{2,2}$
- $k_1 = k_2 = 1$:
  $\boldsymbol{y}_h \in S_{6h}^{5,5} \oplus S_{6h}^{5,5}$, and $w_h \in S_{6h}^{5,5}$
- $k_1 = 6,\ k_2 = 3$:
  $\boldsymbol{y}_h \in S_{6h}^{9,9} \oplus S_{6h}^{9,9}$, and $w_h \in S_{6h}^{9,9}$



$k_1 = k_2 = 1$



$k_1 = 6,\ k_2 = 3$

Example 2-1. Adaptive refinement, $u_h \in S_h^{2,2}$, $\mathbf{y}_h \in S_{6h}^{5,5} \oplus S_{6h}^{5,5}$, $w_h \in S_{6h}^{5,5}$

# ref.	$\lVert \nabla_x e \rVert^2_Q$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	$I_{\rm eff}(\overline{\rm M}^{\rm II})$	$\lVert\!\lvert e \rvert\!\rVert^2_{s,h}$	$I_{\rm eff}(\overline{\rm M}^{\rm I}_{s,h})$	$\lVert\!\lvert e \rvert\!\rVert^2_{\mathcal{L}}$	$I_{\rm eff}(\mathbb{E}\mathrm{d})$	e.o.c.
2	4.8004e-3	2.56	1.00	4.8380e-3	2.92	2.9450e-1	1.00	1.38
4	4.8027e-4	2.72	1.00	4.8083e-4	2.85	1.0100e-1	1.00	2.93
6	5.9177e-5	3.64	1.02	5.9191e-5	3.75	3.8929e-2	1.00	2.17
8	9.4871e-6	3.64	1.05	9.4873e-6	3.67	1.7264e-2	1.00	2.66

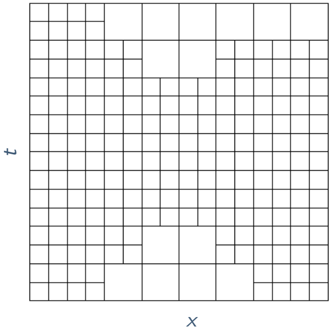
Comparison estimates' error control efficiency.

# ref.	d.o.f.		$t_{\rm as}$		$t_{\rm sol}$		$t_{\rm e/w}$	
	u_h	$\mathbf{y}_h$	u_h	$\mathbf{y}_h$	u_h	$\mathbf{y}_h$	$\lVert\!\lvert e \rvert\!\rVert^2_{\mathcal{L}}$	$\mathbb{E}\mathrm{d}$
2	254	169	6.99e-1	2.23e0	2.97e-3	6.50e-3	2.47e-1	2.61e-1
4	2535	169	4.63e0	2.04e0	1.41e-1	5.99e-3	2.61e0	2.59e0
6	18343	169	2.53e1	1.76e0	1.51e0	4.74e-3	1.59e1	1.50e1
8	105304	356	3.03e2	1.46e1	1.57e1	2.85e-2	8.48e1	8.25e1
			20 :	1	550 :	1	1.03 :	1

Comparison estimates' time efficiency.

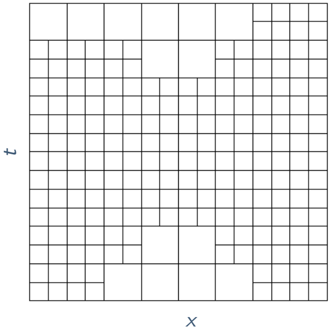
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



ref. 1

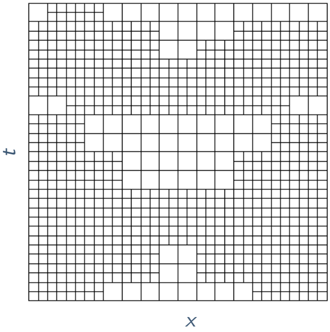
mesh obtained by ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



ref. 1

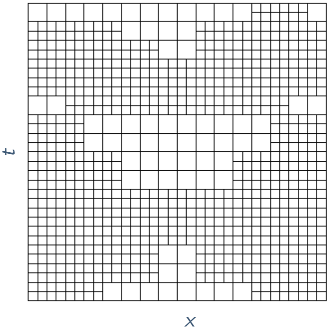
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



ref. 2

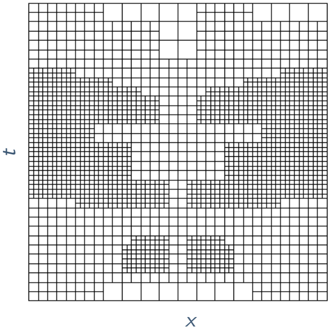
mesh obtained by ref. based on error indicator $\|\mathbf{y}_h - \nabla_x u_h\|^2_Q$



ref. 2

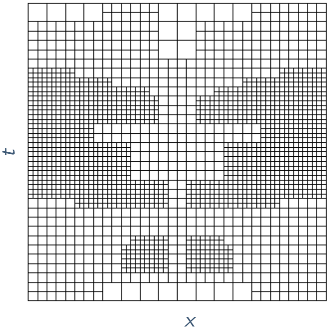
Example 2-1. Comparison of meshes $\mathbb{M}_{\rm BULK}(0.4)$

mesh obtained by ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



ref. 3

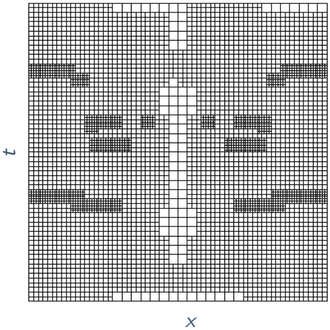
mesh obtained by ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



ref. 3

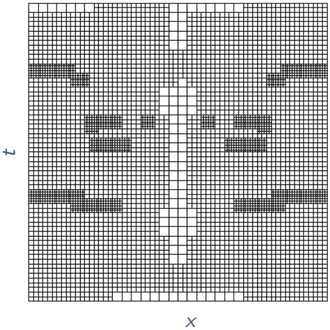
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



ref. 4

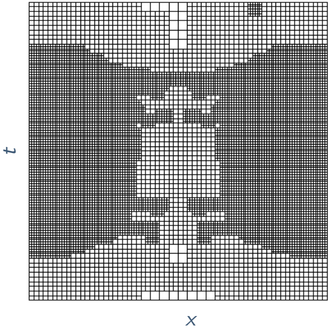
mesh obtained by ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



ref. 4

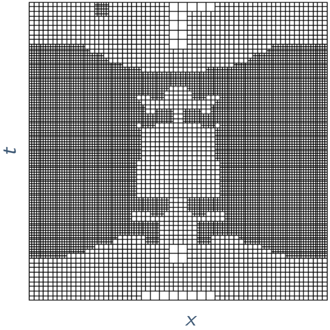
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on true error $\|\nabla_x (u - u_h)\|^2_Q$



ref. 5

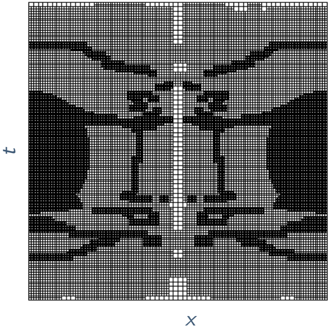
mesh obtained by ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



ref. 5

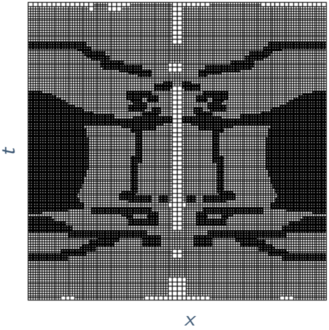
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on true error $\|\nabla_x(u - u_h)\|_Q^2$



ref. 6

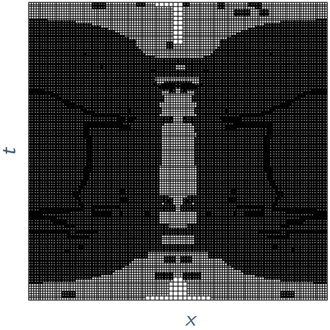
mesh obtained by ref. based on error indicator $\|\mathbf{y}_h - \nabla_x u_h\|_Q^2$



ref. 6

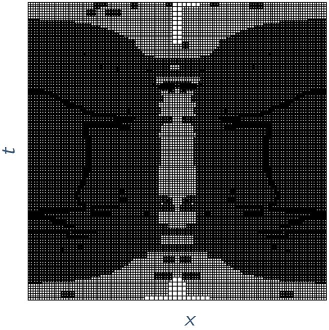
Example 2-1. Comparison of meshes $\mathbb{M}_{\rm BULK}(0.4)$

mesh obtained by ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



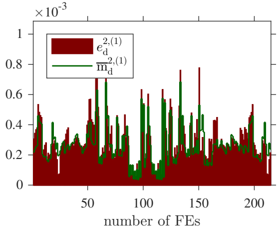
ref. 7

mesh obtained by ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$

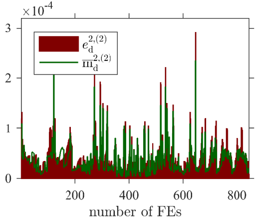


ref. 7

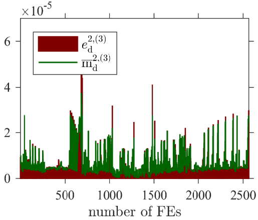
Example 2-1. Local distribution of $e_{\mathrm{d}}^2 := \|\nabla_x e\|^2_Q$ and $\overline{\mathrm{m}}_{\mathrm{d}} := \|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



ref. 1



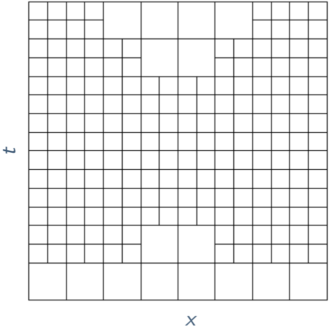
ref. 2



ref. 3

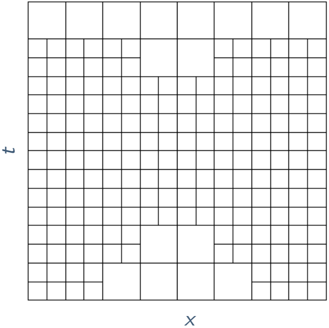
Example 2-1. Comparison of meshes $\mathbb{M}_{\rm BULK}(0.4)$

mesh obtained by ref. based on error-norm induced by solution operator $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 1

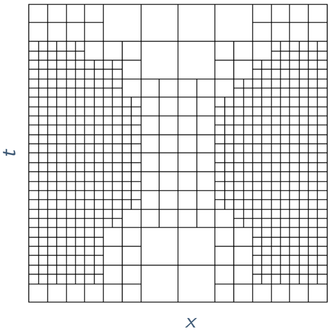
mesh obtained by ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 1

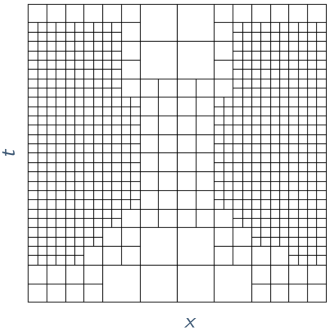
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on error-norm induced by solution operator $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 2

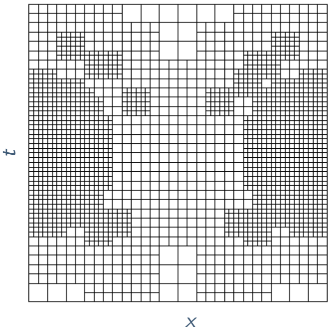
mesh obtained by ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 2

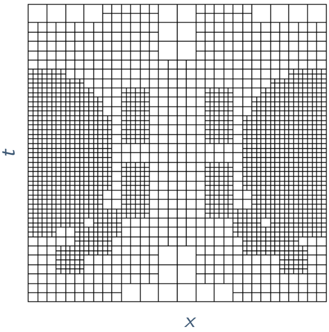
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on error-norm induced by solution operator $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 3

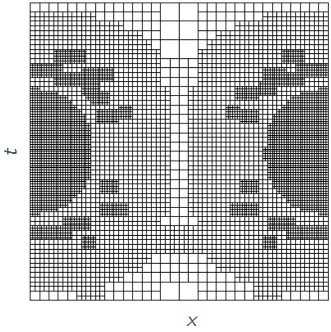
mesh obtained by ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 3

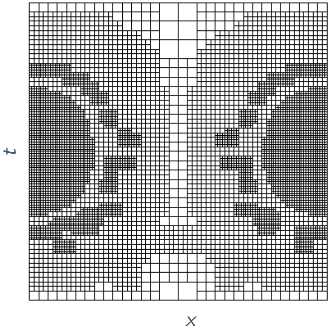
Example 2-1. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

mesh obtained by ref. based on error-norm induced by solution operator $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 4

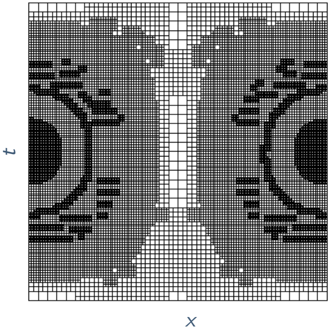
mesh obtained by ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 4

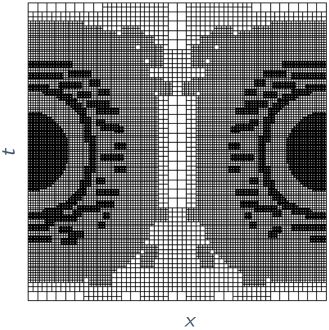
Example 2-1. Comparison of meshes $\mathbb{M}_{\text{BULK}}(0.4)$

mesh obtained by ref. based on error-norm induced by solution operator $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 5

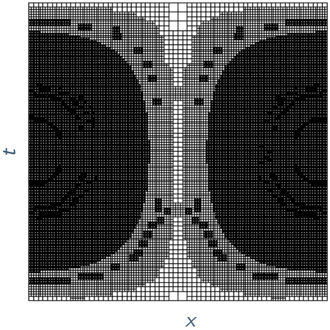
mesh obtained by ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 5

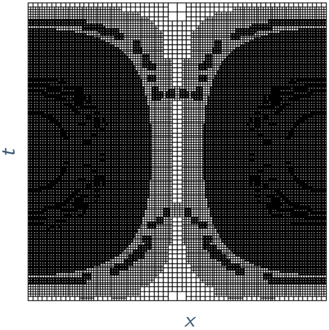
Example 2-1. Comparison of meshes $\mathbb{M}_{\rm BULK}(0.4)$

mesh obtained by ref. based on error-norm induced by solution operator $\|u - u_h\|^2_{\mathcal{L},Q}$



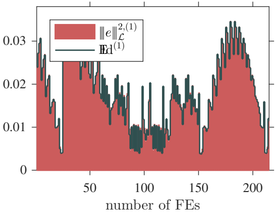
ref. 6

mesh obtained by ref. based on error identity $\mathbb{E}\mathrm{d}$

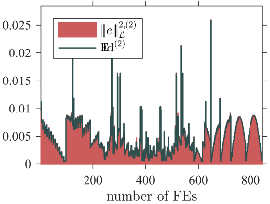


ref. 6

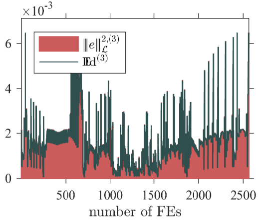
Example 2-1. Local distribution of $\|e\|^2_{\mathcal{L},Q}$ and $\mathbb{E}\mathrm{d}$



ref. 1



ref. 2



ref. 3

Example 2-2. Adaptive ref., $u_h \in S_h^{2,2}$, $\mathbf{y}_h \in S_{6h}^{9,9} \oplus S_{6h}^{9,9}$, $w_h \in S_{6h}^{9,9}$

# ref.	$\|\nabla_x e\|_Q$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	$\|\|\| e \|\|\|^2_{s,h}$	$I_{\rm eff}(\overline{\rm M}^{\rm I}_{s,h})$	$\|\|\| e \|\|\|^2_{\mathcal{L}}$	$I_{\rm eff}(\mathbb{E}\rm d)$	e.o.c.
2	1.3932e-1	1.14	1.3932e-1	2.08	3.1026e1	1.00	2.30
3	3.9705e-2	1.18	3.9705e-2	1.62	1.6010e1	1.00	2.21
4	1.3382e-2	1.26	1.3383e-2	1.55	8.7447e0	1.00	1.95
5	4.7457e-3	1.21	4.7457e-3	1.33	5.2254e0	1.00	1.81
6	1.7020e-3	1.37	1.7020e-3	1.45	3.4119e0	1.00	2.07

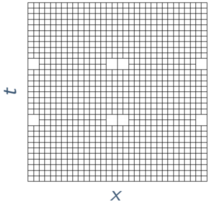
Comparison estimates' error control efficiency.

# ref.	d.o.f.			$t_{\rm as}$			$t_{\rm sol}$			$t_{\rm e/w}$	
	u_h	$\mathbf{y}_h$	w_h	u_h	$\mathbf{y}_h$	w_h	u_h	$\mathbf{y}_h$	w_h	$\|\|\| e \|\|\|^2_{\mathcal{L}}$	$\mathbb{E}\rm d$
1	324	625	625	5.49e-1	6.00e1	3.11e1	8.64e-3	4.61e-2	2.67e-2	1.51e1	7.20e-1
2	1104	625	625	1.97e0	5.98e1	2.60e1	5.87e-2	5.08e-2	3.10e-2	2.15e1	2.90e0
3	3427	625	625	7.72e0	5.89e1	2.97e1	3.27e-1	5.21e-2	3.33e-2	4.14e1	1.00e1
4	10521	625	625	3.00e1	5.87e1	4.45e1	1.68e0	5.32e-2	2.66e-2	1.07e2	3.25e1
5	32610	625	625	9.70e1	5.68e1	6.68e1	3.02e0	5.86e-2	5.78e-2	3.10e2	1.06e2
6	87639	625	625	3.92e2	5.74e1	6.60e1	1.17e1	5.40e-2	8.29e-2	8.05e2	2.87e2
				6.82 :	1 :	1.15	216 :	1 :	1.53	2.8 :	1

Comparison estimates' time efficiency.

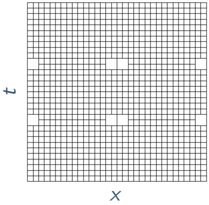
Example 2-2. Comparison of meshes $\mathbb{M}_{\rm BULK}(0.4)$

ref. based on true error $\|\nabla_x (u - u_h)\|^2_Q$



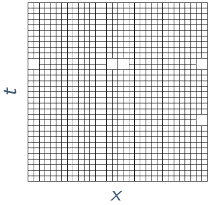
ref. 1

ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



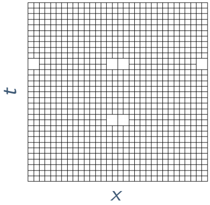
ref. 1

ref. based on error measured by $\mathcal{L}$-norm $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 1

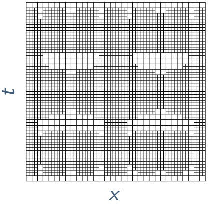
ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 1

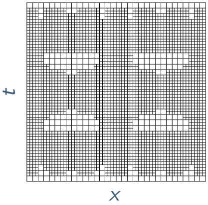
Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on true error $\|\nabla_x (u - u_h)\|^2_Q$



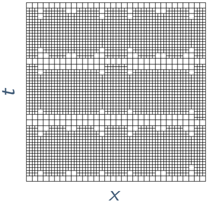
ref. 2

ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



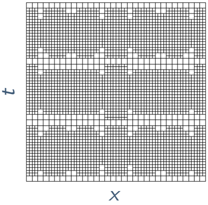
ref. 2

ref. based on error measured by $\mathcal{L}$-norm $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 2

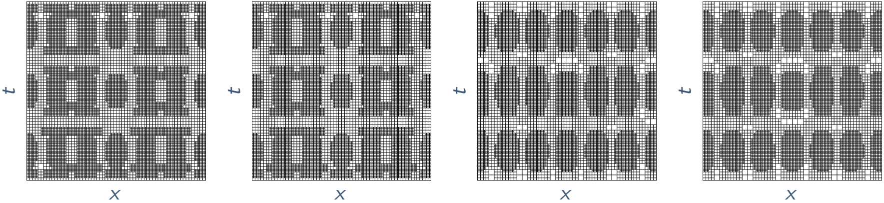
ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 2

Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

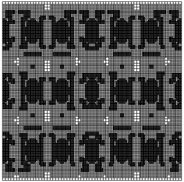
ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$	ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$	ref. based on error measured by $\mathcal{L}$-norm $\|u - u_h\|^2_{\mathcal{L},Q}$	ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 3	ref. 3	ref. 3	ref. 3

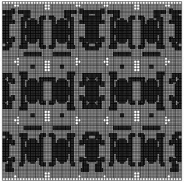
Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on
true error
$\|\nabla_x(u - u_h)\|^2_Q$



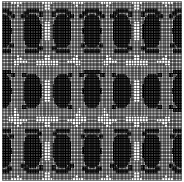
ref. 4

ref. based on
error indicator
$\|\mathbf{y}_h - \nabla_x u_h\|^2_Q$



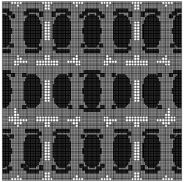
ref. 4

ref. based on
error measured by $\mathcal{L}$-norm
$\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 4

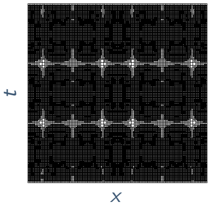
ref. based on
error identity
$\mathbb{E}\mathrm{d}$



ref. 4

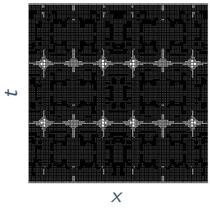
Example 2-2. Comparison of meshes $\mathbb{M}_{\rm BULK}(0.4)$

ref. based on true error $\|\nabla_x (u - u_h)\|^2_Q$



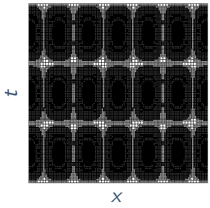
ref. 5

ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



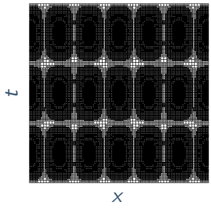
ref. 5

ref. based on error measured by $\mathcal{L}$-norm $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 5

ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 5

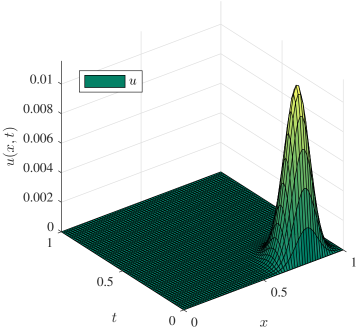
Example 3. Sharp local Gaussian jump

Given data:

- $\Omega = (0,1)$, $T = 1$
- $u = (x^2 - 2\,x)\,(t^2 - t)\, e^{-100\,|(x,t)-(0.8,0.05)|}$
- $f = \dots$
- $u_D = 0$

Discretization:

- $u_h \in S_h^{2,2}$
- $\mathbf{y}_h \in S_h^{3,3} \oplus S_h^{3,3}$
- $w_h \in S_h^{3,3}$



Example 3. Adaptive refinement, $u_h \in S_h^{2,2}$, $\boldsymbol{y}_h \in S_h^{3,3} \oplus S_h^{3,3}$ and $w_h \in S_h^{3,3}$

# ref.	$\|\nabla_x e\|_Q^2$	$I_{\rm eff}(\overline{\rm M}^{\rm I})$	$I_{\rm eff}(\overline{\rm M}^{\rm II})$	$\|\|\|e\|\|\|_{s,h}^2$	$I_{\rm eff}(\overline{\rm M}^{\rm I}_{s,h})$	$\|\|\|e\|\|\|_{\mathcal{L}}^2$	$I_{\rm eff}(\mathbb{E}{\rm d})$	e.o.c.
2	2.3860e-4	2.98	1.57	2.3862e-4	3.46	5.2907e-2	1.00	15.08
3	7.9064e-5	3.46	1.64	7.9097e-5	3.73	2.7679e-2	1.00	4.29
4	1.9474e-5	2.33	1.32	1.9475e-5	2.73	1.4184e-2	1.00	3.47
5	6.0613e-6	2.71	1.38	6.0614e-6	2.98	7.6247e-3	1.00	2.48
6	2.2038e-6	2.66	1.31	2.2039e-6	2.84	4.5977e-3	1.00	1.95

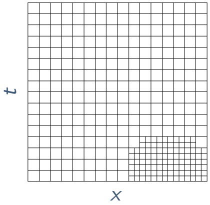
Comparison estimates' error control efficiency.

# ref.	d.o.f.		$t_{\rm as}$		$t_{\rm sol}$		$t_{\rm e/w}$	
	u_h	$\boldsymbol{y}_h$	u_h	$\boldsymbol{y}_h$	u_h	$\boldsymbol{y}_h$	$\|\|\|e\|\|\|_{\mathcal{L}}^2$	$\mathbb{E}{\rm d}$
1	324	722	1.59e-1	5.53e-1	3.11e-3	3.83e-3	1.49e-1	1.35e-1
2	400	872	2.33e-1	9.42e-1	4.22e-3	5.32e-3	2.23e-1	2.06e-1
3	669	1394	4.98e-1	2.46e0	1.08e-2	9.50e-3	4.36e-1	4.23e-1
4	1501	2952	1.51e0	7.72e0	4.36e-2	4.74e-2	1.12e0	1.10e0
5	3848	7418	3.67e0	2.06e1	2.10e-1	1.73e-1	3.07e0	2.92e0
6	10854	20744	1.31e1	5.52e1	1.04e0	7.29e-1	8.89e0	8.52e0
			1 :	4.21	1.42 :	1	1.043 :	1

Comparison estimates' time efficiency.

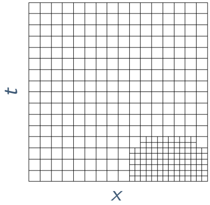
Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on
true error
$\|\nabla_x(u - u_h)\|^2_Q$



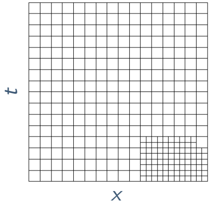
ref. 1

ref. based on
error indicator
$\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



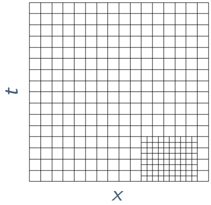
ref. 1

ref. based on
error measured by $\mathcal{L}$-norm
$\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 1

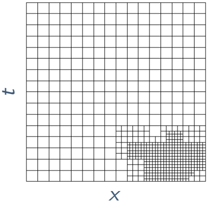
ref. based on
error identity
$\mathbb{E}\mathrm{d}$



ref. 1

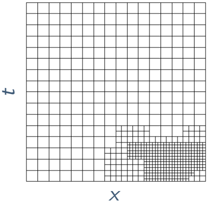
Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



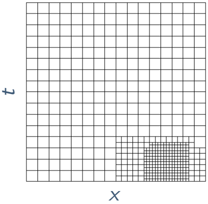
ref. 2

ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



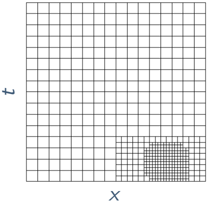
ref. 2

ref. based on error measured by $\mathcal{L}$-norm $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 2

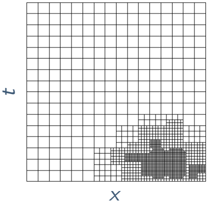
ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 2

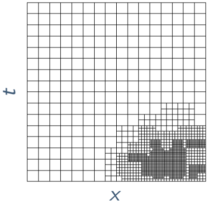
Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on true error
$\|\nabla_x(u-u_h)\|^2_Q$



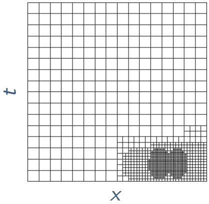
ref. 3

ref. based on error indicator
$\|\boldsymbol{y}_h-\nabla_x u_h\|^2_Q$



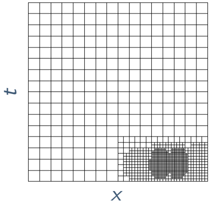
ref. 3

ref. based on error measured by $\mathcal{L}$-norm
$\|u-u_h\|^2_{\mathcal{L},Q}$



ref. 3

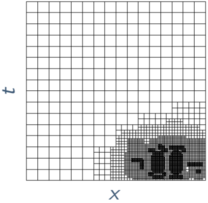
ref. based on error identity
$\mathbb{E}\mathrm{d}$



ref. 3

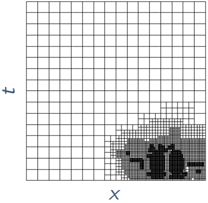
Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



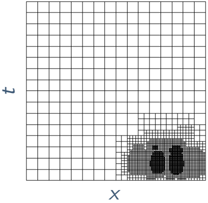
ref. 4

ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



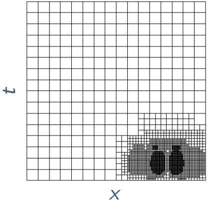
ref. 4

ref. based on error measured by $\mathcal{L}$-norm $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 4

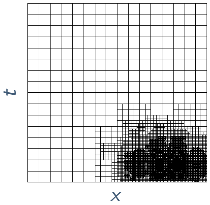
ref. based on error identity $\mathbb{E}\mathrm{d}$



ref. 4

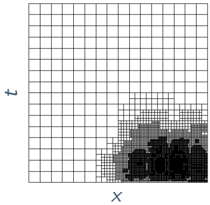
Example 2-2. Comparison of meshes $\mathbb{M}_{\mathrm{BULK}}(0.4)$

ref. based on true error $\|\nabla_x(u - u_h)\|^2_Q$



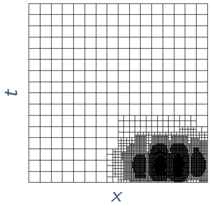
ref. 5

ref. based on error indicator $\|\boldsymbol{y}_h - \nabla_x u_h\|^2_Q$



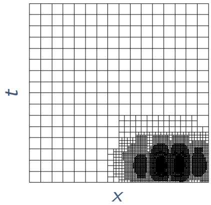
ref. 5

ref. based on error measured by $\mathcal{L}$-norm $\|u - u_h\|^2_{\mathcal{L},Q}$



ref. 5

ref. based on error identity $\mathbb{E}\mathrm{d}$



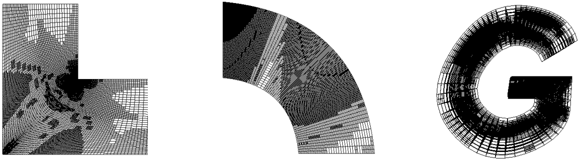
ref. 5

Conclusions and roadmap on future work

1. Developing **efficient algorithms of the flux reconstruction**:
   - improving assembling time, in particular, for the THB-splines
   - studying solvers and preconditioners for the system

$$\left(\beta^{-1}\, C^2_{\mathrm{F}\Omega} \mathrm{Div}_{\mathrm{h}}^{(\mathrm{d})} + \mathrm{M}_{\mathrm{h}}^{(\mathrm{d})}\right) \underline{\mathbf{y}}_h = -\beta^{-1}\, C^2_{\mathrm{F}\Omega} \mathrm{z}_h + \mathrm{g}_h.$$

2. Extending the **space-time scheme** and corresponding **error estimates**:
   - localisation
   - moving in time domain Ω
   - extension to the multi-patch discretisation
   - treating the discontinuity in time



THANK YOU FOR YOUR ATTENTION!

U. Langer, S. Matculevich, and S. Repin. A posteriori error estimates for space- time IgA approximations to parabolic initial boundary value problems, arXiv.org, math.NA/1612.08998, 2016.

U. Langer, S. Matculevich, and S. Repin. Functional type error control for stabilised space-time IgA approximations to parabolic problems, RICAM-Report 2017-14.

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U. Langer, S. Matculevich, and S. Repin. Functional type error estimates and identities for space-time IgA approximations to I-BVP of parabolic type, RICAM-Report 2017 (in preparation).