An Inverse Fresnel Problem in Global Navigation Satellite System Reflectometry.

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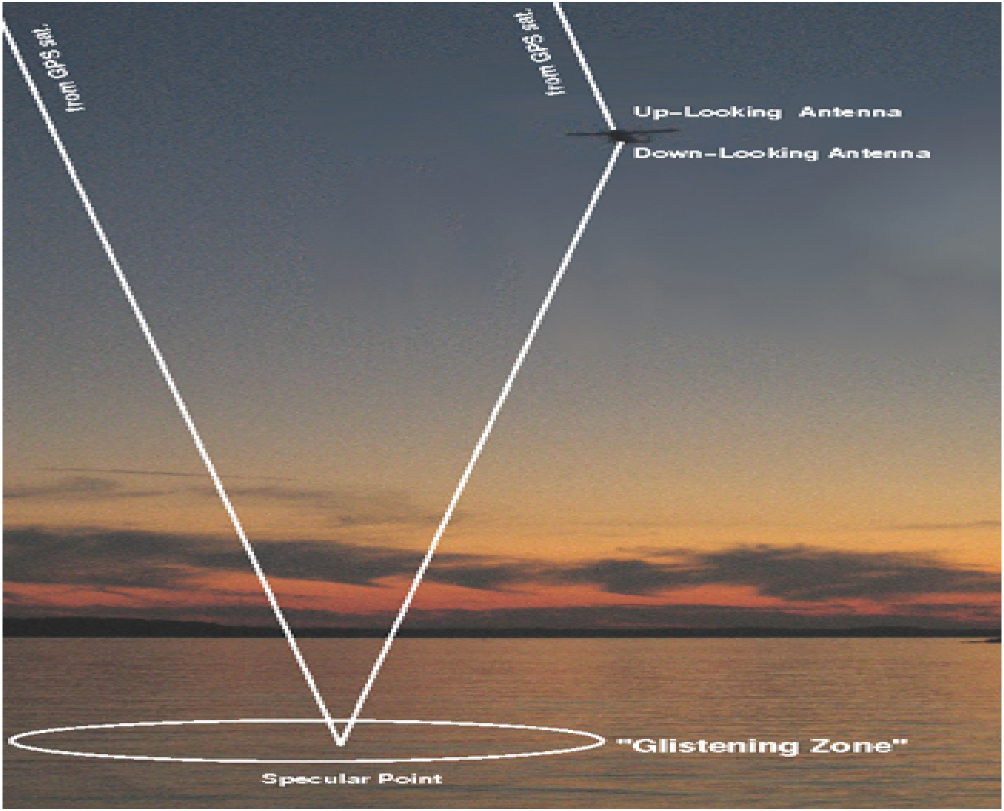
1 Global Navigation Satellite System Reflectometry.

- The main goal of GNSS-R is to derive information on the properties of a portion of soil (e.g., soil moisture, snow depth, wave configurations, ...), by remote sensing; that is, by analyzing signals emitted by GNSS satellites, and the reflected signals captured by an antenna.

- Example: Moisture increases the dielectric constant of the soil medium;

- Dielectric constant can be retrieved from measurements of reflectivity or transmissivity of surface;

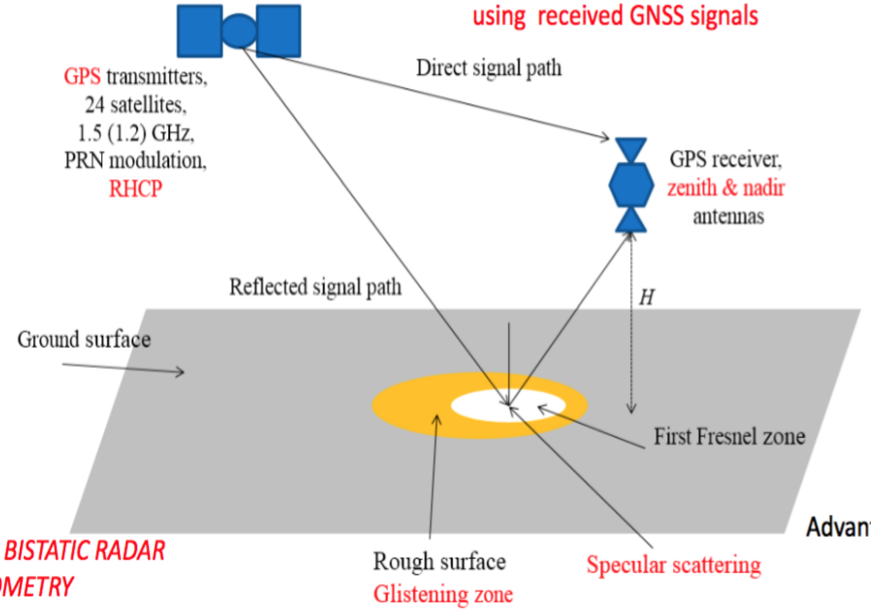
- To recover dielectric constant, one has to solve an inverse problem concerning the Fresnel coefficients.


from GPS sat.
from GPS sat.
Up-Looking Antenna
Down-Looking Antenna
"Glistening Zone"
Specular Point

GNSS-Reflectometry

Passive bi-static radar

Detect the surface characteristics by using received GNSS signals



Advantages

GPS BISTATIC RADAR GEOMETRY

APPLICATIONS:

(GNSS-R, cont.)

- Again: The GPS receiver measures a number of quantities, related to the perpendicular and parallel polarization of the signals.

- These quantities depend principally on the value of the incidence angle θ, and on the dielectric constant ε.

- The dielectric constant is intrinsic of the soil, and provides information on its composition and properties.

- For "dispersive" soils, $\varepsilon \in \mathbb{C}$; for non-dispersive ones, $\varepsilon \in \mathbb{R}_{>0}$.

- In fact, assume $\Re(\varepsilon) > 1$; i.e., the soil is denser than the air (for which, by convention, $\varepsilon = 1$).

- Also, here, neglect scattering due to the 'roughness' of the soil.

2 The Fresnel Coefficients.

- For a smooth, perfectly flat, non-magnetic surface, the Fresnel reflection coefficients are a combination of the horizontal and vertical polarization coefficients

$$\Gamma_n = \frac{\cos\theta - \sqrt{\varepsilon - \sin^2\theta}}{\cos\theta + \sqrt{\varepsilon - \sin^2\theta}}, \tag{2.1}$$

$$\Gamma_p = \frac{\varepsilon\cos\theta - \sqrt{\varepsilon - \sin^2\theta}}{\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta}}. \tag{2.2}$$

- Often, only measurements of moduli $|\Gamma_n|$, $|\Gamma_p|$, or of combinations such as $\frac{1}{2}|\Gamma_n - \Gamma_p|$ or $\frac{1}{2}|\Gamma_n + \Gamma_p|$ (circular polarization).

- In each case, the goal is to recover the value of ε from the available measurements on Γ_n and Γ_p, via system (2.1)+(2.2).

3 Background.

- Maxwell's equations (linear).

- Assume: Time-harmonic dependence, yielding "elliptic" equations for the fields; e.g.,

$$\Delta E + k^2 E = 0 \,. \tag{3.1}$$

- In fact, a family of such equations, parametrized by t:

$$E(t,x) = \mathrm{e}^{j\omega t} E_0(x) \qquad (j^2 := -1) \,. \tag{3.2}$$

- Assume: Plane waves:

$$E_0(x) = E_0 \, \mathrm{e}^{-j\,k\,u\cdot x} \,, \qquad E_0 \in \mathbb{R}^3 \,, \quad |u| = 1 \,. \tag{3.3}$$

- Note: E and H orthogonal ($E \cdot H = 0$).

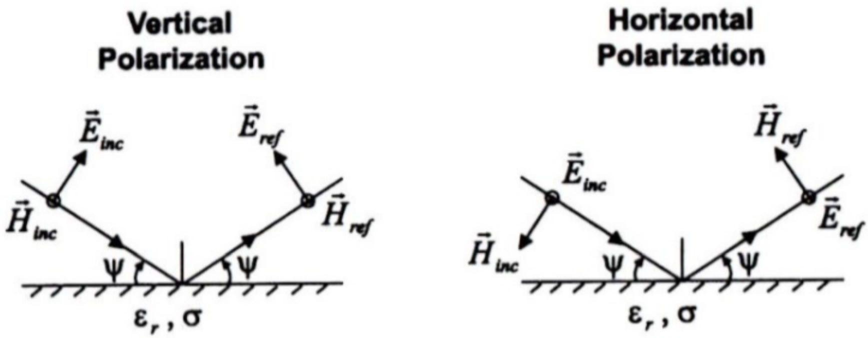
(Background, cont.)

- Plane wave incident onto a plane boundary, assumed to be the (x, y)-plane.

- Parallel (or horizontal) polarization: E orthogonal to the (x, z)-plane, H parallel to the (x, z)-plane.

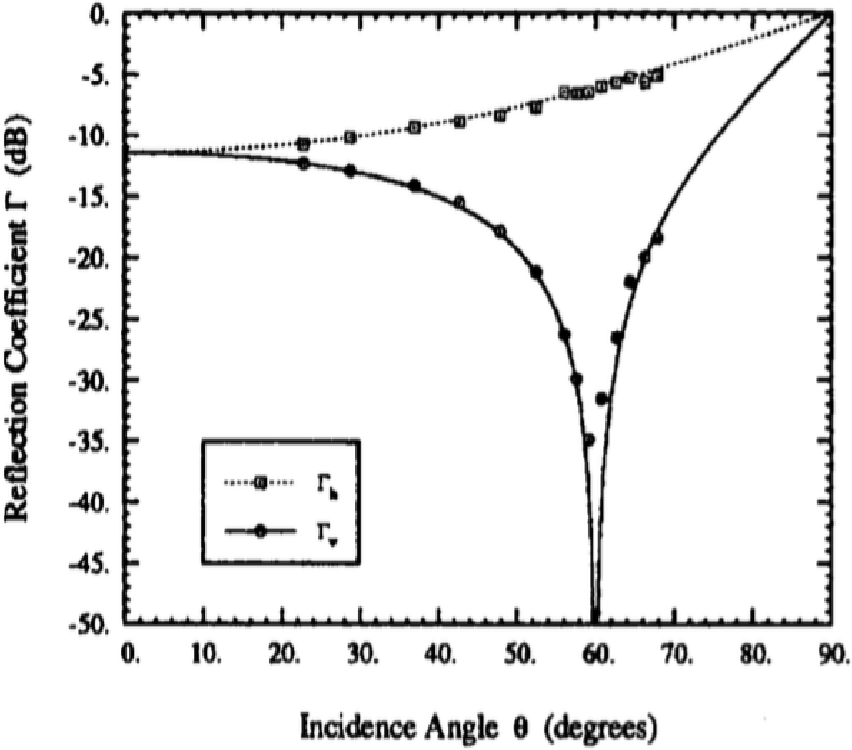
- Perpendicular (or vertical) polarization: viceversa, i.e. H orthogonal to the (x, z)-plane, E parallel to the (x, z)-plane.

- Imposing the continuity of the tangential components of the fields across the boundary $z = 0$, and then implementing Snell's laws, deduce the Fresnel system (2.1)+(2.2).



4 Goal.

- Assume have measurements of $\gamma_n := |\Gamma_n|$ and $\gamma_p := |\Gamma_p|$, with $0 < \gamma_p \leq \gamma_n < 1$ (see figure 4).



0.
-5.
-10.
-15.
-20.
-25.
-30.
-35.
-40.
-45.
-50.
0.
10.
20.
30.
40.
50.
60.
70.
80.
90.
Reflection Coefficient Γ (dB)
Incidence Angle θ (degrees)
Γ_h
Γ_v

(Goal, cont.)

- Continuous curves of figure are of γ_n and γ_p; such curves are all of same shape, for each $\varepsilon \in I\!R_{>1}$.

- Given incidence angle $\theta \in \left]0, \frac{\pi}{2}\right[$, and such measured values γ_n, $\gamma_p \in]0, 1[$, find $\varepsilon \in \mathbb{C}$, with $\Re(\varepsilon) > 1$, solution of the system

$$\left|\frac{\cos\theta - \sqrt{\varepsilon - \sin^2\theta}}{\cos\theta + \sqrt{\varepsilon - \sin^2\theta}}\right| = \gamma_n \, , \tag{4.1}$$

$$\left|\frac{\varepsilon\cos\theta - \sqrt{\varepsilon - \sin^2\theta}}{\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta}}\right| = \gamma_p \, . \tag{4.2}$$

5 Immediate Remarks.

- In the literature, system (4.1)+(4.2) seems to be solved numerically, even though exact, algebraic solution is (almost) elementary.

- If soil is known to be essentially non-dispersive, i.e. $|\Im(\varepsilon)| << 1$, then one solves (4.1)+(4.2) for $\varepsilon \in I\!R_{>1}$.

- However, in this case system is over-determined, and can be solved only under suitable compatibility conditions.

- More specifically, looking for real solutions $\varepsilon \in I\!R_{>1}$:

  - (4.1) yields $\varepsilon_n = \varphi(\gamma_n, \theta)$; (4.2) yields $\varepsilon_p = \psi(\gamma_p, \theta)$;
  - So, need to make sure that:
    1) $\varepsilon_n = \varepsilon_p =: \varepsilon$; 2) ε is independent of θ.

- In all cases, find "optimal" strategy to find ε.

6 Real Solutions of (4.1).

Theorem 6.1 *1) For all* $\theta \in \left[0, \frac{\pi}{2}\right[$ *and all corresponding* $\gamma_n \in \left]0,1\right[$ *(as measured), there exists a unique solution* $\varepsilon = \varepsilon_n > 1$ *of equation* (4.1)*, given by*

$$\boxed{\varepsilon_n = 1 + \frac{4\,\gamma_n\,\cos^2\theta}{(1-\gamma_n)^2}}\,. \tag{6.1}$$

2) This solution is independent of θ *if and only if the measured values of* γ_n *satisfy the following condition: There is* $\alpha > 1$ *such that*

$$\gamma_n(\theta) = \frac{\sqrt{\alpha - \sin^2\theta} - \cos\theta}{\sqrt{\alpha - \sin^2\theta} + \cos\theta} \tag{6.2}$$

for all $\theta \in \left[0, \frac{\pi}{2}\right[$*. In this case,* $\varepsilon_n(\theta) \equiv \alpha$*.* ◇

- Solution (6.1) is immediate.
- Condition (6.2) certainly not surprising, as it essentially is the definition of γ_n itself.

7 The Brewster Angle.

- When $\varepsilon \in I\!R_{>0}$, the numerator of (2.2) can change sign: at angle

$$\theta_B = \theta_B(\varepsilon) = \arctan(\sqrt{\varepsilon})\ . \tag{7.1}$$

- θ_B called BREWSTER angle.

- $\varepsilon > 1 \iff \theta_B > \frac{\pi}{4}$.

- It is in fact often observed in measurements that there indeed is an angle $\tilde{\theta}$ such that, correspondingly, $\gamma_p \approx 0$ (recall figure 4). In this case, $\tilde{\theta}$ is taken as an approximation of θ_B.

8 Real Solutions of (4.2).

Solution of (4.2) in three steps:

1. Algebraic solution [slightly less immediate than for (4.1)]. In contrast with (4.1), find three different solutions to (4.2):

   1.1 A solution ε_p^0 defined for all $\theta \in \left]0, \frac{\pi}{2}\right[$, and

   1.2 Two other solutions, ε_p^1 and ε_p^2, defined in a smaller interval $\left[\theta_0, \frac{\pi}{2}\right[\subset \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$

   - In the common interval $\left[\theta_0, \frac{\pi}{2}\right[$, $\varepsilon_p^0 \geq \varepsilon_p^1 \geq \varepsilon_p^2 \geq 1$.

2. Condition for the existence of a solution ε_p of (4.2) independent of θ. Again not surprisingly, this condition is essentially the definition of γ_p itself.

3. Obtain this constant solution ε_p by suitably patching together the three above different solutions of (4.2).

(Real Solutions of (4.2), cont.)

Theorem 8.1 *Assume there is* $\theta_B \in \ \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$ *such that, correspondingly,* $\gamma_p = 0$*. For* $\gamma_p \in [0, 1[$*, set*

$$\lambda_p := \frac{1+\gamma_p}{1-\gamma_p}\,. \tag{8.1}$$

- λ_p *depends on* θ*, via* γ_p*, and* $\lambda_p \geq 1$*. Then:*

1) For all $\theta \in \left[0, \frac{\pi}{2}\right[$ *and all corresponding* $\gamma_p \in [0, 1[$ *(as measured), there exists a solution* $\varepsilon_p^0 > 1$ *of problem* (4.2)*, given by*

$$\boxed{\varepsilon_p^0 = \frac{\lambda_p^2}{2\,\cos^2\theta}\left(1+\sqrt{1-\frac{\sin^2(2\theta)}{\lambda_p^2}}\right)}\,, \tag{8.2}$$

with

$$\varepsilon_p^0 \geq \tan^2\theta \qquad \forall\,\theta \in \left[0, \tfrac{\pi}{2}\right[\,. \tag{8.3}$$

Thus, ε_p^0 *cannot be independent of* θ *in all of* $\left[0, \frac{\pi}{2}\right[$*.*

(Theorem 8.1, cont.)

2) Assume that

$$\lambda_p(\theta)\ \sin(2\theta) \leq 1 \tag{8.4}$$

(see below). Then, for all $\theta \in \left[\theta_B, \frac{\pi}{2}\right[$ *and all corresponding* $\gamma_p \in [0,1[$*, there is a solution* $\varepsilon_p^1 > 1$ *of problem* (4.2)*, given by*

$$\boxed{\varepsilon_p^1 = \frac{1}{2\,\lambda_p^2\ \cos^2\theta}\ \left(1 + \sqrt{1 - \lambda_p^2\ \sin^2(2\theta)}\right)}\,. \tag{8.5}$$

This solution satisfies the conditions

$$2\ \sin^2\theta \leq \varepsilon_p^1 \leq \tan^2\theta \qquad \forall\, \theta \in \left[\theta_B, \tfrac{\pi}{2}\right[\,. \tag{8.6}$$

Thus, again, ε_p^1 *cannot be independent of* θ *in all of* $\left[\theta_B, \frac{\pi}{2}\right[$ *(unless* $\varepsilon_p^1 \geq 2$ *for all* $\theta \in \left[\theta_B, \frac{\pi}{2}\right[$*, in which case it may be independent of* θ*).*

(Theorem 8.1, cont.)

3) In addition, problem (4.2) *also has, in* $\left]\theta_B, \frac{\pi}{2}\right[$*, a solution* $\varepsilon_p^2 > 1$*, given by*

$$\boxed{\varepsilon_2^p = \frac{1}{2\,\lambda_p^2\,\cos^2\theta}\left(1 - \sqrt{1 - \lambda_p^2\,\sin^2(2\theta)}\right)}\,. \tag{8.7}$$

This solution satisfies the condition

$$\varepsilon_p^2 \leq 2\,\sin^2\theta \qquad \forall\,\theta \in \left[\theta_B, \tfrac{\pi}{2}\right[\,. \tag{8.8}$$

$\diamond$

- Condition (8.4) is necessary for the existence of a solution $\varepsilon > 1$ of (4.2) (not necessarily independent of θ) in $\left[0, \frac{\pi}{2}\right[$.

9 Constant Real Solutions to (4.2).

- If $\tan^2\theta_B \geq 2$, define

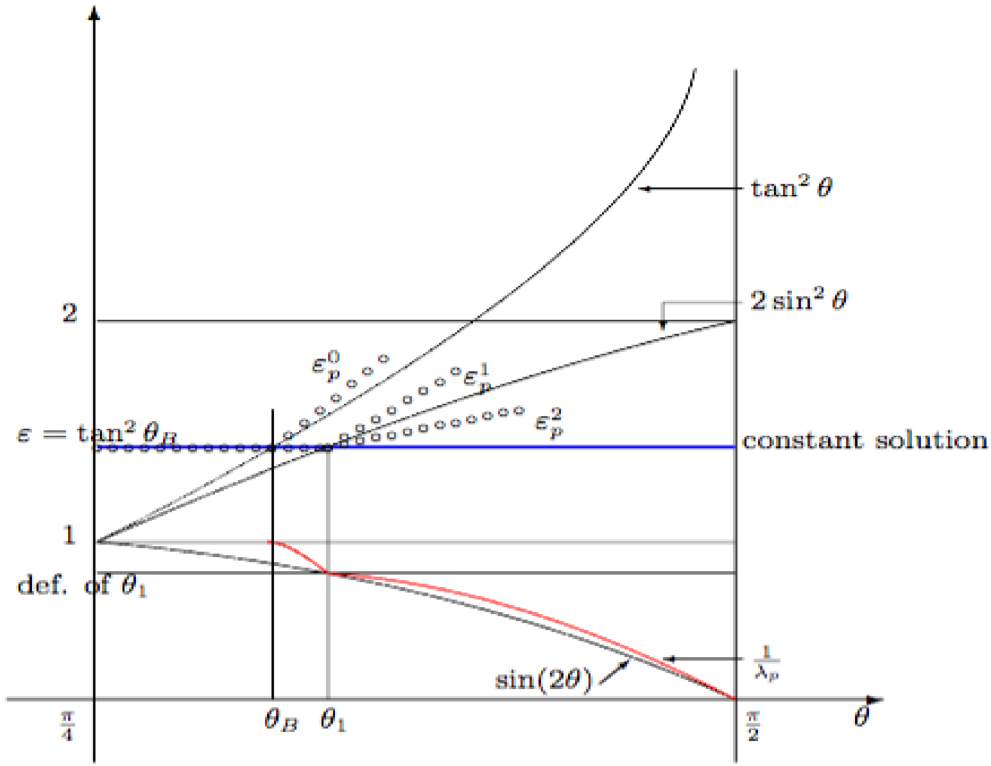
$$\varepsilon_p = \begin{cases} \varepsilon_0^p & \text{if} \quad 0 \leq \theta \leq \theta_B \ , \\ \varepsilon_1^p & \text{if} \quad \theta_B \leq \theta < \frac{\pi}{2} \ . \end{cases} \tag{9.1}$$

- If $1 < \tan^2\theta_B < 2$, define θ_1 by the identity

$$2\,\sin^2\theta_1 = \tan^2\theta_B \tag{9.2}$$

(See figure 9), and

$$\varepsilon_p = \begin{cases} \varepsilon_0^p & \text{if} \quad 0 \leq \theta \leq \theta_B \ , \\ \varepsilon_1^p & \text{if} \quad \theta_B \leq \theta \leq \theta_1 \ , \\ \varepsilon_2^p & \text{if} \quad \theta_1 < \theta < \frac{\pi}{2} \ . \end{cases} \tag{9.3}$$



$\tan^2\theta$
$2\sin^2\theta$
2
ε_p^0
ε_p^1
ε_p^2
$\varepsilon = \tan^2\theta_B$
constant solution
1
def. of θ_1
$\frac{1}{\lambda_p}$
$\sin(2\theta)$
$\frac{\pi}{4}$
θ_B
θ_1
$\frac{\pi}{2}$
θ

Theorem 9.1 *Define* $\varepsilon_p : \left[0, \frac{\pi}{2}\right[\to [1, +\infty[$ *by* (9.1) *if* $\tan^2\theta_B \geq 2$, *or by* (9.3) *if* $1 < \tan^2\theta_B < 2$. *Then:*

1) ε_p *is a continuous solution of* (4.2)*;*

2) ε_p *is independent of* θ *if and only if the measured values of* γ_p *satisfy the following condition: There is* $\beta > 1$ *such that*

$$\gamma_p(\theta) = \left| \frac{\beta\cos\theta - \sqrt{\beta - \sin^2\theta}}{\beta\cos\theta + \sqrt{\beta - \sin^2\theta}} \right| \tag{9.4}$$

for all $\theta \in \left[0, \frac{\pi}{2}\right[$ *(compare to* (2.2)*). In this case,* $\varepsilon_p(\theta) \equiv \beta = \tan^2\theta_B$.

◇

10 Common Real Solutions to (4.1) and (4.2).

Theorem 10.1 *Let* ε_n *and* ε_p *be as in theorems 6.1 and 8.1. Then:*
1) $\varepsilon_n(\theta) = \varepsilon_p(\theta)$ *on all of* $\left[0, \frac{\pi}{2}\right[$ *if and only if the compatibility condition*

$$\lambda_n^2 \cos^2\theta + \sin^2\theta = \begin{cases} \lambda_n \lambda_p & \text{if} \quad 0 \le \theta \le \theta_B \, , \\ \frac{\lambda_n}{\lambda_p} & \text{if} \quad \theta_B \le \theta < \frac{\pi}{2} \, , \end{cases} \tag{10.1}$$

holds in $\left[0, \frac{\pi}{2}\right[$, *together with the additional conditions* $\boxed{\gamma_p \le \gamma_n^2}$ *if* $\theta_B \le \theta \le \theta_1$, *or* $\boxed{\gamma_n^2 \le \gamma_p}$ *if* $\theta_1 \le \theta < \frac{\pi}{2}$. *In this case, the common solution* $\varepsilon_n = \varepsilon_p =: \varepsilon_c$ *is given by*

$$\boxed{\varepsilon_c = \begin{cases} \lambda_n \lambda_p & \text{if} \quad 0 \le \theta \le \theta_B \, , \\ \frac{\lambda_n}{\lambda_p} & \text{if} \quad \theta_B \le \theta < \frac{\pi}{2} \, , \end{cases}} \, . \tag{10.2}$$

2) This common solution ε_c *is independent of* θ *if and only if* γ_n *and* γ_p *are of the form* (6.2) *and* (9.4), *with* $\alpha = \beta$*; in this case,* $\varepsilon_c = \tan^2\theta_B$.

◇

11 Complex Solutions of (4.1)+(4.2).

[Work in progress!] $\Longrightarrow$ Preliminary, partial results.

- Given $\gamma_n,\, \gamma_p \in \,]0,1[$ as measured, and $\theta \in \left[0, \frac{\pi}{2}\right[$, define

$$\mu_n \;:=\; \frac{1+\gamma_n^2}{1-\gamma_n^2}\,, \qquad \mu_p := \frac{1+\gamma_p^2}{1-\gamma_p^2}\,, \tag{11.1}$$

$$C = C(\theta) \;:=\; (\mu_n^2 - 1)\cos^2\theta - \mu_n\,\mu_p + 1\,. \tag{11.2}$$

- Note: $1 < \mu_p \le \mu_n, \quad \mu_n < \lambda_n < 2\,\mu_n, \quad \mu_p < \lambda_p < 2\,\mu_p$.

- Note: $C(0) > 0 > C\left(\frac{\pi}{2}\right)$; so, there is $\theta_0 \in \left]0, \frac{\pi}{2}\right[$ such that $C(\theta_0) = 0$.

- Note:

$$\theta_0 < \;[=\,,\; >]\, \frac{\pi}{4} \iff \gamma_p > \;[=\,,\; <]\, \gamma_n^2\,. \tag{11.3}$$

(Complex Solutions to (4.1)+(4.2), cont.)

Strategy:

- Set $\varepsilon = x + j\,y\,,\qquad \sqrt{\varepsilon - \sin^2\theta} = u + j\,v.$

- Replacing into (4.1)+(4.2) obtain

$$v^2 = -\,(u^2 - 2\,(\mu_n \cos\theta)u + \cos^2\theta) =: -\,P(u)\,, \quad (11.4)$$

$$x = u^2 - v^2 + \sin^2\theta\,, \quad (11.5)$$

$$y = 2\,u\,|v|\,. \quad (11.6)$$

- Note: ε is a solution $\iff$ $\bar{\varepsilon}$ is a solution.

- Thus, determine u. Need $P(u) < 0$; $\Longrightarrow$ compatibility conditions

$$\frac{1}{\lambda_n}\cos\theta < u < \lambda_n \cos\theta \quad (11.7)$$

($\Longrightarrow$ $u > 0$; in fact, $\boxed{\mu_n \cos\theta < u < \lambda_n \cos\theta}$ to have $\Re(\varepsilon) > 1$).

(Complex Solutions to (4.1)+(4.2), cont.)

Theorem 11.1 1) *If* $\gamma_n^2 = \gamma_p$*, let* $\theta_* := \arccos\sqrt{\frac{\lambda_n}{2\mu_n}}$*. Then, problem* (4.1)+(4.2) *has a solution* $\varepsilon \in \mathbb{C}$ *for all* $\theta \in \left]\theta_*, \frac{\pi}{2} - \theta_*\right[$*, given by*

$$\boxed{u = \frac{1}{2\,\mu_n\,\cos\theta}}\,. \tag{11.8}$$

In addition, if $\theta = \frac{\pi}{4}$*, problem* (4.1)+(4.2) *has also infinitely many complex conjugate solutions*

$$\varepsilon = \varepsilon(r) = (r^2 - \lambda_p\,r + 1) \pm j\,r\sqrt{2\,\lambda_p\,r - r^2 - 1}\,, \tag{11.9}$$

parametrized by $r \in \left]\frac{1}{\lambda_n}, \lambda_n\right[$*. In particular, the solution*(*11.8*) *corresponds to the value* $r = \frac{1}{\lambda_n}$ *in* (*11.9*).

(Theorem 11.1, cont.)

2) *If* $\gamma_n^2 \neq \gamma_p$, *there exist angles* $\theta_i \in \left]0, \frac{\pi}{2}\right[$, $1 \leq i \leq 4$, *with*

$$0 < \theta_1 < \theta_2 < \tfrac{\pi}{4} < \theta_3 < \theta_4 < \tfrac{\pi}{2}\,, \tag{11.10}$$

such that problem (4.1)+(4.2) *has a solution* $\varepsilon \in \mathbb{C}$ *if and only if* $\theta \in \left]\theta_1, \theta_2\right[$ *or* $\theta \in \left]\theta_3, \theta_4\right[$. *This solution is given by*

$$\boxed{u = \frac{(\mu_n - \mu_p)\,\cos(2\theta)}{2\,C\,\cos\theta}}\,, \tag{11.11}$$

where C *is as in (11.2).* ◇

- Note: $v = 0 \implies y = 0 \implies \varepsilon = \varepsilon_n$ of (6.1).
- Note: $\theta = 0 \implies u = \frac{1}{2\,\mu_n} \implies P(u) = \frac{1}{4\,\mu_n^2} > 0 \implies v = \Im(\varepsilon) \notin I\!R$: no.

12 How good are the above results?

- One control situation, with experimental data from:
De Roo - Ulaby, *Bistatic Specular Scattering from Rough Dielectric Surfaces*, IEEE Trans. on Antennas and Propagation, 42/2, Feb. 1994.

- Soil is sand, for which $\boxed{\varepsilon = 3 + 0.05\, j}$ is known.

- In their paper, $\Im(\varepsilon)$ is neglected ($|\Im(\varepsilon)| << \Re(\varepsilon)$).

- Brewster angle would be $\theta = \arctan(\sqrt{3}) = 60^0$; in fact, in paper, $\tilde{\theta}_B \approx 60^0$.

(How good are the above results?, cont.)

eps from Fig. 5; "True" epsilon = 3; theta = angle in degrees; gamma_n and gamma_p from table (in dB); epsilon_n from formula (6.1); epsilon_p from formula (9.3); epsilon_c from formula (10.2); % errors = ABS{eps - 3}/3.							
theta	gamma_n	epsilon_n	gamma_p	epsilon_p	epsilon_c	error_c	tan^2(theta)
0	0.26915	3.01557	0.26915	3.01562	3.01565	0.00520	0
23	0.29854	3.05641	0.24831	3.09418	3.07421	0.02474	0.18079
29	0.31623	3.06958	0.22909	3.06834	3.06900	0.02300	0.30726
36.5	0.35481	3.20312	0.19952	3.07529	3.14671	0.04890	0.54754
42.5	0.36728	2.99478	0.16788	3.08821	3.03292	0.01097	0.83966
48	0.39811	2.96812	0.12882	3.07684	3.00983	0.00328	1.23346
53.5	0.42170	2.78456	0.08912	3.23317	2.93947	0.02018	1.82364
58	0.48978	3.11332	0.03162	3.10549	3.11057	0.03686	2.56107
60 [B]	0.50119	3.01434	0.00316 [0]	3.05711	3.02860	0.00953	3.00000
62.5	0.51880	2.91082	0.05012	2.73259	2.85500	0.04834	3.69017
66	0.56234	2.94272	0.10593	2.75306	2.88495	0.03802	5.04468
68	0.57544	2.79198	0.12589	3.10664	2.88091	0.03970	

(How good are the above results?, cont.)

eps from Fig. 5;
"True" epsilon = 3 + 0.05 j;
theta = angle in degrees;
gamma_n and gamma_p from table (in dB);
complex epsilon from formula (11.11) for u;
% error = SQRT{ (Re(eps)-3)^2 + Im(eps) - 0.05)^2 }/3.

theta	gamma_n	epsilon_n	gamma_p	eps complex	error complex	tan^2(theta)
23	0.29854	3.05641	0.24831			0.18079
29	0.31623	3.06958	0.22909	3.03658 + 0.31496 j	0.08916	0.30726
36.5	0.35481	3.20312	0.19952			0.54754
42.5	0.36728	2.99478	0.16788	u > u_{+}		0.83966
48	0.39811	2.96812	0.12882			1.23346
53.5	0.42170	2.78456	0.08912	u < u_{-}		1.82364
58	0.48978	3.11332	0.03162			2.56107
60	0.50119	3.01434	0.00316 [0]	3.01356 + 0.05525 j	0.00485	3.00000
62.5	0.51880	2.91082	0.05012			3.69017
66	0.56234	2.94272	0.10593	2.82754 + 0.54092	0.17344	5.04468
68	0.57544	2.79198	0.12589			

13 A few open questions.

- Find additional conditions for $\varepsilon \in \mathbb{C}$ to be independent of θ.

- By theorem 9.1, optimal strategy to solve (4.1)+(4.2) for $\varepsilon \in I\!R_{>1}$ is to "wait" till position $\theta = \theta_B$: $\boxed{\varepsilon = tan^2\theta_B}$; but:

  1. Can one afford to wait that long, in actual measurements?

  2. θ_B is defined only for non-dispersive soils; otherwise?

  3. Even if measure $\gamma_p \approx 0$ at some θ_p, can conclude $\Re(\varepsilon) \approx \tan^2\theta_p$, $\Im(\varepsilon) \approx 0$? With what degree of confidence? $\Longrightarrow$ Control of error !

- Solve analogous problems when γ_n and γ_p are replaced by $\frac{1}{2}|\Gamma_n - \Gamma_p|$ and (¿ or ?) $\frac{1}{2}|\Gamma_n + \Gamma_p|$.

- Incorporate roughness of soil in model (typically by Root Mean Square of microscopic peaks and valleys). ...... etc.

14 What we REALLY would like to do.

- Revisit the model when at least one of the media (e.g., the soil) is non-linear. That is, for instance, $D = \varepsilon(E)$ (ε monotone ?).

- Emitted signal in air may be time-harmonic; but non-linearity of soil destroys the time-harmonicity of the reflected signal.

- DO THE FRESNEL FORMULAS STILL HOLD?

- At least as a 0-order approximation (of what, exactly)?

- What would replace the Fresnel formulas?

- How to give a more realistic model, which should include scattering from the terrain?

- ......

15 Conclusions.

- Equations (4.1) and (4.2) are explicitly solvable.
- (4.2) only solvable in specific ranges of θ.
- For non-dispersive soils, best strategy is $\boxed{\varepsilon = \tan^2\theta_B}$.
- Likewise, if it is known that soil is almost non-dispersive, and can observe an angle θ_p such that $\gamma_p \approx 0$, then $\boxed{\varepsilon \approx \tan^2\theta_p + 0\,j}$.
- Values of ε from De Roo & Ulaby's measurements via the explicit formulas match "true" values with error not exceeding 1%.

τὸ τέλος·

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