Functional A Posteriori Error Estimates (FAPEE) for Electro-Magneto Statics (EMS) . . . and more

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RMMM 8 - Berlin 2017

Matheon Workshop - Reliable Methods of Mathematical Modeling

August 2 2017

Humboldt Universität zu Berlin, Germany

Model Problem: Electro-Magneto-Static Maxwell Equations

setting: Hilbert/L^2-based Sobolev spaces
geometry: $\Omega \subset \mathbb{R}^3$ bounded domain with weak Lipschitz boundary $\Gamma = \partial\Omega$

$$\operatorname{rot} E = J \quad \text{in } \Omega \tag{1}$$

$$-\operatorname{div} \varepsilon E = j \quad \text{in } \Omega \tag{2}$$

$$\nu \times E = 0 \quad \text{at } \Gamma_t \tag{3}$$

$$\nu \cdot \varepsilon E = 0 \quad \text{at } \Gamma_n \tag{4}$$

non-trivial kernel: $\mathcal{H}_{\mathtt{D},\varepsilon} = \{H \in \mathsf{L}^2 : \operatorname{rot} H = 0, \operatorname{div} \varepsilon H = 0, \nu \times H|_{\Gamma_t} = 0, \nu \cdot \varepsilon H|_{\Gamma_n} = 0\}$
additional condition on Dirichlet/Neumann fields for uniqueness:

$$\pi_{\mathtt{D}} E = H \in \mathcal{H}_{\mathtt{D},\varepsilon} \tag{5}$$

well known: (1)-(5) uniquely solvable by Helmholtz decompositions and Friedrichs/Poincaré/Maxwell type estimates for given right hand sides F, G, H

aim: functional a posteriori error estimates (FAPEE) for electro-magneto statics (EMS) – simple and easy estimates

Model Problem: Electro-Magneto-Static Maxwell Equations

literature: not very much

for $\underline{\operatorname{rot}\mathring{\operatorname{rot}}+1}$ (second order toy equation):

residual based:	Beck/Hoppe/Hiptmair/Wohlmuth ('00)
	Creusé/Nicaise/... ('03)
	Schöberl ('08)
functional:	Anjam/Neittaanmäki/Repin ('07)
	Anjam/P. ('16, '17)

for $\underline{\operatorname{rot}\mathring{\operatorname{rot}}}$ (second order equation):

residual based:	Creusé/Nicaise/... ('14, '15)
equilibrated:	Braess/Schöberl ('08)
functional:	P./Repin ('09)

for $\underline{\text{first order } \mathring{\operatorname{rot}}}/\underline{\operatorname{div}\text{-system}}$ of EMS: nothing (to the best of our knowledge)

Model Problem: Main Result for Electro-Magneto-Static Maxwell Equations

Theorem (sharp upper bounds)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming!) and* $e := E - \tilde{E}$. *Then*

$$
\begin{aligned}
|e|^2_{\mathsf{L}^2_\varepsilon} &= |\pi_\nabla e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathrm{rot}} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathrm{D}} e|^2_{\mathsf{L}^2_\varepsilon} \\
&= \min_{\Phi \in \mathsf{H}_{\Gamma_n}(\operatorname{div}\varepsilon)} \Big(c_{\mathrm{fp}} |\operatorname{div}\varepsilon\Phi + j|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \Big)^2 && \boxed{\textit{reg. } (-\nabla_{\Gamma_t} \operatorname{div}_{\Gamma_n} + 1)\textit{-prbl. in } \mathsf{H}_{\Gamma_n}(\operatorname{div})} \\
&\quad + \min_{\Phi \in \mathsf{H}_{\Gamma_t}(\mathrm{rot})} \Big(c_{\mathrm{m}} |\operatorname{rot}\Phi - J|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \Big)^2 && \boxed{\textit{reg. } (\operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} + 1)\textit{-prbl. in } \mathsf{H}_{\Gamma_t}(\mathrm{rot})} \\
&\quad + \min_{\phi \in \mathsf{H}^1_{\Gamma_t}, \Psi \in \mathsf{H}_{\Gamma_n}(\mathrm{rot})} |\nabla\phi + \varepsilon^{-1} \operatorname{rot}\Psi + \tilde{E} - H|^2_{\mathsf{L}^2_\varepsilon}.
\end{aligned}
$$

$\boxed{\textit{cpld. } (-\operatorname{div}_{\Gamma_n} \nabla_{\Gamma_t})\text{-}(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})\textit{-sys. in } \mathsf{H}^1_{\Gamma_t}\text{-}\mathsf{H}_{\Gamma_n}(\mathrm{rot})}$

Remark

- $(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})$*-prbl. (Dirichlet/Neumann fields err.) needs saddle point form..*
- Ω *top. trv.* $\Rightarrow \pi_{\mathrm{D}} = 0$ *and* $\mathsf{H}_{\Gamma_t}(\operatorname{rot} 0) = \nabla \mathsf{H}^1_{\Gamma_t}$ *and* $\mathsf{H}_{\Gamma_n}(\operatorname{div} 0) = \operatorname{rot} \mathsf{H}_{\Gamma_n}(\mathrm{rot})$
- $\boxed{\Omega \textit{ convex (and } \varepsilon = \mu = 1) \textit{ and } \Gamma_t = \Gamma \textit{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \operatorname{diam}_\Omega / \pi}$

  $\Rightarrow$ *all constants known*

Underlying Structure of the Model Problem

∇-rot-div-complex (de Rham complex):

$$\{0\} \text{ or } \mathbb{R} \;\underset{\pi}{\overset{\iota}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\operatorname{div}_{\Gamma_n}\varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2_{\varepsilon} \;\underset{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}}\; \mathsf{L}^2 \;\underset{\iota}{\overset{\pi}{\rightleftarrows}}\; \mathbb{R} \text{ or } \{0\}$$

unbounded, densely defined, closed, linear operators with adjoints

$$\nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t} \subset \mathsf{L}^2 \to \mathsf{L}^2_{\varepsilon}, \quad -\operatorname{div}_{\Gamma_n}\varepsilon = (\nabla_{\Gamma_t})^* : \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \subset \mathsf{L}^2_{\varepsilon} \to \mathsf{L}^2$$

$$\operatorname{rot}_{\Gamma_t} : \mathsf{R}_{\Gamma_t} \subset \mathsf{L}^2_{\varepsilon} \to \mathsf{L}^2, \quad \varepsilon^{-1}\operatorname{rot}_{\Gamma_n} = (\operatorname{rot}_{\Gamma_t})^* : \mathsf{R}_{\Gamma_n} \subset \mathsf{L}^2 \to \mathsf{L}^2_{\varepsilon}$$

$$\operatorname{div}_{\Gamma_t} : \mathsf{D}_{\Gamma_t} \subset \mathsf{L}^2 \to \mathsf{L}^2, \quad -\nabla_{\Gamma_n} = (\operatorname{div}_{\Gamma_t})^* : \mathsf{H}^1_{\Gamma_n} \subset \mathsf{L}^2 \to \mathsf{L}^2$$

sometimes:

$\mathsf{R} = H(\operatorname{rot}) = H(\operatorname{curl})$

$\mathsf{D} = H(\operatorname{div})$

complex: 'range $\subset$ kernel' ($\operatorname{rot}\nabla = 0$, $\operatorname{div}\operatorname{rot} = 0$)

$$\nabla_{\Gamma_t}\mathsf{H}^1_{\Gamma_t} \subset \mathsf{R}_{\Gamma_t,0}, \quad \operatorname{rot}_{\Gamma_t}\mathsf{R}_{\Gamma_t} \subset \mathsf{D}_{\Gamma_t,0}, \quad \operatorname{div}_{\Gamma_t}\mathsf{D}_{\Gamma_t} \subset N(\pi) = \mathsf{L}^2 \text{ or } \mathsf{L}^2_{\perp}$$

crucial: compact embeddings (Rellich's selection theorem & Weck's selection theorem)

$$\mathsf{H}^1 \hookrightarrow \mathsf{L}^2, \qquad \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1}\mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2$$

$\Rightarrow$ Helmholtz decompositions, closed ranges, continuous inverses, and Friedrichs/Poincaré/Maxwell type estimates $\checkmark$

Weck's selection theorem (Weck '74, (Habil.) stimulated by Rolf Leis)
(Weber '80, Picard '84, Costabel '90, Witsch '93, Jochmann '97, Kuhn '99, Picard/Weck/Witsch '01, Bauer/P./Schomburg '16, '17)

Abstract Formulation

$$\begin{aligned} \operatorname{rot} E &= J && \text{in } \Omega \\ -\operatorname{div} \varepsilon E &= j && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n \\ \pi_{\mathrm{D}} E &= H \in \mathcal{H}_{\mathrm{D},\varepsilon} \end{aligned}$$

$$\wr$$

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= J \\ -\operatorname{div}_{\Gamma_n} \varepsilon E &= j \\ \pi_{\mathrm{D}} E &= H \in \mathcal{H}_{\mathrm{D},\varepsilon} \end{aligned}$$

$$(\mathrm{A}_i := \operatorname{rot}_{\Gamma_t}, \quad \mathrm{A}_i^* = \varepsilon^{-1} \operatorname{rot}_{\Gamma_n}) \qquad \wr \ (x := E) \qquad (\mathrm{A}_{i-1} := \nabla_{\Gamma_t}, \quad \mathrm{A}_{i-1}^* = -\operatorname{div}_{\Gamma_n} \varepsilon)$$

$$\begin{aligned} \mathrm{A}_i x &= f \\ \mathrm{A}_{i-1}^* x &= g \\ \pi_i x &= h \in \mathcal{H}_i := N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*) \end{aligned}$$

General or Abstract Problem

setting: unbounded, densely defined, closed, linear operators with adjoints

$$\mathrm{A}_i : D(\mathrm{A}_i) \subset \mathrm{H}_i \to \mathrm{H}_{i+1}, \quad \mathrm{A}_i^* : D(\mathrm{A}_i^*) \subset \mathrm{H}_{i+1} \to \mathrm{H}_i, \quad i \in \mathbb{Z}$$

complex:

$$\dots \rightleftarrows \mathrm{H}_{i-2} \underset{\mathrm{A}_{i-2}^*}{\overset{\mathrm{A}_{i-2}}{\rightleftarrows}} \boxed{\mathrm{H}_{i-1} \underset{\mathrm{A}_{i-1}^*}{\overset{\mathrm{A}_{i-1}}{\rightleftarrows}} \boxed{\mathrm{H}_i} \underset{\mathrm{A}_i^*}{\overset{\mathrm{A}_i}{\rightleftarrows}} \mathrm{H}_{i+1}} \underset{\mathrm{A}_{i+1}^*}{\overset{\mathrm{A}_{i+1}}{\rightleftarrows}} \mathrm{H}_{i+2} \rightleftarrows \dots$$

complex property: 'range $\subset$ kernel' ($\boxed{\mathrm{A}_i \mathrm{A}_{i-1} = 0} \quad \Leftrightarrow \quad \mathrm{A}_{i-1}^* \mathrm{A}_i^* = 0$)

$$\boxed{R(\mathrm{A}_{i-1}) \subset N(\mathrm{A}_i)} \quad \Leftrightarrow \quad R(\mathrm{A}_i^*) \subset N(\mathrm{A}_{i-1}^*)$$

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\boxed{\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h,}$$

where $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}_{i-1}^*)$ and $h \in \mathcal{H}_i$ with kernel $\mathcal{H}_i := N(\mathrm{A}_i) \cap N(\mathrm{A}_{i-1}^*)$

Toolbox

Hodge/Helmholtz/Weyl decompositions:

$$\mathsf{H}_i = N(\mathrm{A}_i) \oplus_{\mathsf{H}_i} \overline{R(\mathrm{A}_i^*)}, \qquad \mathsf{H}_{i+1} = N(\mathrm{A}_i^*) \oplus_{\mathsf{H}_{i+1}} \overline{R(\mathrm{A}_i)}$$

$\Rightarrow$ reduce A_i, A_i^* to $N(\mathrm{A}_i)^\perp$, $N(\mathrm{A}_i^*)^\perp$

$\Rightarrow$ injective reduced operators $\mathcal{A}_i$, $\mathcal{A}_i^*$

$\Rightarrow$ $\mathcal{A}_i^{-1}$, $(\mathcal{A}_i^*)^{-1}$ exist

$\Rightarrow$ complex for $\mathcal{A}_i$, $\mathcal{A}_i^*$

crucial: compact embeddings

$$D(\mathcal{A}_i) \hookrightarrow \mathsf{H}_i \qquad \big(\Leftrightarrow \quad D(\mathcal{A}_i^*) \hookrightarrow \mathsf{H}_{i+1} \big)$$

$$\Rightarrow \begin{cases} \text{(general) Friedrichs/Poincaré/Maxwell type estimates} \\ \qquad \forall\, \varphi \in D(\mathcal{A}_i) \quad |\varphi|_{\mathsf{H}_i} \le \boxed{c_i}\, |\mathrm{A}_i\varphi|_{\mathsf{H}_{i+1}} \\ \qquad \forall\, \psi \in D(\mathcal{A}_i^*) \quad |\psi|_{\mathsf{H}_{i+1}} \le \boxed{c_i}\, |\mathrm{A}_i^*\psi|_{\mathsf{H}_i} \\ \text{closed ranges } R(\mathrm{A}_i) = R(\mathcal{A}_i),\ R(\mathrm{A}_i^*) = R(\mathcal{A}_i^*) \\ \text{continuous and compact invers operators } \mathcal{A}_i^{-1},\ (\mathcal{A}_i^*)^{-1} \\ \text{lots of Helmholtz type decompositions} \end{cases}$$

best world: $D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*) \hookrightarrow \mathsf{H}_i \Rightarrow \quad D(\mathcal{A}_i) \hookrightarrow \mathsf{H}_i$

Abstract Problem and Goal

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}^*_{i-1})$ s.t.

$$\mathrm{A}_i x = f$$
$$\mathrm{A}^*_{i-1} x = g$$
$$\pi_i x = h$$

Theorem (solution theory)

unique solution (cont. dpd. on data) $\Leftrightarrow$ $f \in R(\mathrm{A}_i)$, $g \in R(\mathrm{A}^*_{i-1})$ *and* $h \in \mathcal{H}_i$

Proof.

$x = \mathcal{A}_i^{-1} f + (\mathcal{A}^*_{i-1})^{-1} g + h$ □

goal: functional a posteriori error estimates (fapee) 'in the spirit of Repin'

For $\tilde{x} \in \mathrm{H}_i$ (very non-conforming!) estimate $|x - \tilde{x}|_{\mathrm{H}_i}$ in terms of $\tilde{x}$, and f, g, h, and c_i.

Upper Bounds

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t. $\mathrm{A}_i x = f$, $\mathrm{A}_{i-1}^* x = g$, $\pi_i x = h$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_i$

def., dcmp. err. $e := x - \tilde{x} = \pi_{\mathrm{A}_{i-1}} e + \pi_i e + \pi_{\mathrm{A}_i^*} e \in \mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}_i^*)$

Theorem (sharp upper bounds I)

Let $\tilde{x} \in \mathrm{H}_i$ and $e := x - \tilde{x}$. Then

$$|e|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_i e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2,$$

$$|\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}_{i-1}^*)} \big(c_{i-1} |\mathrm{A}_{i-1}^* \phi - g|_{\mathrm{H}_{i-1}} + |\phi - \tilde{x}|_{\mathrm{H}_i} \big),$$

reg. $(\mathrm{A}_{i-1}\mathrm{A}_{i-1}^* + 1)$*-prbl. in* $D(\mathrm{A}_{i-1}^*)$

$$|\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i} = \min_{\varphi \in D(\mathrm{A}_i)} \big(c_i |\mathrm{A}_i \varphi - f|_{\mathrm{H}_{i+1}} + |\varphi - \tilde{x}|_{\mathrm{H}_i} \big),$$

reg. $(\mathrm{A}_i^*\mathrm{A}_i + 1)$*-prbl. in* $D(\mathrm{A}_i)$

$$|\pi_i e|_{\mathrm{H}_i} = |\pi_i \tilde{x} - h|_{\mathrm{H}_i} = \min_{\substack{\xi \in D(\mathrm{A}_{i-1}),\\ \zeta \in D(\mathrm{A}_i^*)}} |\mathrm{A}_{i-1}\xi + \mathrm{A}_i^* \zeta + \tilde{x} - h|_{\mathrm{H}_i}.$$

cpld. $(\mathrm{A}_{i-1}^*\mathrm{A}_{i-1})$*-*$(\mathrm{A}_i\mathrm{A}_i^*)$*-sys. in* $D(\mathcal{A}_{i-1})$*-*$D(\mathcal{A}_i^*)$

Remark

Even $\pi_i e = h - \pi_i \tilde{x}$ and the minima are attained at

$$\hat{\phi} = \pi_{\mathrm{A}_{i-1}} e + \tilde{x}, \quad \hat{\varphi} = \pi_{\mathrm{A}_i^*} e + \tilde{x}, \quad \mathrm{A}_{i-1}\hat{\xi} + \mathrm{A}_i^* \hat{\zeta} = (\pi_i - 1)\tilde{x}.$$

Upper Bounds (with less computations)

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}^*_{i-1})$ s.t. $\mathrm{A}_i x = f, \quad \mathrm{A}^*_{i-1} x = g, \quad \pi_i x = h$

'very' conforming 'approximation' of x: $\tilde{x} \in D(\mathrm{A}_i) \cap D(\mathrm{A}^*_{i-1})$

setting $\phi = \varphi = \tilde{x}$ in latter theorem (or directly by Poincaré type estimates) $\Rightarrow$

$$\begin{aligned}|e|^2_{D(\mathrm{A}_i)\cap D(\mathrm{A}^*_{i-1})} &= |\pi_{\mathrm{A}_{i-1}} e|^2_{\mathrm{H}_i} + |\pi_i e|^2_{\mathrm{H}_i} + |\pi_{\mathrm{A}^*_i} e|^2_{\mathrm{H}_i} + |\mathrm{A}_i e|^2_{\mathrm{H}_{i+1}} + |\mathrm{A}^*_{i-1} e|^2_{\mathrm{H}_{i-1}} \\ &\leq |\pi_i \tilde{x} - h|^2_{\mathrm{H}_i} + (1 + c_i^2)|\mathrm{A}_i \tilde{x} - f|^2_{\mathrm{H}_{i+1}} + (1 + c_{i-1}^2)|\mathrm{A}^*_{i-1}\tilde{x} - g|^2_{\mathrm{H}_{i-1}}\end{aligned}$$

let

$$\mathcal{F}(\tilde{x}) := |\pi_i \tilde{x} - h|^2_{\mathrm{H}_i} + |\mathrm{A}_i \tilde{x} - f|^2_{\mathrm{H}_{i+1}} + |\mathrm{A}^*_{i-1}\tilde{x} - g|^2_{\mathrm{H}_{i-1}}$$

least squares functional of the problem. set $c_{m,i} := \max\{c_i, c_{i-1}\}$

Corollary (error equivalence with least squares functional)

Let $\tilde{x} \in D(\mathrm{A}_i) \cap D(\mathrm{A}^*_{i-1})$ and $e := x - \tilde{x}$. Then

$$\frac{1}{1 + c_{m,i}^2} |e|^2_{D(\mathrm{A}_i)\cap D(\mathrm{A}^*_{i-1})} \leq \mathcal{F}(\tilde{x}) \leq |e|^2_{D(\mathrm{A}_i)\cap D(\mathrm{A}^*_{i-1})}.$$

note $|e|^2_{D(\mathrm{A}_i)\cap D(\mathrm{A}^*_{i-1})} = \mathcal{F}(\tilde{x}) + |\pi_{\mathrm{A}_{i-1}} e|^2_{\mathrm{H}_i} + |\pi_{\mathrm{A}^*_i} e|^2_{\mathrm{H}_i}$

note: also partial results for (part. conf. approx.) $\tilde{x} \in D(\mathrm{A}_i)$ or $\tilde{x} \in D(\mathrm{A}^*_{i-1})$

Upper Bounds (without harmonic fields)

problem: find $x \in D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t. $\mathrm{A}_i x = f, \quad \mathrm{A}_{i-1}^* x = g, \quad \pi_i x = h$

'very' non-conforming 'approximation' of x: $\tilde{x} \in \mathrm{H}_i$ 1 with $\tilde{x} = \mathrm{A}_{i-1}\tilde{y} + \mathrm{A}_i^*\tilde{z} + h$

reasonable assumption (by num. method): $e = x - \tilde{x} \in R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} R(\mathrm{A}_i^*) \perp_{\mathrm{H}_i} \mathcal{H}_i$

$\Rightarrow \quad e = \pi_{\mathrm{A}_{i-1}} e + \pi_{\mathrm{A}_i^*} e \in R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} R(\mathrm{A}_i^*)$

$\Rightarrow$ no error in the 'harmonic fields' part $|\pi_i e|_{\mathrm{H}_i}$

Theorem (sharp upper bounds II)

Let $\tilde{x} \in \mathrm{H}_i$ and $e := x - \tilde{x} \perp_{\mathrm{H}_i} \mathcal{H}_i$. Then

$$|e|_{\mathrm{H}_i}^2 = |\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i}^2 + |\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i}^2,$$

$$|\pi_{\mathrm{A}_{i-1}} e|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}_{i-1}^*)} \big(c_{i-1}|\mathrm{A}_{i-1}^*\phi - g|_{\mathrm{H}_{i-1}} + |\phi - \tilde{x}|_{\mathrm{H}_i}\big),$$

$$|\pi_{\mathrm{A}_i^*} e|_{\mathrm{H}_i} = \min_{\varphi \in D(\mathrm{A}_i)} \big(c_i|\mathrm{A}_i\varphi - f|_{\mathrm{H}_{i+1}} + |\varphi - \tilde{x}|_{\mathrm{H}_i}\big).$$

no (computation of) projector π_i onto $\mathcal{H}_i$ needed!

Lower Bounds

recall problem: $\boxed{\text{find } x \in D(\mathrm{A}_i) \cap D(\mathrm{A}^*_{i-1}) \quad \text{s.t.} \quad \mathrm{A}_i x = f, \quad \mathrm{A}^*_{i-1} x = g, \quad \pi_i x = h}$

'very' non-conforming 'approximation' of x: $\boxed{\tilde{x} \in \mathrm{H}_i}$

error $\boxed{e = x - \tilde{x}} = \pi_{\mathrm{A}_{i-1}} e + \pi_i e + \pi_{\mathrm{A}^*_i} e \in \mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}^*_i)$

Theorem (sharp lower bounds)

Let $\tilde{x} \in \mathrm{H}_i$ *and* $e := x - \tilde{x}$. *Then*

$$|e|^2_{\mathrm{H}_i} = |\pi_{\mathrm{A}_{i-1}} e|^2_{\mathrm{H}_i} + |\pi_i e|^2_{\mathrm{H}_i} + |\pi_{\mathrm{A}^*_i} e|^2_{\mathrm{H}_i} \geq |\pi_{\mathrm{A}_{i-1}} e|^2_{\mathrm{H}_i} + |\pi_{\mathrm{A}^*_i} e|^2_{\mathrm{H}_i},$$

$$|\pi_{\mathrm{A}_{i-1}} e|^2_{\mathrm{H}_i} = \max_{\phi \in D(\mathrm{A}_{i-1})} \Big(2\langle g, \phi \rangle_{\mathrm{H}_{i-1}} - \langle 2\tilde{x} + \mathrm{A}_{i-1}\phi, \mathrm{A}_{i-1}\phi \rangle_{\mathrm{H}_i} \Big),$$

$$|\pi_{\mathrm{A}^*_i} e|^2_{\mathrm{H}_i} = \max_{\varphi \in D(\mathrm{A}^*_i)} \Big(2\langle f, \varphi \rangle_{\mathrm{H}_{i+1}} - \langle 2\tilde{x} + \mathrm{A}^*_i\varphi, \mathrm{A}^*_i\varphi \rangle_{\mathrm{H}_i} \Big),$$

$$\pi_i e = h - \pi_i \tilde{x}.$$

Remark

The maxima are attained at $\phi \in D(\mathrm{A}_{i-1})$ *with* $\mathrm{A}_{i-1}\phi = \pi_{\mathrm{A}_{i-1}} e$ *and* $\varphi \in D(\mathrm{A}^*_i)$ *with* $\mathrm{A}^*_i \varphi = \pi_{\mathrm{A}^*_i} e$.

Abstract Problem and Goal

problem: find $x \in D(\mathrm{A}_i^* \mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)$ s.t.

$$\begin{aligned} \mathrm{A}_i^* \mathrm{A}_i x &= f \\ \mathrm{A}_{i-1}^* x &= g \\ \pi_i x &= h \end{aligned}$$

equivalent mixed formulation ($y := \mathrm{A}_i x$):
find pair $(x, y) \in \big(D(\mathrm{A}_i) \cap D(\mathrm{A}_{i-1}^*)\big) \times \big(\underbrace{D(\mathrm{A}_i^*) \cap R(\mathrm{A}_i)}_{=D(\mathcal{A}_i^*)}\big)$ s.t.

$$\begin{aligned} \mathrm{A}_i x &= y, & \mathrm{A}_{i+1} y &= 0 \\ \mathrm{A}_{i-1}^* x &= g, & \mathrm{A}_i^* y &= f \\ \pi_i x &= h, & \pi_{i+1} y &= 0 \end{aligned}$$

cont. solution theory $\surd$: $x = \mathcal{A}_i^{-1} y + (\mathcal{A}_{i-1}^*)^{-1} g + h$ and $y = (\mathcal{A}_i^*)^{-1} f$
goal: functional a posteriori error estimates 'in the spirit of Repin'

for $(\tilde{x}, \tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$ (very non-conforming!)

estimate $|(x, y) - (\tilde{x}, \tilde{y})|_{\mathrm{H}_i \times \mathrm{H}_{i+1}}$ in terms of $\tilde{x}$, and $\tilde{y}$, f, g, h, and c_i

Upper Bounds

Theorem (sharp upper bounds)

Let $(\tilde{x}, \tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$ *and* $e := (x, y) - (\tilde{x}, \tilde{y}) \in \mathrm{H}_i \times \mathrm{H}_{i+1}$.
Then $\pi_i e_x = h - \pi_i \tilde{x}$ *and* $(1 - \pi_{\mathrm{A}_i}) e_y = -(1 - \pi_{\mathrm{A}_i}) \tilde{y}$ *and*

$$|e_x|^2_{\mathrm{H}_i} = |\pi_{\mathrm{A}_{i-1}} e_x|^2_{\mathrm{H}_i} + |\pi_i e_x|^2_{\mathrm{H}_i} + |\pi_{\mathrm{A}_i^*} e_x|^2_{\mathrm{H}_i},$$

$$|e_y|^2_{\mathrm{H}_{i+1}} = |\pi_{\mathrm{A}_i} e_y|^2_{\mathrm{H}_{i+1}} + |(1 - \pi_{\mathrm{A}_i}) e_y|^2_{\mathrm{H}_{i+1}}$$

as well as

$$|\pi_{\mathrm{A}_i} e_y|_{\mathrm{H}_{i+1}} = \min_{\theta \in D(\mathrm{A}_i^*)} \big(c_i |\mathrm{A}_i^* \theta - f|_{\mathrm{H}_i} + |\theta - \tilde{y}|_{\mathrm{H}_{i+1}} \big),$$

reg. $(\mathrm{A}_i \mathrm{A}_i^* + 1)$*-prbl. in* $D(\mathrm{A}_i^*)$

$$|(1 - \pi_{\mathrm{A}_i}) e_y|_{\mathrm{H}_{i+1}} = |(1 - \pi_{\mathrm{A}_i}) \tilde{y}|_{\mathrm{H}_{i+1}} = \min_{\xi \in D(\mathrm{A}_i)} |\mathrm{A}_i \xi - \tilde{y}|_{\mathrm{H}_{i+1}},$$

$(\mathrm{A}_i^* \mathrm{A}_i)$*-prbl. in* $D(\mathcal{A}_i)$

$$|\pi_{\mathrm{A}_{i-1}} e_x|_{\mathrm{H}_i} = \min_{\phi \in D(\mathrm{A}_{i-1}^*)} \big(c_{i-1} |\mathrm{A}_{i-1}^* \phi - g|_{\mathrm{H}_{i-1}} + |\phi - \tilde{x}|_{\mathrm{H}_i} \big),$$

reg. $(\mathrm{A}_{i-1} \mathrm{A}_{i-1}^* + 1)$*-prbl. in* $D(\mathrm{A}_{i-1}^*)$

$$|\pi_{\mathrm{A}_i^*} e_x|_{\mathrm{H}_i} = \min_{\substack{\varphi \in D(\mathrm{A}_i), \\ \psi \in D(\mathrm{A}_i^*)}} \big(|\varphi - \tilde{x}|_{\mathrm{H}_i} + c_i |\mathrm{A}_i \varphi - \psi|_{\mathrm{H}_{i+1}} + c_i^2 |\mathrm{A}_i^* \psi - f|_{\mathrm{H}_i} \big).$$

cpld. $(\mathrm{A}_i^* \mathrm{A}_i + 1)$-$(\mathrm{A}_i \mathrm{A}_i^* + 1)$*-sys. in* $D(\mathrm{A}_i)$-$D(\mathrm{A}_i^*)$

Lower Bounds

...

Electro-Magneto-Static Maxwell

$\Omega \subset \mathbb{R}^3$ bounded domain with weak Lipschitz boundary $\Gamma = \partial\Omega$

$$\begin{aligned} \operatorname{rot}_{\Gamma_t} E &= J \in \operatorname{rot} \mathsf{R}_{\Gamma_t} && \text{in } \Omega \\ -\operatorname{div}_{\Gamma_n} \varepsilon E &= j \in \operatorname{div} \mathsf{D}_{\Gamma_n} = \mathsf{L}^2 \text{ or } \mathsf{L}^2_{\perp} && \text{in } \Omega \\ \nu \times E &= 0 && \text{at } \Gamma_t \\ \nu \cdot \varepsilon E &= 0 && \text{at } \Gamma_n \\ \pi_{\mathtt{D}} E &= H \in \mathcal{H}_{\mathtt{D},\varepsilon} = \mathsf{R}_{\Gamma_t,0} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n,0} \end{aligned}$$

$\Rightarrow E \in \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n}$

$$\mathrm{A}_{i-1} := \nabla_{\Gamma_t} : \mathsf{H}^1_{\Gamma_t} \subset \mathsf{L}^2 \to \mathsf{L}^2_{\varepsilon}, \qquad \boxed{\mathrm{A}_i := \operatorname{rot}_{\Gamma_t}} : \mathsf{R}_{\Gamma_t} \subset \mathsf{L}^2_{\varepsilon} \to \mathsf{L}^2$$

$$\boxed{\mathrm{A}^*_{i-1} = -\operatorname{div}_{\Gamma_n} \varepsilon} : \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \subset \mathsf{L}^2_{\varepsilon} \to \mathsf{L}^2, \qquad \mathrm{A}^*_i = \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} : \mathsf{R}_{\Gamma_n} \subset \mathsf{L}^2 \to \mathsf{L}^2_{\varepsilon}$$

Electro-Magneto-Static Maxwell

compact embeddings:

$$D(\mathcal{A}_{i-1}) \hookrightarrow \mathrm{H}_{i-1} \quad \Leftrightarrow \quad \mathrm{H}^1_{\Gamma_t} \hookrightarrow \mathrm{L}^2 \quad \text{(Rellich's selection theorem)}$$

$$D(\mathcal{A}_i) \hookrightarrow \mathrm{H}_i \quad \Leftrightarrow \quad \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n} \subset \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathrm{D}_{\Gamma_n} \hookrightarrow \mathrm{L}^2_\varepsilon \quad \text{(Weck's selection theorem)}$$

$c_{i-1} = c_{\mathtt{fp}}$ (Friedrichs/Poincaré constant) and $c_i = c_{\mathtt{m}}$ (Maxwell constant)

$$\forall\, \varphi \in D(\mathcal{A}_{i-1}) \quad |\varphi|_{\mathrm{H}_{i-1}} \le c_{i-1} |\mathrm{A}_{i-1}\varphi|_{\mathrm{H}_i} \quad \Leftrightarrow \quad \forall\, \varphi \in \mathrm{H}^1_{\Gamma_t} \quad |\varphi|_{\mathrm{L}^2} \le c_{\mathtt{fp}} |\nabla \varphi|_{\mathrm{L}^2_\varepsilon}$$

$$\forall\, \phi \in D(\mathcal{A}^*_{i-1}) \quad |\phi|_{\mathrm{H}_i} \le c_{i-1} |\mathrm{A}^*_{i-1}\phi|_{\mathrm{H}_{i-1}} \quad \Leftrightarrow \quad \forall\, \Phi \in \varepsilon^{-1}\mathrm{D}_{\Gamma_n} \cap \nabla \mathrm{H}^1_{\Gamma_t} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathtt{fp}} |\operatorname{div} \varepsilon \Phi|_{\mathrm{L}^2}$$

$$\forall\, \varphi \in D(\mathcal{A}_i) \quad |\varphi|_{\mathrm{H}_i} \le c_i |\mathrm{A}_i\varphi|_{\mathrm{H}_{i+1}} \quad \Leftrightarrow \quad \forall\, \Phi \in \mathrm{R}_{\Gamma_t} \cap \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n} \quad |\Phi|_{\mathrm{L}^2_\varepsilon} \le c_{\mathtt{m}} |\operatorname{rot} \Phi|_{\mathrm{L}^2}$$

$$\forall\, \psi \in D(\mathcal{A}^*_i) \quad |\psi|_{\mathrm{H}_{i+1}} \le c_i |\mathrm{A}^*_i\psi|_{\mathrm{H}_i} \quad \Leftrightarrow \quad \forall\, \Psi \in \mathrm{R}_{\Gamma_n} \cap \operatorname{rot} \mathrm{R}_{\Gamma_t} \quad |\Psi|_{\mathrm{L}^2} \le c_{\mathtt{m}} |\operatorname{rot} \Psi|_{\mathrm{L}^2_\varepsilon}$$

Helmholtz decomposition:

$$\mathrm{H}_i = R(\mathrm{A}_{i-1}) \oplus_{\mathrm{H}_i} \mathcal{H}_i \oplus_{\mathrm{H}_i} R(\mathrm{A}^*_i) \quad \Leftrightarrow \quad \mathrm{L}^2_\varepsilon = \nabla \mathrm{H}^1_{\Gamma_t} \oplus_{\mathrm{L}^2_\varepsilon} \mathcal{H}_{\mathrm{D},\varepsilon} \oplus_{\mathrm{L}^2_\varepsilon} \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n}$$

orthonormal projectors:

$$\pi_{\mathrm{A}_{i-1}} : \mathrm{H}_i \to R(\mathrm{A}_{i-1}), \qquad \pi_{\mathrm{A}^*_i} : \mathrm{H}_i \to R(\mathrm{A}^*_i), \qquad \pi_i : \mathrm{H}_i \to \mathcal{H}_i$$

$$\Leftrightarrow \quad \pi_{\nabla_{\Gamma_t}} : \mathrm{L}^2_\varepsilon \to \nabla \mathrm{H}^1_{\Gamma_t}, \qquad \pi_{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}} : \mathrm{L}^2_\varepsilon \to \varepsilon^{-1} \operatorname{rot} \mathrm{R}_{\Gamma_n}, \qquad \pi_{\mathrm{D}} : \mathrm{L}^2_\varepsilon \to \mathcal{H}_{\mathrm{D},\varepsilon}$$

Electro-Magneto-Static Maxwell: Upper Bounds

Theorem (sharp upper bounds I)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming!) and* $e := E - \tilde{E}$. *Then*

$$
\begin{aligned}
|e|^2_{\mathsf{L}^2_\varepsilon} &= |\pi_{\nabla_{\Gamma_t}} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\mathcal{D}} e|^2_{\mathsf{L}^2_\varepsilon} \\
&= \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \big(c_{\mathrm{fp}} |\operatorname{div} \varepsilon\Phi + j|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \big)^2 \\
&\quad + \min_{\Phi \in \mathsf{R}_{\Gamma_t}} \big(c_{\mathrm{m}} |\operatorname{rot} \Phi - J|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon} \big)^2 \\
&\quad + \min_{\phi \in \mathsf{H}^1_{\Gamma_t}, \Psi \in \mathsf{R}_{\Gamma_n}} |\nabla\phi + \varepsilon^{-1} \operatorname{rot} \Psi + \tilde{E} - H|^2_{\mathsf{L}^2_\varepsilon}.
\end{aligned}
$$

reg. $(-\nabla_{\Gamma_t} \operatorname{div}_{\Gamma_n} + 1)$*-prbl. in* D_{Γ_n}

reg. $(\operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} + 1)$*-prbl. in* R_{Γ_t}

cpld. $(-\operatorname{div}_{\Gamma_n} \nabla_{\Gamma_t})$-$(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})$*-sys. in* $\mathsf{H}^1_{\Gamma_t}$-R_{Γ_n}

Remark

- $(\operatorname{rot}_{\Gamma_t} \operatorname{rot}_{\Gamma_n})$*-prbl. needs saddle point formulation.*
- Ω *top. trv.* $\Rightarrow \pi_{\mathcal{D}} = 0$ *and* $\mathsf{R}_{\Gamma_t,0} = \nabla \mathsf{H}^1_{\Gamma_t}$ *and* $\mathsf{D}_{\Gamma_n,0} = \operatorname{rot} \mathsf{R}_{\Gamma_n}$
- $\boxed{\Omega \textit{ convex and } \varepsilon = \mu = 1 \textit{ and } \Gamma_t = \Gamma \textit{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathrm{f}} \le c_{\mathrm{m}} \le c_{\mathrm{p}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$

Electro-Magneto-Static Maxwell: Upper Bounds

reas. asspt. (by num. meth.): $\mathsf{L}^2_\varepsilon \ni \tilde{E} = \nabla\tilde{u} + \varepsilon^{-1}\operatorname{rot}\tilde{V} + H, \quad \tilde{u} \in \mathsf{H}^1_{\Gamma_t}, \ \tilde{V} \in \mathsf{R}_{\Gamma_n}$

$\Rightarrow \quad e = E - \tilde{E} = \pi_{\nabla_{\Gamma_t}} e + \pi_{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}} e \in \nabla\mathsf{H}^1_{\Gamma_t} \oplus_{\mathsf{L}^2_\varepsilon} \varepsilon^{-1}\operatorname{rot}\mathsf{R}_{\Gamma_n} \perp_{\mathsf{L}^2_\varepsilon} \mathcal{H}_{\mathtt{D},\varepsilon}$

$\Rightarrow$ no error in the 'Dirichlet/Neumann fields' part $|\pi_{\mathtt{D}} e|_{\mathsf{L}^2_\varepsilon}$

Theorem (sharp upper bounds II)

Let $\boxed{\tilde{E} \in \mathsf{L}^2_\varepsilon}$ *(very non-conforming!) and* $e := E - \tilde{E}$. *Then*

$$|e|^2_{\mathsf{L}^2_\varepsilon} = |\pi_{\nabla_{\Gamma_t}} e|^2_{\mathsf{L}^2_\varepsilon} + |\pi_{\varepsilon^{-1}\operatorname{rot}_{\Gamma_n}} e|^2_{\mathsf{L}^2_\varepsilon}$$

$$= \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \Big(c_{\mathtt{fp}}|\operatorname{div}\varepsilon\Phi + j|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon}\Big)^2$$

$\boxed{\textit{reg. } (-\nabla_{\Gamma_t}\operatorname{div}_{\Gamma_n} + 1)\textit{-prbl. in } \mathsf{D}_{\Gamma_n}}$

$$+ \min_{\Phi \in \mathsf{R}_{\Gamma_t}} \Big(c_{\mathtt{m}}|\operatorname{rot}\Phi - J|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon}\Big)^2$$

$\boxed{\textit{reg. } (\operatorname{rot}_{\Gamma_n}\operatorname{rot}_{\Gamma_t} + 1)\textit{-prbl. in } \mathsf{R}_{\Gamma_t}}$

Remark

- *no computation of projector* $\pi_{\mathtt{D}}$ *onto* $\mathcal{H}_{\mathtt{D},\varepsilon}$ *needed!*
- $\boxed{\Omega \textit{ convex (and } \varepsilon = \mu = 1) \textit{ and } \Gamma_t = \Gamma \textit{ or } \Gamma_n = \Gamma \Rightarrow c_{\mathtt{f}} \le c_{\mathtt{m}} \le c_{\mathtt{p}} \le \frac{\operatorname{diam}_\Omega}{\pi}}$

  $\Rightarrow$ *everything is computable!*

Dirichlet/Neumann Laplace

$\Omega \subset \mathbb{R}^3$ bounded domain with weak Lipschitz boundary $\Gamma = \partial\Omega$

$$\begin{aligned} -\operatorname{div} \varepsilon \nabla u = f \in \mathrm{L}^2 & \qquad \text{in } \Omega \\ u = 0 & \qquad \text{at } \Gamma_t \\ \nu \cdot \varepsilon \nabla u = 0 & \qquad \text{at } \Gamma_n \end{aligned}$$

$$\Leftrightarrow \qquad \begin{array}{lll} \nabla u = E \in \nabla \mathrm{H}^1_{\Gamma_t} & \operatorname{rot} E = 0 \in \operatorname{rot} \mathrm{R}_{\Gamma_t} & \text{in } \Omega \\ & -\operatorname{div} \varepsilon E = f \in \mathrm{L}^2 \text{ or } \mathrm{L}^2_{\perp} & \text{in } \Omega \\ u = 0 & \nu \times E = 0 & \text{at } \Gamma_t \\ & \nu \cdot \varepsilon E = 0 & \text{at } \Gamma_n \\ & \pi_{\mathtt{D}} E = 0 \in \mathcal{H}_{\mathtt{D},\varepsilon} & \end{array}$$

$$\Rightarrow (u, E) \in \mathrm{H}^1_{\Gamma_t} \times \left(\varepsilon^{-1}\mathrm{D}_{\Gamma_n} \cap \nabla \mathrm{H}^1_{\Gamma_t}\right)$$

$$\boxed{\mathrm{A}_i := \nabla_{\Gamma_t}} : \mathrm{H}^1_{\Gamma_t} \subset \mathrm{L}^2 \to \mathrm{L}^2_{\varepsilon}, \qquad \mathrm{A}_{i+1} := \operatorname{rot}_{\Gamma_t} : \mathrm{R}_{\Gamma_t} \subset \mathrm{L}^2_{\varepsilon} \to \mathrm{L}^2$$

$$\boxed{\mathrm{A}_i^* = -\operatorname{div}_{\Gamma_n} \varepsilon} : \varepsilon^{-1}\mathrm{D}_{\Gamma_n} \subset \mathrm{L}^2_{\varepsilon} \to \mathrm{L}^2, \qquad \mathrm{A}_{i+1}^* = \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} : \mathrm{R}_{\Gamma_n} \subset \mathrm{L}^2 \to \mathrm{L}^2_{\varepsilon}$$

Dirichlet/Neumann Laplace: Upper Bounds

Theorem (sharp upper bounds)

Let $(\tilde{u}, \tilde{E}) \in \mathsf{L}^2 \times \mathsf{L}^2_\varepsilon$ *(very non-conforming!) and* $e := (u, E) - (\tilde{u}, \tilde{E}) \in \mathsf{L}^2 \times \mathsf{L}^2_\varepsilon$. *Then* $\pi_i = 0$, $\pi_{-\operatorname{div}_{\Gamma_n}\varepsilon} = \mathrm{id}$ *or* $\pi_{\mathsf{L}^2_\perp}$ *and* $(1 - \pi_{\nabla_{\Gamma_t}})e_E = -(1 - \pi_{\nabla_{\Gamma_t}})\tilde{E}$ *and*

$$|e_E|^2_{\mathsf{L}^2_\varepsilon} = |\pi_{\nabla_{\Gamma_t}} e_E|^2_{\mathsf{L}^2_\varepsilon} + |(1 - \pi_{\nabla_{\Gamma_t}})e_E|^2_{\mathsf{L}^2_\varepsilon},$$

$$|\pi_{\nabla_{\Gamma_t}} e_E|_{\mathsf{L}^2_\varepsilon} = \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \big(c_{\mathrm{fp}}|\operatorname{div}\varepsilon\Phi + f|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon}\big),$$

$$|(1 - \pi_{\nabla_{\Gamma_t}})e_E|_{\mathsf{L}^2_\varepsilon} = |(1 - \pi_{\nabla_{\Gamma_t}})\tilde{E}|_{\mathsf{L}^2_\varepsilon} = \min_{\varphi \in \mathsf{H}^1_{\Gamma_t}} |\nabla\varphi - \tilde{E}|_{\mathsf{L}^2_\varepsilon},$$

$$|e_u|_{\mathsf{L}^2} = \min_{\substack{\varphi \in \mathsf{H}^1_{\Gamma_t},\\ \Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}}} \big(|\varphi - \tilde{u}|_{\mathsf{L}^2} + c_{\mathrm{fp}}|\nabla\varphi - \Phi|_{\mathsf{L}^2_\varepsilon} + c^2_{\mathrm{fp}}|\operatorname{div}\varepsilon\Phi + f|_{\mathsf{L}^2}\big).$$

Remark

- $\tilde{E} \in \mathsf{L}^2_\varepsilon$ *approx. of* $\nabla u \Rightarrow$ *applicable to any DG-method*
- *If* $\tilde{u} \in \mathsf{H}^1_{\Gamma_t}$ *conf. approx. and* $\tilde{E} = \nabla\tilde{u}$*, then* $e_E = \nabla(u - \tilde{u}) \in \nabla\mathsf{H}^1_{\Gamma_t}$.

  $\Rightarrow \quad |e_E|_{\mathsf{L}^2_\varepsilon} = |\pi_{\nabla_{\Gamma_t}} e_E|_{\mathsf{L}^2_\varepsilon} = \min_{\Phi \in \varepsilon^{-1}\mathsf{D}_{\Gamma_n}} \big(c_{\mathrm{fp}}|\operatorname{div}\varepsilon\Phi + f|_{\mathsf{L}^2} + |\Phi - \tilde{E}|_{\mathsf{L}^2_\varepsilon}\big)$

  (well known old estimate for conforming approximations!)

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^3$ bounded weak Lipschitz domain

Electro/Magneto-Static Maxwell with mixed boundary conditions
∇-rot-div-complex (symmetry!, de Rham complex):

$$\{0\} \text{ or } \mathbb{R} \; \underset{\pi}{\overset{\iota}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{-\operatorname{div}_{\Gamma_n} \varepsilon}{\overset{\nabla_{\Gamma_t}}{\rightleftarrows}} \; \mathsf{L}^2_\varepsilon \; \underset{\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}}{\overset{\operatorname{rot}_{\Gamma_t}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{-\nabla_{\Gamma_n}}{\overset{\operatorname{div}_{\Gamma_t}}{\rightleftarrows}} \; \mathsf{L}^2 \; \underset{\iota}{\overset{\pi}{\rightleftarrows}} \; \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{array}{llllllll} \nabla_{\Gamma_t} u = A & \text{in } \Omega & | \quad \operatorname{rot}_{\Gamma_t} E = J & \text{in } \Omega & | \quad \operatorname{div}_{\Gamma_t} H = k & \text{in } \Omega & | \quad \pi v = b & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega & | \quad -\nabla_{\Gamma_n} v = B & \text{in } \Omega \end{array}$$

related sos

$$\begin{array}{llllll} -\operatorname{div}_{\Gamma_n} \varepsilon \nabla_{\Gamma_t} u = j & \text{in } \Omega & | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} \operatorname{rot}_{\Gamma_t} E = K & \text{in } \Omega & | \quad -\nabla_{\Gamma_n} \operatorname{div}_{\Gamma_t} H = B & \text{in } \Omega \\ \pi u = a & \text{in } \Omega & | \quad -\operatorname{div}_{\Gamma_n} \varepsilon E = j & \text{in } \Omega & | \quad \varepsilon^{-1} \operatorname{rot}_{\Gamma_n} H = K & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\nabla_{\Gamma_t}) \cap D(\pi) = D(\nabla_{\Gamma_t}) = \mathsf{H}^1_{\Gamma_t} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

$$D(\operatorname{rot}_{\Gamma_t}) \cap D(-\operatorname{div}_{\Gamma_n} \varepsilon) = \mathsf{R}_{\Gamma_t} \cap \varepsilon^{-1} \mathsf{D}_{\Gamma_n} \hookrightarrow \mathsf{L}^2_\varepsilon \qquad \text{(Weck's selection theorem)}$$

$$D(\operatorname{div}_{\Gamma_t}) \cap D(\varepsilon^{-1} \operatorname{rot}_{\Gamma_n}) = \mathsf{D}_{\Gamma_t} \cap \mathsf{R}_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Weck's selection theorem)}$$

$$D(\nabla_{\Gamma_n}) \cap D(\pi) = D(\nabla_{\Gamma_n}) = \mathsf{H}^1_{\Gamma_n} \hookrightarrow \mathsf{L}^2 \qquad \text{(Rellich's selection theorem)}$$

Weck's selection theorem for weak Lip. dom. and mixed bc: Bauer/P./Schomburg ('16)

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^N$ bd w. Lip. dom. or Ω Riemannian manifold with cpt cl. and Lip. boundary Γ

Generalized Electro/Magneto-Static Maxwell with mixed boundary conditions
d-d-complex (symmetry!, de Rham complex):

$$\{0\} \text{ or } \mathbb{R} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathrm{L}^{2,0} \underset{-\delta^1_{\Gamma_n}}{\overset{\mathrm{d}^0_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^{2,1} \underset{-\delta^2_{\Gamma_n}}{\overset{\mathrm{d}^1_{\Gamma_t}}{\rightleftarrows}} \cdots \underset{-\delta^{q+1}_{\Gamma_n}}{\overset{\mathrm{d}^q_{\Gamma_t}}{\rightleftarrows}} \cdots \underset{-\delta^N_{\Gamma_n}}{\overset{\mathrm{d}^{N-1}_{\Gamma_t}}{\rightleftarrows}} \mathrm{L}^{2,N} \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathbb{R} \text{ or } \{0\}$$

related fos

$$\begin{aligned} \mathrm{d}^q_{\Gamma_t} E &= F && \text{in } \Omega \\ -\delta^q_{\Gamma_n} E &= G && \text{in } \Omega \end{aligned}$$

related sos

$$\begin{aligned} -\delta^{q+1}_{\Gamma_n} \mathrm{d}^q_{\Gamma_t} E &= F && \text{in } \Omega \\ -\delta^q_{\Gamma_n} E &= G && \text{in } \Omega \end{aligned}$$

includes: EMS rot / div, Laplacian, rot rot, and more...
corresponding compact embeddings:

$$D(\mathrm{d}^q_{\Gamma_t}) \cap D(\delta^q_{\Gamma_n}) \hookrightarrow \mathrm{L}^{2,q} \qquad \text{(Weck's selection theorems)}$$

Weck's selection theorem for Lip. manifolds and mixed bc: Bauer/P./Schomburg ('17)

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

Elasticity
$\operatorname{sym}\nabla$-$\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top$-$\operatorname{Div}_{\mathbb{S}}$-complex (symmetry!):

$$\{0\} \underset{\pi}{\overset{\iota}{\rightleftarrows}} L^2 \underset{-\operatorname{Div}_{\mathbb{S}}}{\overset{\operatorname{sym}\nabla_\Gamma}{\rightleftarrows}} L^2_{\mathbb{S}} \underset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top}{\overset{\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^\top}{\rightleftarrows}} L^2_{\mathbb{S}} \underset{-\operatorname{sym}\nabla}{\overset{\operatorname{Div}_{\mathbb{S},\Gamma}}{\rightleftarrows}} L^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} RM$$

related fos ($\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^\top$, $\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top$ first order operators!)

$$\begin{array}{rl|rl|rl|rl} \operatorname{sym}\nabla_\Gamma v = M & \text{in } \Omega & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^\top M = F & \text{in } \Omega & \operatorname{Div}_{\mathbb{S},\Gamma} N = g & \text{in } \Omega & \pi v = r & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & -\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top N = G & \text{in } \Omega & -\operatorname{sym}\nabla v = M & \text{in } \Omega \end{array}$$

related sos ($\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top \operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^\top$ second order operator!)

$$\begin{array}{rl|rl|rl} -\operatorname{Div}_{\mathbb{S}}\operatorname{sym}\nabla_\Gamma v = f & \text{in } \Omega & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top \operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^\top M = G & \text{in } \Omega & -\operatorname{sym}\nabla\operatorname{Div}_{\mathbb{S},\Gamma} N = M & \text{in } \Omega \\ \pi v = 0 & \text{in } \Omega & -\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$D(\operatorname{sym}\nabla_\Gamma) \cap D(\pi) = D(\nabla_\Gamma) = H^1_\Gamma \hookrightarrow L^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

$$D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S},\Gamma}^\top) \cap D(\operatorname{Div}_{\mathbb{S}}) \hookrightarrow L^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\operatorname{Div}_{\mathbb{S},\Gamma}) \cap D(\operatorname{Rot}\operatorname{Rot}_{\mathbb{S}}^\top) \hookrightarrow L^2_{\mathbb{S}} \qquad \text{(new selection theorem)}$$

$$D(\pi) \cap D(\operatorname{sym}\nabla) = D(\nabla) = H^1 \hookrightarrow L^2 \qquad \text{(Rellich's selection theorem and Korn ineq.)}$$

two new selection theorems for strong Lip. dom.: P./Zulehner ('17)

More First and Second Order Systems (FOS & SOS)

$\Omega \subset \mathbb{R}^3$ bounded strong Lipschitz domain

General Relativity or Biharmonic Equation
$\nabla\nabla$-$\mathrm{Rot}_{\mathbb{S}}$-$\mathrm{Div}_{\mathbb{T}}$-complex (no symmetry!):

$$\{0\} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathsf{L}^2 \underset{\operatorname{div}\operatorname{Div}_{\mathbb{S}}}{\overset{\nabla\nabla_{\Gamma}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{S}} \underset{\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}}{\overset{\operatorname{Rot}_{\mathbb{S},\Gamma}}{\rightleftarrows}} \mathsf{L}^2_{\mathbb{T}} \underset{-\operatorname{dev}\nabla}{\overset{\operatorname{Div}_{\mathbb{T},\Gamma}}{\rightleftarrows}} \mathsf{L}^2 \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathsf{RT}$$

related fos ($\nabla\nabla_\Gamma$, $\operatorname{div}\operatorname{Div}_{\mathbb{S}}$ first order operators!)

$$\begin{array}{rl|rl|rl|rl} \nabla\nabla_\Gamma u = M & \text{in } \Omega & \operatorname{Rot}_{\mathbb{S},\Gamma} M = F & \text{in } \Omega & \operatorname{Div}_{\mathbb{T},\Gamma} N = g & \text{in } \Omega & \pi v = r & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega & -\operatorname{dev}\nabla v = T & \text{in } \Omega \end{array}$$

related sos ($\operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma = \Delta^2_\Gamma$ second order operator!)

$$\begin{array}{rl|rl|rl} \operatorname{div}\operatorname{Div}_{\mathbb{S}}\nabla\nabla_\Gamma u = \Delta^2_\Gamma u = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}}\operatorname{Rot}_{\mathbb{S},\Gamma} M = G & \text{in } \Omega & -\operatorname{dev}\nabla\operatorname{Div}_{\mathbb{T},\Gamma} N = T & \text{in } \Omega \\ \pi u = 0 & \text{in } \Omega & \operatorname{div}\operatorname{Div}_{\mathbb{S}} M = f & \text{in } \Omega & \operatorname{sym}\operatorname{Rot}_{\mathbb{T}} N = G & \text{in } \Omega \end{array}$$

corresponding compact embeddings:

$$\begin{aligned} D(\nabla\nabla_\Gamma) \cap D(\pi) = D(\nabla\nabla_\Gamma) = \mathsf{H}^2_\Gamma &\hookrightarrow \mathsf{L}^2 && \text{(Rellich's selection theorem)} \\ D(\operatorname{Rot}_{\mathbb{S},\Gamma}) \cap D(\operatorname{div}\operatorname{Div}_{\mathbb{S}}) &\hookrightarrow \mathsf{L}^2_{\mathbb{S}} && \text{(new selection theorem)} \\ D(\operatorname{Div}_{\mathbb{T},\Gamma}) \cap D(\operatorname{sym}\operatorname{Rot}_{\mathbb{T}}) &\hookrightarrow \mathsf{L}^2_{\mathbb{T}} && \text{(new selection theorem)} \\ D(\pi) \cap D(\operatorname{dev}\nabla) = D(\operatorname{dev}\nabla) = D(\nabla) = \mathsf{H}^1 &\hookrightarrow \mathsf{L}^2 && \text{(Rellich's selection theorem and Korn type ineq.)} \end{aligned}$$

two new selection theorems for strong Lip. dom.: P./Zulehner ('16)

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