

A Functional Analytic Perspective to the div-curl Lemma

1. Motivation

to appear in JOT,
arXiv: 1703.09593

2. An abstract point of view

3. The application to the div-curl lemma

1. Definition. (H -convergence, ~ 70-80s, Tartar, Murat)

$\Omega \subseteq \mathbb{R}^d$ open, bdd, $(a_n)_n \in (L^2(\Omega)^d) \ni a_\infty$.
 $\exists c > 0 \forall n \in \mathbb{N} : \Re a_n, \Re a_\infty \geq c, (a_n)_n$ bdd.

The $a_n \xrightarrow{H} a_\infty \Leftrightarrow$
 $\forall f \in H^{-1}(\Omega) :$

$$\left[\begin{array}{l} \forall n \in \mathbb{N} : \\ u_n \in H_0^1(\Omega) \\ -\operatorname{div} a_n \operatorname{grad} u_n = f \end{array} \right] \Rightarrow \begin{array}{l} u_n \xrightarrow{H} u_\infty \in H_0^1(\Omega) \\ a_n \operatorname{grad} u_n \rightharpoonup a_\infty \operatorname{grad} u_\infty \\ -\operatorname{div} a_\infty \operatorname{grad} u_\infty = f. \end{array}$$

Example. $a_n = a(n \cdot)$ for some $a \in L^\infty(\mathbb{R}^d)^{d \times d}$,
 $a_0 \in \mathbb{C}^{d \times d}$ $a \in [0, 1]^d$ -periodic.

Q: Does "energy" (i.e. $\langle a_n \operatorname{grad} u_n, \operatorname{grad} u_n \rangle \xrightarrow{n \rightarrow \infty} \langle a_\infty \operatorname{grad} u_\infty, \operatorname{grad} u_\infty \rangle$)
converge? Interesting: $(a_n \operatorname{grad} u_n, \operatorname{grad} u_n) \rightharpoonup$ weakly only.

A: Yes, D-C-L:
Flux

Thm. (Tartar - Murat '78) $(u_n)_n, (v_n)_n$ weakly con. in $L^2(\Omega)^d$ (2)

$(\operatorname{div} u_n)_n, ((\operatorname{curl} v_n)_n$ rel. compact in H^{-1} .

Then

$$\forall \varphi \in C_c^\infty(\Omega): \lim_{n \rightarrow \infty} \int \varphi \langle u_n, v_n \rangle \rightarrow \int \varphi \langle \lim_n u_n, \lim_n v_n \rangle$$

A to Q: $\operatorname{div} \operatorname{grad} u_n = -f$, and $\operatorname{grad} u_n = 0$.

2/ An abstract p.o.v.:

Proposition. Let $(\varphi_n)_n, (\psi_n)_n$ in H.f. $H \ni \varphi_\infty, \psi_\infty$

$$\varphi_n \rightharpoonup \varphi_\infty, \psi_n \rightharpoonup \psi_\infty \quad (n \rightarrow \infty)$$

$$\Rightarrow \langle \varphi_n, \psi_n \rangle \rightarrow \langle \varphi_\infty, \psi_\infty \rangle.$$

Proof. $(\varphi_n)_n$ bdd (UBP), w.r. $\varphi_\infty = 0$.

So

$$|\langle \varphi_n, \psi_n \rangle| \leq |\langle \varphi_n, \psi_n - \psi_\infty \rangle| + |\langle \varphi_n, \psi_\infty \rangle|$$

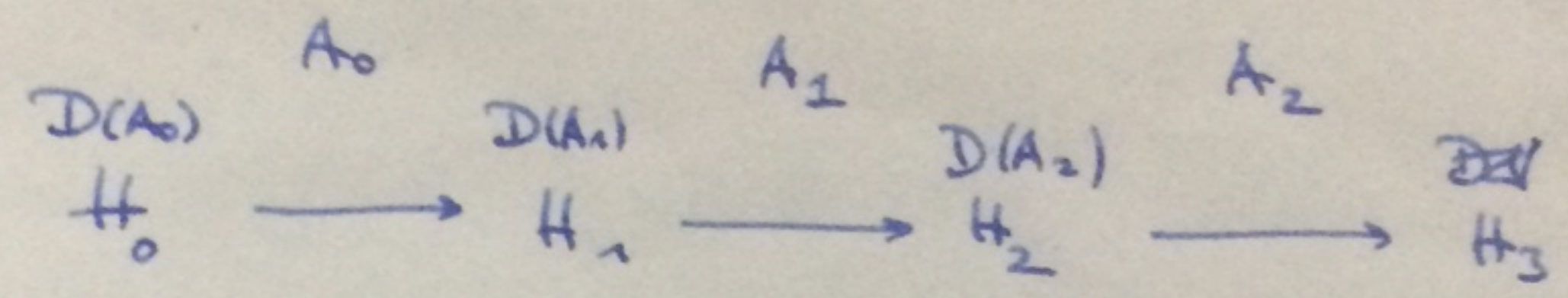
$$\leq \underbrace{\sup_k \|\varphi_k\|}_{\rightarrow 0} \underbrace{\|\psi_n - \psi_\infty\|}_{\rightarrow 0} + \underbrace{|\langle \varphi_n, \psi_\infty \rangle|}_{\rightarrow 0}$$

□

This is ^{one} ~~the~~ CORE-ARGUMENT!

The above proposition must be hidden somewhere in the assumptions.

Recall: sequences of operators in Hilbert spaces

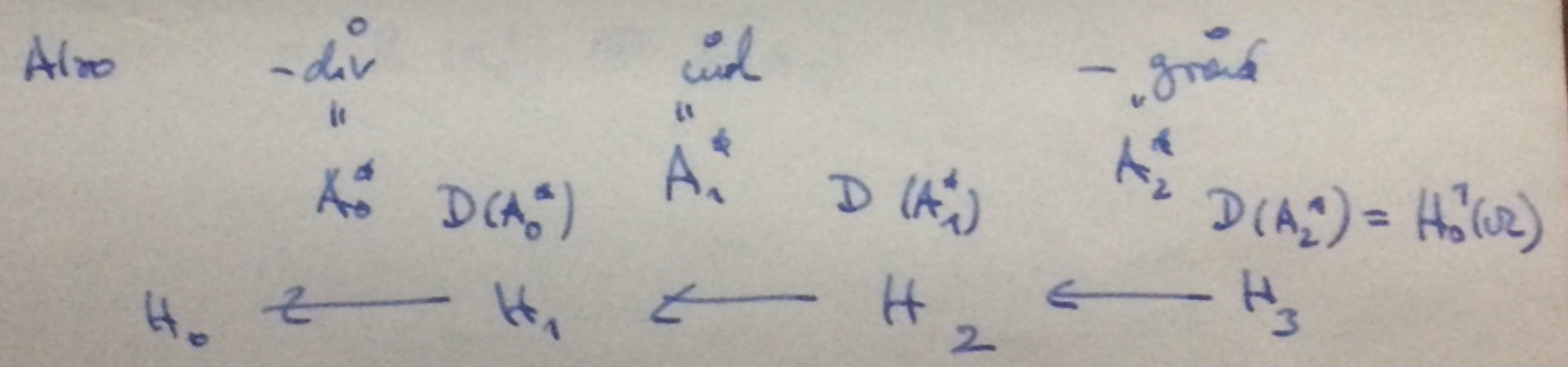


A_0, A_1, A_2 closed densely defined

$$R(A_0) \subseteq N(A_1), \quad R(A_1) \subseteq N(A_2)$$

$$A_0 = \text{grad} \quad A_1 = \text{curl} \quad A_2 = \text{div}$$

$$H_0 = L^2(\Omega) \quad H_1 = L^2(\Omega)^3 \quad H_2 = L^2(\Omega)^3$$



Ω nice: $\bullet R(A_j)$ closed ($\Rightarrow R(A_j^*)$ closed)

$$\bullet A_j : D(A_j) \cap N(A_j)^\perp \xrightarrow{N(A_j)^\perp} R(A_j)$$

continuously invertible!

$$\bullet \text{let } \varphi \in D(A_j) \text{ then } \mathcal{U}_j \left(\begin{array}{c} \perp \\ \mathcal{U}_j^* A_j \varphi = \begin{array}{c} \pi_{R(A_j)} \varphi \\ N(A_j)^\perp \end{array} \end{array} \right)$$

(Similarly for "distributional" variants)

3/ The D-C-L:

$$\dim N(A_0^*) \cap N(A_1) < \infty$$

(2nd Thm (17)) Let $(u_n)_n, (v_n)_n$ weakly converge in H_1 ,

$(A_0^* u_n)_n, (A_1 v_n)_n$ strongly convergent to ~~$D(A_0)$~~ , ~~$D(A_1)$~~ .

Then

$$\langle u_n, v_n \rangle \longrightarrow \langle \lim_{n \rightarrow \infty} u_n, \lim_{n \rightarrow \infty} v_n \rangle$$

Proof. $H_1 = R(A_0) \oplus N(A_0^*)$

$$= R(A_0) \oplus (N(A_0^*) \cap (N(A_1) \oplus R(A_1^*)))$$

$$= R(A_0) \oplus N(A_0^*) \cap N(A_1) \oplus R(A_1^*)$$

$$= R(A_1^*) \oplus N(A_1)$$

So,

$$\langle u_n, v_n \rangle = \langle \overbrace{\pi_{R(A_0)} u_n}^{\text{strongly conv.}} + \overbrace{\pi_{N(A_0^*) \cap N(A_1)} u_n}^{\text{strongly conv.}}, \pi_{R(A_1^*)} v_n + \pi_{N(A_1)} v_n \rangle$$

$\begin{array}{c} \underbrace{\pi_{R(A_0)} u_n}_{-N(A_0^*)^\perp} \quad \perp \quad \pi_{N(A_0^*) \cap N(A_1)} u_n \\ \hline \pi_{R(A_1^*)} u_n \quad \pi_{N(A_1)} v_n \end{array}$

$$\longrightarrow \langle \lim u_n, \lim v_n \rangle \quad \square$$

$$\langle \underbrace{\pi_{R(A_1^*)} v_n}_{N(A_1)^\perp} \rangle$$

strongly conv.