A Functional Analytic Perspective to the div-curl Lemma

1./ Motivation

to appear in JOT ;
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2./ An abstract point of view

3./ The application to the div-curl lemma

1./ __Definition.__ (H-convergence, ~70-80s, __Tartar__, Murat)

$\Omega \subseteq \mathbb{R}^d$ open, bdd, $(a_n)_n$ in $L(L^2(\Omega)^d) \ni a_\infty$.

$\exists c > 0 \, \forall n \in \mathbb{N}$: $\operatorname{Re} a_n, \operatorname{Re} a_\infty \geq c$, $(a_n)_n$ bdd.

Then

$$a_n \xrightarrow{H} a_\infty \; :\Longleftrightarrow$$

$\forall f \in H^{-1}(\Omega)$:

$$\left[\forall n \in \mathbb{N}: \begin{array}{c} u_n \in H_0^1(\Omega) \\ -\operatorname{div} a_n \operatorname{grad} u_n = f \end{array} \right] \Longrightarrow \begin{array}{l} u_n \longrightarrow u_\infty \in H_0^1(\Omega) \\ a_n \operatorname{grad} u_n \longrightarrow a_\infty \operatorname{grad} u_\infty \\ -\operatorname{div} a_\infty \operatorname{grad} u_\infty = f. \end{array}$$

__Example.__ $a_n = a(n\,\cdot)$ for some $a \in L^\infty(\mathbb{R}^d)^{d\times d}$,
a $[0,1)^d$-periodic.

$a_\infty \in \mathbb{C}^{d\times d}$

__Q:__ Does "energy" (e.g. $\langle a_n \operatorname{grad} u_n, \operatorname{grad} u_n \rangle \xrightarrow{n\to\infty} \langle a_\infty \operatorname{grad} u_\infty, \operatorname{grad} u_\infty \rangle$) converge? Interestingly: $\left.\begin{array}{l} a_n \operatorname{grad} u_n \\ \operatorname{grad} u_n \end{array}\right\} \xrightarrow{?}$ weakly only.

__A:__ Yes, D-C-L:

Then

<u>Thm.</u> (Tartar - Murat '78) $(u_n)_n$, $(v_n)_n$ weakly con. in $L^2(\Omega)^d$,

$(\operatorname{div} u_n)_n$, $(\operatorname{curl} v_n)_n$ rel. compact in H^{-1}.

Then

$$\forall \varphi \in C_c^\infty(\Omega): \lim_{n\to\infty} \int \varphi \langle u_n, v_n \rangle \longrightarrow \int \varphi \langle \lim_n u_n, \lim_n v_n \rangle$$

A to Q: ~~ob~~ $\operatorname{div} a_n \operatorname{grad} u_n = -f$, $\operatorname{curl} \operatorname{grad} u_n = 0$.

2.) An abstract p.o.v.:

<u>Proposition.</u> Let $(\varphi_n)_n$, $(\psi_n)_n$ in H.s. $H \ni \varphi_\infty, \psi_\infty$

$$\varphi_n \rightharpoonup \varphi_\infty, \quad \psi_n \longrightarrow \psi_\infty \quad (n \to \infty)$$

$$\Longrightarrow \quad \langle \varphi_n, \psi_n \rangle \longrightarrow \langle \varphi_\infty, \psi_\infty \rangle.$$

<u>Proof.</u> $(\varphi_n)_n$ bdd (UBP), w.r. $\varphi_\infty = 0$.

So

$$|\langle \varphi_n, \psi_n \rangle| \leq |\langle \varphi_n, \psi_n - \psi_\infty \rangle| + |\langle \varphi_n, \psi_\infty \rangle|$$

$$\leq \sup_k \|\varphi_k\| \underbrace{\|\psi_n - \psi_\infty\|}_{\to 0} + \underbrace{|\langle \varphi_n, \psi_\infty \rangle|}_{\to 0} \quad \square$$

$\boxed{\text{This is one CORE-ARGUMENT!}}$

The above proposition must be hidden somewhere in the ~~app~~ assumptions.

Recall: sequences of operators in Hilbert spaces

$$\overset{D(A_0)}{H_0} \xrightarrow{\quad A_0 \quad} \overset{D(A_1)}{H_1} \xrightarrow{\quad A_1 \quad} \overset{D(A_2)}{H_2} \xrightarrow{\quad A_2 \quad} \overset{\cancel{D(A)}}{H_3}$$

A_0, A_1, A_2 closed densely defined

$R(A_0) \subseteq N(A_1)$, $R(A_1) \subseteq N(A_2)$

$A_0 = \operatorname{grad}$ $\quad A_1 = \operatorname{curl}$ $\quad A_2 = \operatorname{div}$

$H_0 = L^2(\Omega)$ $\quad H_1 = L^2(\Omega)^3$ $\quad H_2 = L^2(\Omega)^3$

Also $\quad -\mathring{\operatorname{div}} = A_0^*$ $\quad \mathring{\operatorname{curl}} = A_1^*$ $\quad -\mathring{\operatorname{grad}} = A_2^*$

$$H_0 \xleftarrow{\quad A_0^* \quad} \overset{D(A_0^*)}{H_1} \xleftarrow{\quad A_1^* \quad} \overset{D(A_1^*)}{H_2} \xleftarrow{\quad A_2^* \quad} \overset{D(A_2^*) = H_0^1(\Omega)}{H_3}$$

Ω nice: • $R(A_j)$ closed ($\Rightarrow R(A_j^*)$ closed)

• $\mathcal{A}_j : D(A_j) \cap N(A_j)^\perp \subseteq \overset{N(A_j)^\perp}{\cancel{H_j}} \longrightarrow R(A_j)$

continuously invertible!

• Let $\varphi \in D(A_j)$ then $\cancel{\mathcal{A}_j}$ $\boxed{\mathcal{A}_j^{-1} A_j \varphi = \overset{\pi_{R(A_j^*)}}{\pi_{N(A_j)^\perp}} \varphi}$

(similarly for "distributional" variants)

3/ <u>The D-C-L:</u>

$$\dim N(A_0^*) \cap N(A_1) < \infty$$

<u>(Rh Thm</u> ('17) Let $(u_n)_n$, $(v_n)_n$ weakly converge in H_1, $(A_0^* u_n)_n$ $(A_1 v_n)_n$ strongly convergent in ~~$D(A_0^*)$~~), ~~$D(A_1)$~~).

Then

$$\langle u_n, v_n \rangle \longrightarrow \langle \lim_{n\to\infty} u_n, \lim_{n\to\infty} v_n \rangle$$

<u>Proof.</u>

$$\begin{aligned} H_1 &= R(A_0) \oplus N(A_0^*) \\ &= R(A_0) \oplus \Big(N(A_0^*) \cap \big(N(A_1) \oplus R(A_1^*)\big)\Big) \\ &= R(A_0) \oplus N(A_0^*) \cap N(A_1) \oplus R(A_1^*) \\ &= R(A_1^*) \oplus N(A_1) \end{aligned}$$

So,

$$\langle u_n, v_n \rangle = \langle \overbrace{\pi_{\substack{R(A_0) \\ = N(A_0^*)^\perp}} u_n}^{\text{strongly conv.}} + \overbrace{\pi_{\substack{N(A_0^*) \cap N(A_1) \\ \perp}} u_n}^{\text{strongly conv.}} + \pi_{R(A_1^*)} u_n \,,\ \pi_{N(A_1)} v_n + \underbrace{\pi_{\underbrace{R(A_1^*)}_{N(A_1)^\perp}} v_n}_{\text{strongly conv.}} \rangle$$

$$\longrightarrow \langle \lim u_n, \lim v_n \rangle \quad \square$$