Time-Multipatch Discontinuous Galerkin Space-Time Isogeometric Analysis of Parabolic Evolution Problems

Ulrich Langer
Institute of Computational Mathematics (NuMa)
Johannes Kepler University Linz

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (ÖAW)
Linz, Austria

AANMPDE 2017, Paleochora, Crete, 2 - 6 October 2017

Joint work with my collaborators

- Christoph Hofer (JKU, DK)
- Martin Neumüller (JKU, NuMa)
- Ioannis Toulopoulos (RICAM, CM4PDE)

Main results have just been published in

[1] U. Langer, M. Neumüller, I. Toulopoulos. Multipatch Space-Time Isogeometric Analysis of Parabolic Diffusion Problems. In *LSSC 2017 proceedings*, ed by I. Lirkov and S. Margenov, LNCS Springer series, 2017.

[2] C. Hofer, U. Langer, M. Neumüller, I. Toulopoulos. Time-multipatch discontinuous Galerkin space-time isogeometric analysis of parabolic evolution problems. *RICAM Report No. 2017-26*, and submitted.

FWF — GEOMETRY NFN S117 SIMULATION — G+SMO — JKU JOHANNES KEPLER UNIVERSITÄT LINZ — Institute of NuMa Computational Mathematics — Doctoral Program Computational Mathematics Numerical Analysis and Symbolic Computation

Outline

Parabolic Initial-Boundary Value Model Problem

Let us consider the IBVP problem: Find $u : \overline{Q} \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t u - \Delta u = f \quad \text{in} \quad Q := \Omega \times (0, T), \\ u = u_D := 0 \quad \text{on} \quad \Sigma := \partial\Omega \times (0, T), \\ u = u_0 \quad \text{on} \quad \overline{\Sigma}_0 := \overline{\Omega} \times \{0\}, \end{aligned} \tag{1}$$

as the typical model problem for a linear parabolic evolution equation posed in the space-time cylinder $\overline{Q} = \overline{\Omega} \times [0, T]$.

Our space-time technology can be generalized to more general parabolic equations like

$$-\mathrm{div}_x(A(x,t)\nabla u) + b(x,t) \cdot \nabla_x u + c(x,t)\partial_t u + a(x,t)u = f,$$

eddy-current problems, non-linear problems etc.

Parabolic Initial-Boundary Value Model Problem

Let us consider the IBVP problem: Find $u : \overline{Q} \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t u - \Delta u = f \quad &\text{in} \quad Q := \Omega \times (0, T), \\ u = u_D := 0 \quad &\text{on} \quad \Sigma := \partial\Omega \times (0, T), \\ u = u_0 \quad &\text{on} \quad \overline{\Sigma}_0 := \overline{\Omega} \times \{0\}, \end{aligned} \tag{1}$$

as the typical model problem for a linear parabolic evolution equation posed in the space-time cylinder $\overline{Q} = \overline{\Omega} \times [0, T]$.
Our space-time technology can be generalized to more general parabolic equations like

$$-\mathrm{div}_x(A(x,t)\nabla u) + b(x,t) \cdot \nabla_x u + c(x,t)\partial_t u + a(x,t)u = f,$$

eddy-current problems, non-linear problems etc.

Standard Weak Space-Time Variational Formulation

Find $u \in H_0^{1,0}(Q)$ such that

$$a(u,v) = \ell(v) \quad \forall v \in H_{0,\overline{0}}^{1,1}(Q), \tag{2}$$

with the bilinear form

$$a(u,v) = -\int_Q u(x,t)\partial_t v(x,t)dxdt + \int_Q \nabla_x u(x,t)\cdot\nabla_x v(x,t)dxdt$$

and the linear form

$$\ell(v) = \int_Q f(x,t)v(x,t)dxdt + \int_\Omega u_0(x)v(x,0)dx,$$

see Monograph by Ladyzhenskaya, Solonnikov & Uralceva (1967) or Lecture Notes by Ladyzhenskaya (1973) for solvability and regularity results !

Some References to

Time-parallel methods:

Gander (2015): Nice historical overview on 50 years time-parallel method

Parareal introduced by Lions, Maday, Turinici (2001)

Time-parallel multigrid: Hackbusch (1984),...,Vandewalle (1993),...

Gander & Neumüller (2014): smart time-parallel multigrid :

15 667 822 592 space-time dofs in 30 sec on 262 144 cores at Vulcan - BlueGene/Q, LLNL
perfect week and strong scalings

Space-time methods for parabolic evolution problems

Babuška & Janik (1989,1990), Behr (2008), Schwab & Stevenson (2009), Neumüller & Steinbach (2011), Neumüller (2013), Andreev (2013), Bank & Metti (2013), Mollet (2014), Urban & Patera (2014), Schwab & Stevenson (2014), Bank & Vassilevski (2014), Karabelas & Neumüller (2015), **Langer & Moore & Neumüller (2016)**, Bank & Vassilevski & Zikatanov (2016), Larsson & Molteni (2017), ...

Some References to

Time-parallel methods:

Gander (2015): Nice historical overview on 50 years time-parallel method
Parareal introduced by Lions, Maday, Turinici (2001)
Time-parallel multigrid: Hackbusch (1984),...,Vandewalle (1993),...
Gander & Neumüller (2014): smart time-parallel multigrid :

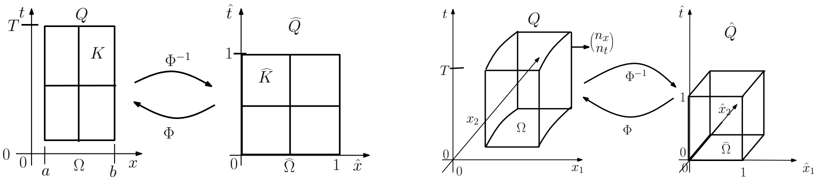
15 667 822 592 space-time dofs in 30 sec on 262 144 cores at Vulcan - BlueGene/Q, LLNL
perfect week and strong scalings

Space-time methods for parabolic evolution problems

Babuška & Janik (1989,1990), Behr (2008), Schwab & Stevenson (2009), Neumüller & Steinbach (2011), Neumüller (2013), Andreev (2013), Bank & Metti (2013), Mollet (2014), Urban & Patera (2014), Schwab & Stevenson (2014), Bank & Vassilevski (2014), Karabelas & Neumüller (2015), **Langer & Moore & Neumüller (2016)**, Bank & Vassilevski & Zikatanov (2016), Larsson & Molteni (2017), ...

Single-Patch Space-Time IgA: LMN'16

[3] U. Langer, S. Moore, M. Neumüller. Space-time isogeometric analysis of parabolic evolution equations. *Comput. Methods Appl. Mech. Engrg.*, v. 306, pp. 342–363, 2016.



Space-Time IgA paraphernalia: $Q \subset \mathbb{R}^{d+1}$; $d = 1$ (l) and $d = 2$ (r).

In this talk: Generalization to the multipatch case !

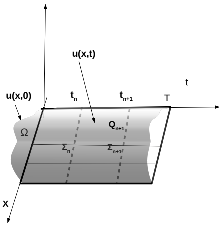
Outline

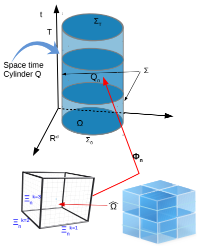
Time Multi-Patch Decomposition of Q

We decompose the space-time cylinder $Q = \Omega \times (0, T)$ into N non-overlapping space-time subcylinder $Q_n = \Omega \times (t_{n-1}, t_n) = \Phi_n(\widehat{Q})$, $n = 1, 2, \ldots, N$, such that

$$\overline{Q} = \bigcup_{n=1}^{N} \overline{Q}_n$$

with the time faces $\Sigma_n = \overline{Q}_{n+1} \cap \overline{Q}_n = \Omega \times \{t_n\}$, $\Sigma_N = \Sigma_T$:





Time Multipatch dG IgA spaces V_{0h}

We look for an approximate solution u_h to the IBVP (2) in the globally discontinuous, but patch-wise smooth IgA (B-spline, NURBS) spaces

$$V_{0h} = \{v_h \in H_0^{1,0}(Q) : v_h^n := v_h|_{Q_n} \in \mathbb{B}_{\mathbf{\Xi}_n^{d+1}}(Q_n), n = 1, \ldots, N\}$$
$$= \{v_h \in L_2(Q) : v_h^n \in V_{0h}^n, n = 1, \ldots, N\} = \mathrm{span}\{\varphi_i\}_{i \in \mathcal{I}},$$
$$V_{0h}^n = \{v_h^n \in \mathbb{B}_{\mathbf{\Xi}_n^{d+1}}(Q_n) : v_h^n = 0 \text{ on } \Sigma\} = \mathrm{span}\{\varphi_{n,i}\}_{i \in \mathcal{I}_n}$$

where $\mathbb{B}_{\mathbf{\Xi}_n^{d+1}}(Q_n)$ is the smooth (depending of the polynomial degrees and multiciplicity of the knots) IgA space corresponding to the knot vector

$$\mathbf{\Xi}_n^{d+1} = \mathbf{\Xi}_n^{d+1}(n_1^n, .., n_{d+1}^n; p_1^n, .., p_{d+1}^n) = \ldots$$

Stable Time Multipatch dG IgA Scheme

Multiplying the PDE by $v_h + \theta_n h_n \partial_t v_h$, integrating over Q_n, integrating by parts, suming over n, and using that the jumps $[|u|]$ across Σ_n are 0 at the solution $u \in H^{2,1}(Q)$, we get the multi-patch space-time scheme: Find $u_h \in V_{0h}$ such that

$$a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{0h},$$

where

$$a_h(u_h, v_h) = \sum_{n=1}^{N} \int_{Q_n} (\partial_t u_h v_h + \theta_n h_n \partial_t u_h \partial_t v_h + \nabla_x u_h \nabla_x v_h + \theta_n h_n \nabla_x u_h \cdot \nabla_x \partial_t v_h)\, dxdt$$

$$+ \sum_{n=1}^{N} \int_{\Sigma_{n-1}} [\![u_h^{n-1}]\!]\, v_{h,+}^{n-1}\, dx,$$

$$\ell_h(v_h) = \sum_{n=1}^{N} \int_{Q_n} f\,[v_h + \theta_n h_n \partial_t v_h]\, dxdt + \int_{\Sigma_0} u_0 v_{h,+}^0\, dx.$$

Space-Time IgA Scheme and System of IgA Eqns

Hence, we look for the solution $u_h \in V_{0h}$ of the IgA scheme

$$a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V_{0h} \tag{3}$$

in the form of

$$u_h(x,t) = u_h(x_1,\ldots,x_d,x_{d+1}) = \textstyle\sum_{i\in\mathcal{I}} u_i\varphi_i(x,t)$$

where $\mathbf{u}_h := [u_i]_{i\in\mathcal{I}} \in \mathbb{R}^{N_h=|\mathcal{I}|}$ is the unknown solution vector of control points defined by the solution of the linear system

$$\mathbf{L}_h\mathbf{u}_h = \mathbf{f}_h \tag{4}$$

with huge, non-symmetric, but **positive definite** system matrix $\mathbf{L}_h$.

$V_{0h}-$Coercivity of the bilinear form $a_h(\cdot,\cdot)$

We now introduce the mesh-dependent dG norm

$$\|v\|_{dG}^2 = \sum_{n=1}^{N} \left(\|\nabla_x v\|_{L_2(Q_n)}^2 + \theta_n\, h_n\, \|\partial_t v\|_{L_2(Q_n)}^2 + \frac{1}{2} \|[\![v]\!]^{n-1}\|_{L_2(\Sigma_{n-1})}^2 \right) + \frac{1}{2} \|v\|_{L_2(\Sigma_N)}^2.$$

Lemma (Coercivity / Ellipticity on V_{0h})

The bilinear form $a_h(\cdot,\cdot) : V_{0h} \times V_{0h} \to \mathbb{R}$ *is* $V_{0h}-$*coercive wrt the norm* $\|\cdot\|_h$*, i.e., there exists a constant* $\mu_c = 1/2$ *such that*

$$a_h(v_h, v_h) \geq \mu_c \|v_h\|_h^2, \quad \forall v_h \in V_{0h}. \tag{5}$$

provided that $\theta_n \leq c_{inv,0}^{-2}$*, where* $\|v_h\|_{L_2(\partial E)}^2 \leq c_{inv,0} h_n^{-1} \|v_h\|_{L_2(E)}^2$

This lemma immediately yields **uniqueness** and **existence** of the solution $u_h \in V_{0h}$ and $\mathbf{u}_h \in \mathbb{R}^{N_h}$ of (9) and (4), respectively.

Uniform Boundedness of $a_h(\cdot,\cdot)$ on $V_{0h,*} \times V_{0h}$

Let us introduce the space $V_{0h,*} = H_0^{1,0}(Q) \cap H^{2,1}(Q) + V_{0h}$ equipped with the norm

$$\|v\|_{dG,*} = \Big(\|v\|_{dG}^2 + \sum_{n=1}^{N} (\theta_n h_n)^{-1} \|v\|_{L_2(Q_n)}^2 + \sum_{n=2}^{N} \|v_-^{n-1}\|_{L_2(\Sigma_{n-1})}^2\Big)^{\frac{1}{2}}. \qquad (6)$$

Lemma (Boundedness)

The bilinear form $a_h(\cdot,\cdot)$ is uniformly bounded on $V_{0h,*} \times V_{0h}$:

$$|a_h(u, v_h)| \le \mu_b \|u\|_{dG,*} \|v_h\|_{dG}, \ \forall\, u \in V_{0h,*}, \forall\, v_h \in V_{0h}, \qquad (7)$$

with $\mu_b = \max(c_{inv,1}\, \theta_{max}, 2)$, where $\theta_{max} = \max_n\{\theta_n\} \le c_{inv,0}^{-2}$. and $c_{inv,k} = c_{inv,k}(p)$ are constants in the inverse inequalities

$$\|\partial_t \partial_{x_i} v_h\|_{L_2(E)}^2 \le c_{inv,1} h_n^{-2} \|\partial_{x_i} v_h\|_{L_2(E)}^2 \quad \textit{and} \quad \|v_h\|_{L_2(\partial E)}^2 \le c_{inv,0} h_n^{-1} \|v_h\|_{L_2(E)}^2.$$

Consistency and Galerkin orthogonality

Lemma (Consitency)

If the solution $u \in H_0^{1,0}(Q)$ *of the variational problem* (2) *belongs to* $H^{2,1}(Q)$, *then it satisfies the consistency identity*

$$a_h(u, v_h) = \ell_h(v_h) \quad \forall\, v_h \in V_{0h}. \tag{8}$$

Lemma (Galerkin orthogonality)

Let $u \in H_0^{1,0}(Q)$ *be the solution the variational problem* (2), *that belongs to* $H^{2,1}(Q)$, *and let* $u_h \in V_{0h}$ *the solution of the space-time dG IgA scheme* (9) *then it holds the Galerkin orthogonality*

$$a_h(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_{0h}. \tag{9}$$

Cea-like Discretization Error Estimate

Theorem (Cea-like Estimate)

Let the exact solution u *of (2) belong to* $H_0^{1,0}(Q) \cap H^{2,1}(Q)$, *and let* u_h *be the solution of the space-time IgA scheme (9). Then it holds the a priori discretization error estimate*

$$\|u - u_h\|_{dG} \le \Big(1 + \frac{\mu_b}{\mu_c}\Big) \inf_{v_h \in V_{0h}} \|u - v_h\|_{dG,*}, \qquad (10)$$

where $\|v\|_{dG,*} = \left(\|v\|_{dG}^2 + \sum_{n=1}^N (\theta_n h_n)^{-1} \|v\|_{L_2(Q_n)}^2 + \sum_{n=2}^N \|v_-^{n-1}\|_{L_2(\Sigma_{n-1})}^2\right)^{\frac{1}{2}}$ *and*
$$\|v\|_{dG} = \left(\sum_{n=1}^{N}\left(\frac{1}{2}\|\nabla_x v\|_{L^2(Q_n)}^2 + \theta_n\, h_n\, \|\partial_t v\|_{L^2(Q_n)}^2 + \frac{1}{2}\|[\![v]\!]\|_{L^2(\Sigma_{n-1})}^2\right) + \frac{1}{2}\|v\|_{L^2(\Sigma_N)}^2\right)^{\frac{1}{2}}.$$

Proof: $\|u - u_h\|_{dG} \le \|u - v_h\|_{dG} + \|v_h - u_h\|_{dG}$

$$\mu_c \|v_h - u_h\|_{dG}^2 \le a_h(v_h - u_h, v_h - u_h) = a_h(v_h - u, v_h - u_h)$$
$$\le \mu_b \|u - v_h\|_{dG,*} \|v_h - u_h\|_{dG} \quad \square$$

Approximation Error Estimate

Theorem (Approximation Theorem)

Let $p_n + 1 \geq \ell_n \geq 2$ and $p_n + 1 \geq m_n \geq 1$ be integers, and let $u \in L_2(Q)$ such that the restriction $u^n := u|_{Q_n}$ belongs to $H^{\ell_n, m_n}(Q_n)$ for $n = 1, \ldots, N$. Then there exists a quasi-interpolant $\Pi_h u \in V_{0h}$ such that

$$\begin{aligned}
\|u - \Pi_h u\|^2_{dG,*} =& \Big(\sum_{n=1}^{N} \Big(\|\nabla_x (u - \Pi_h^n u)\|^2_{L_2(Q_n)} + \theta_n \, h_n \, \|\partial_t (u - \Pi_h^n u)\|^2_{L_2(Q_n)} \\
&+ \frac{1}{2} \|[\![(u - \Pi_h u)^{n-1}]\!]\|^2_{L_2(\Sigma_{n-1})} \Big) + \frac{1}{2} \|u - \Pi_h^N u\|^2_{L_2(\Sigma_N)} \Big) \\
&+ \sum_{n=1}^{N} \frac{1}{\theta_n h_n} \|u - \Pi_h^n u\|^2_{L_2(Q_n)} + \sum_{n=2}^{N} \|(u - \Pi_h^{n-1} u)_-^{n-1}\|^2_{L_2(\Sigma_{n-1})} \\
\leq& \sum_{n=1}^{N} \Big(C_n \Big(h_n^{2(\ell_n - 1)} + \theta_n h_n^{2\ell_n - 1} + h_n^{2m_n - 1} + \theta_n h_n^{2m_n - 1} \Big) \|u\|^2_{H^{\ell_n, m_n}(Q_n)} \\
\leq& \sum_{n=1}^{N} \Big(\tilde{C}_n \Big(h_n^{2(\ell_n - 1)} + h_n^{2(m_n - \frac{1}{2})} \Big) \|u\|^2_{H^{\ell_n, m_n}(Q_n)}
\end{aligned}$$

A priori Discretization Error Estimate

Theorem (A priori Disscretization Error Estimate)

$$\|u-u_h\|_{dG} \leq (1+\frac{\mu_b}{\mu_c})\sum_{n=1}^{N}\left(\tilde{C}_n\left(h_n^{2(\ell_n-1)}+h_n^{2(m_n-\frac{1}{2})}\right)\|u\|^2_{H^{\ell_n,m_n}(Q_n)}\right.$$

We remark that for the case of highly smooth solutions, i.e., $p+1 \leq \min(\ell_n, m_n)$, the above estimate takes the form

$$\begin{aligned}\|u-u_h\|_{dG} &\leq C\sum_{n=1}^{N} h_n^p \|u\|_{H^{p+1,p+1}(Q_n)}\\ &\leq C\, h^p \|u\|_{H^{p+1,p+1}(Q)}\end{aligned}$$

where the last estimate holds if $u \in H^{p+1,p+1}(Q)$ and $h=\max\{h_n\}$ is assumed.

Outline

One huge system of IgA equations

Once the basis is chosen, the IgA scheme (9) can be rewritten as a huge system of algebraic equations of the form

$$\mathbf{L}_h \mathbf{u}_h = \mathbf{f}_h \tag{11}$$

for determining the vector $\mathbf{u}_h = ((u_{1,i})_{i\in\mathcal{I}_1}, \ldots, (u_{N,i})_{i\in\mathcal{I}_N}) \in \mathbb{R}^{N_h}$ of the control points of the IgA solution

$$u_h(x,t) = \sum_{i\in\mathcal{I}_n} u_{n,i}\varphi_{n,i}(x,t),\ (x,t) \in \overline{Q}_n,\ n = 1,\ldots,N,$$

solving the IgA scheme (9). The system matrix $\mathbf{L}_h$ is the usual Galerkin (stiffness) matrix, and $\mathbf{f}_h$ is the rhs (load) vector.

System Matrix $\mathbf{L}_h$

The Galerkin matrix $\mathbf{L}_h$ can be rewritten in the block form

$$\mathbf{L}_h = \begin{pmatrix} \mathbf{A}_1 & & & & \\ -\mathbf{B}_2 & \mathbf{A}_2 & & & \\ & -\mathbf{B}_3 & \mathbf{A}_3 & & \\ & & \ddots & \ddots & \\ & & & -\mathbf{B}_N & \mathbf{A}_N \end{pmatrix},$$

with the matrices

$\mathbf{A}_n := \mathbf{M}_{n,x} \otimes \mathbf{K}_{n,t} + \mathbf{K}_{n,x} \otimes \mathbf{M}_{n,t}$ for $n = 1, \ldots, N$,

$\mathbf{B}_n := \widetilde{\mathbf{M}}_{n,x} \otimes \mathbf{N}_{n,t}$ for $n = 2, \ldots, N$.

Parallel Space-Time Multigrid Solvers

Solve:

$$\mathbf{L}_h \mathbf{u}_h = \mathbf{f}_h, \qquad \text{with} \qquad \mathbf{L}_h = \begin{pmatrix} \mathbf{A}_1 & & & \\ -\mathbf{B}_2 & \mathbf{A}_2 & & \\ & \ddots & \ddots & \\ & & -\mathbf{B}_N & \mathbf{A}_N \end{pmatrix}$$

Smoother: inexact damped block Jacobi ($\omega_t = \frac{1}{2}$):

$$\mathbf{u}_h{}^{(n+1)} = \mathbf{u}_h{}^{(n)} + \omega_t \mathbf{D}_h^{-1} \left[\mathbf{f}_h - \mathbf{L}_h \mathbf{u}_h{}^{(n)} \right] \qquad \text{for } n = 1, 2, \ldots$$

with $\mathbf{D}_h := \operatorname{diag}\{\mathbf{A}_k\}_{k,1,\ldots,N}$

- Parallel w.r.t. time
- Replace $\mathbf{D}_h^{-1}$ by multigrid w.r.t space $\rightarrow$ parallel in space

Outline

Parallel Solver Studies for $d = 3$ and $p = 1$

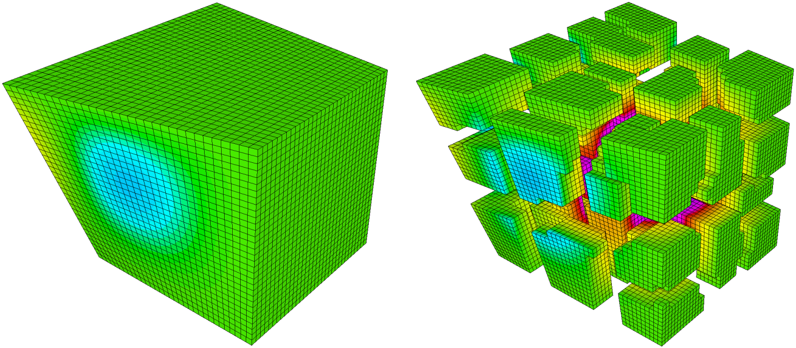


Figure: Computational spatial domain Ω decomposed into 4096 elements (left) and distributed over 32 processors (right). The IgA solution

$$u_h(x,t) \approx u(x,t) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)\sin(\pi t)$$ is plotted at $t = 0.5$.

Parallel Solver Studies for $d = 3$ and $p = 1$

Convergence results in the dG-norm for the regular solution

$$u(x,t) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)\sin(\pi t)$$

as well as iteration numbers and solving times for the parallel space-time multigrid preconditioned GMRES method on Vulcan

N	overall dof	$\|u - u_h\|_{dG}$	eoc	c_x	c_t	cores	iter	time [s]
1	1 125	3.56223E-01	-	1	1	1	1	0.03
2	13 122	1.77477E-01	1.01	1	2	2	13	1.87
4	176 868	8.86255E-02	1.00	1	4	4	15	21.47
8	2 587 464	4.42868E-02	1.00	4	8	32	15	100.48
16	39 546 000	2.21376E-02	1.00	32	16	512	17	94.32
32	618 246 432	1.10675E-02	1.00	256	32	8192	17	162.90
64	9 777 365 568	5.53340E-03	1.00	2048	64	131072	17	211.33

Parallel Solver Studies for $d = 3$ and $p = 1$

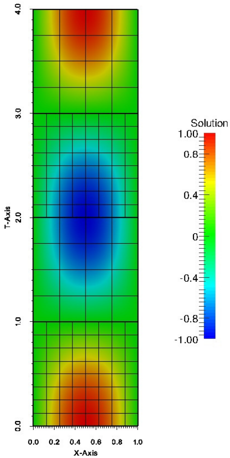
Convergence results in the dG-norm for the low regularity solution

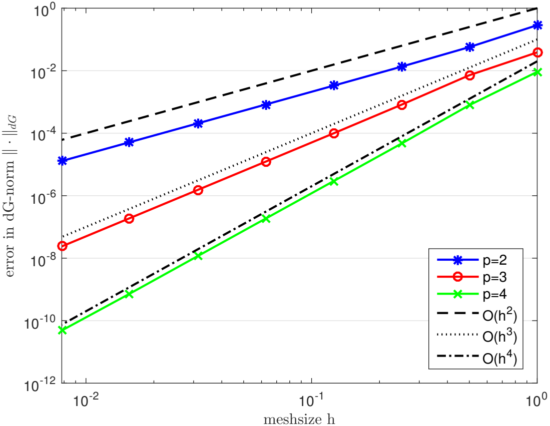
$$u(x,t) = \cos(\beta x_1)\cos(\beta x_2)\cos(\beta x_3)(1-t)^\alpha \in H^{s,\alpha+\frac{1}{2}-\varepsilon}(Q),$$

with $\alpha = 0.75$ and $\beta = 0.3$, for an arbitrary $s \geq 2$ and for an arbitrary small $\varepsilon > 0$, as well as iteration numbers and solving times for the parallel space-time multigrid preconditioned GMRES method on Vulcan BlueGene/Q at LLNL

N	overall dof	$\|u - u_h\|_{dG}$	eoc	c_x	c_t	cores	iter	time [s]
1	1 125	1.58022E-02	-	1	1	1	1	0.03
2	13 122	8.88627E-03	0.83	1	2	2	13	2.00
4	176 868	5.41668E-03	0.71	1	4	4	15	21.48
8	2 587 464	3.33881E-03	0.70	4	8	32	15	100.57
16	39 546 000	2.05545E-03	0.70	32	16	512	17	94.43
32	618 246 432	1.25859E-03	0.71	256	32	8192	17	171.83
64	9 777 365 568	7.65921E-04	0.72	2048	64	131072	17	211.49

(sp cG, mp dG) IgA for $d = 1$ and $p = 2, 3, 4$





(sp cG, mp dG) IgA for $d = 1$ and $p = 2, 3, 4$

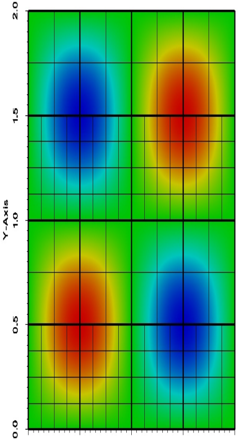
Error in the dG-norm and convergence rate for the exact solution

$$u(x,t) = \sin(\pi x)\sin(\tfrac{\pi}{2}(t+1))$$

and for B-Spline degrees $2, 3$ and 4

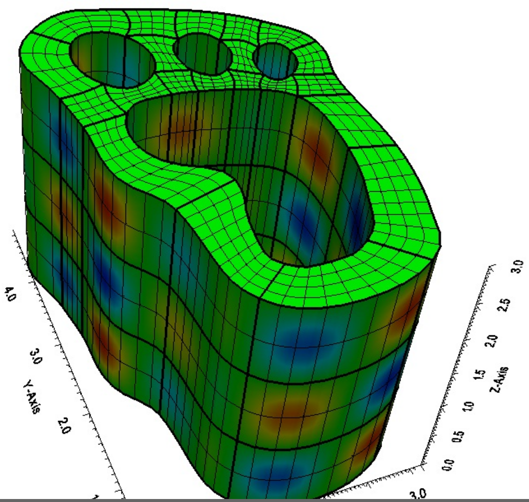
	$p=2$		$p=3$		$p=4$	
refinement	error	eoc	error	eoc	error	eoc
0	2.85633E-02	-	3.85617E-02	-	9.18731E-03	-
1	5.68232E-02	2.33	7.15551E-03	2.43	7.87619E-04	3.54
2	1.34212E-02	2.08	8.11296E-04	3.14	4.62549E-05	4.09
3	3.30721E-03	2.02	9.84754E-05	3.04	2.90675E-06	3.99
4	8.23704E-04	2.01	1.22142E-05	3.01	1.84067E-07	3.98
5	2.05716E-04	2.00	1.52376E-06	3.00	1.16139E-08	3.99
6	5.14138E-05	2.00	1.90375E-07	3.00	7.29917E-10	3.99
7	1.28522E-05	2.00	2.37936E-08	3.00	4.85647E-11	3.91

(mp cG, mp dG) IgA for $d = 1$ and $p = 3, 4$



ref.	$p = 2$ error	$p = 2$ eoc	$p = 3$ error	$p = 3$ eoc
0	0.0724164	-	0.00679753	-
1	0.0155074	2.22	0.000845039	3.01
2	0.00357715	2.12	0.000101469	3.06
3	0.000858059	2.06	1.24268e-05	3.03
4	0.000210051	2.03	1.53787e-06	3.01
5	5.19582e-05	2.02	1.91282e-07	3.01
6	1.29204e-05	2.01	2.38512e-08	3.00

Space mp cG & time mp dG IgA for $d = 2$ and $p = 2$



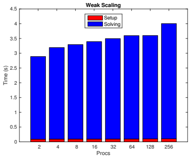
Space mp cG & time mp dG IgA for $d = 2$ and $p = 2$

Numerical experiments :

Parallelization in time - Real Schur

- Degree $p_x, p_t = (3, 3)$, refinement $r_x, r_t = (2, 3)$
- $\theta = 0.01$ and $|t_i - t_{i-1}| = 0.1$;
- $(\mathbf{K}_x + (\alpha_i + |\beta_i|)\mathbf{M}_x)^{-1} \rightsquigarrow$ Direct solver
- Direct Solver: PARDISO
- Tolerance $\varepsilon = 10^{-8}$
- #slaps = #Processors.

#dofs	#slaps	It.	Setup	Solving
48510	2	7	0.088	2.8
97020	4	7	0.092	3.1
194040	8	7	0.093	3.2
388080	16	7	0.093	3.3
776160	32	7	0.094	3.4
1552320	64	7	0.096	3.5
3104640	128	7	0.100	3.5
6209280	256	7	0.104	3.9



Outline

Conclusions & Outlook

- Space-time IgA: space singlepatch cG and time-multipatch dG
- Space-time IgA: space multipatch cG and time-multipatch dG
- Space-time IgA: Fast generation
- Space-time IgA: Fast parallel solvers
- (Functional) a posteriori estimates and THB adaptivity
  $\Longrightarrow$ talk by S. Matculevich
- space-time multipatch dG IgA + adaptivity + fast generation
  + efficient parallel solvers

Conclusions & Outlook

- Space-time IgA: space singlepatch cG and time-multipatch dG
- Space-time IgA: space multipatch cG and time-multipatch dG
- Space-time IgA: Fast generation
- Space-time IgA: Fast parallel solvers
- (Functional) a posteriori estimates and THB adaptivity
  $\Longrightarrow$ talk by S. Matculevich
- space-time multipatch dG IgA + adaptivity + fast generation + efficient parallel solvers

THANK YOU VERY MUCH !